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A NOTE ON P.P. RINGS

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In this note we present some results on divisible p.p. rings. A ring with identity is called (right) principal projective (p.p.) iff every principal (right) ideal is projective. In [3] a divisible ring is defined as a ring with identity in which every non-zero divisor is an unit. Von Neumann regular rings are right p.p. and divisible rings. It is natural to ask whether among right p.p. rings regular rings are characterized by divisible property. It is not so, which can be noted in Small's example [Remark 2]. However an annihilator condition stronger than divisible property characterizes regular rings among right p.p. rings [Theorem 1]. Various authors, namely, Endo, Harada and Small, considered the problem when a right p.p. ring is a left p.p. ring. In this connection, Endo [1] defines that a ring is normal iff its idempotents are in the center. Normal rings are natural generalizations of commutative rings and integral domains. Endo proves that the right p.p. rings are left p.p. and hence in normal case we do not distinguish right or left p.p. condition. In this note we find some equivalent conditions to this normal property. This yields a characterization of strongly regular rings (regular rings without nilpotents) and a description of finite direct sum of division rings.

Notation. If S is a subset of a ring R, then S^r and S^{rl} denote the right annihilator of S and the left annihilator of S^r respectively.

It is well known that if a ring R with identity is a right p.p. ring, then $x^r = eR$ for every x in R, where $e^2 = e$.

Theorem 1. *Let R be a ring with identity. Then the following are equivalent:*

(i) *R is a regular ring.*
(ii) *R is a right* p.p. *ring such that $(Rx)^{rl} = Rx$ for every x in R.*
(iii) *R is a left* p.p. *ring such that $(xR)^{lr} = xR$ for every x in R.*

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PROOF. It suffices to prove (ii) $\Rightarrow$ (i) since (iii) is a symmetric condition to (ii).

$$a \in (Rx)^r \Rightarrow a = 0 \quad \text{if} \quad x^r = 0 .$$

So

$$x^r = 0 \Rightarrow (Rx)^r = 0 \Rightarrow Rx = (Rx)^{rl} = 0^l = R .$$

Now if $Rx \neq R$, then $x^r \neq 0$ and $x^r = eR$, where $e^2 = e$, since R is a right p.p. ring. Also $xe = 0$ which implies $x \in R(1-e)$. But $(Rx)^r \subseteq x^r = eR$. Hence

$$(eR)^l \subseteq (Rx)^{rl} = Rx ,$$

that is, $R(1-e) \subseteq Rx$, which implies $Rx = R(1-e)$. Thus R becomes a regular ring.

REMARK 2. It can be seen from the first two lines in the above proof that the annihilator condition on R implies that R is divisible. For showing right p.p. divisible rings need not be regular, consider Small's example [5; 25]. In this example an element

$$\begin{pmatrix} a & b \\ 0 & f \end{pmatrix}$$

is regular (non-zero divisor) iff a is a unit and $f \neq 0$, and if regular it has an inverse

$$\begin{pmatrix} a' & 1 - a'bf' \\ 0 & f' \end{pmatrix}$$

where $ff' = 1$ and $aa' = 1$. So the ring is divisible and it is also right p.p. since it is shown to be right hereditary. It is not a regular ring since it has nilpotent ideals, namely

$$\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\} .$$

REMARK 3. In Theorem 1 the p.p. condition can't be dropped. For, consider the ring of integers modulo 4. This ring satisfies annihilator condition and it is a local ring.

PROPOSITION 4. *Let R be a right* p.p. *ring. Then the following are equivalent*:

(i) *R is a normal ring.*
(ii) *R has no proper nilpotents.*
(iii) *$x^r = x^l$ for every $x \in R$.*

PROOF. (i) $\Rightarrow$ (ii): Let $x^n = 0$ and $x^{n-1} \neq 0$. Then $x^r \neq 0$. Also

$0 \to x^r \xrightarrow{i} R \xrightarrow{j} xR \to 0$ is an exact sequence of right R-modules, where i is the inclusion mapping and j is defined by $r \to xr$ for every $r \in R$. Since xR is R-projective, x^r is a direct summand. Thus $x^r = eR$, where $e^2 = e$. Now $xe = 0$ and $x^{n-1} \in eR$. Hence $x^{n-1} = ex^{n-1} = x^{n-1}e = 0$, since R is normal. Therefore $x^{n-1} = 0$, a contradiction.

(ii) $\Rightarrow$ (iii): If $x^r = 0$, then $x^l = 0$ by lemma 3 of Endo [1]. Hence $x^r = x^l = 0$. If $x^r \neq 0$, $x^r = eR$ where $e^2 = e$, since R is a right p.p. ring. Since (ii) implies that idempotents in R are in the center [2; 10], $x^l = Re = eR = x^r$.

(iii) $\Rightarrow$ (i): Let e be a non-zero idempotent. Then $e^r = e^l = R(1-e) = (1-e)R$. If $x \in R$, then

$$(1-e)x = [(1-e)x](1-e) \quad \text{and} \quad x(1-e) = (1-e)[x(1-e)] .$$

So $(1-e)x = x(1-e)$ which implies $ex = xe$.

By virtue of Proposition 4 we have

COROLLARY 5. *A ring R with identity is strongly regular iff R is a normal right* p.p. *ring with $(Rx)^{rl} = Rx$ for every $x \in R$.*

Now we shall give a characterization of normal p.p. rings with ascending chain condition.

PROPOSITION 6. *Let R be a normal right Noetherian right* p.p. *ring. Then R is a finite direct sum of integral domains.*

PROOF. If R is indecomposable, then R is an integral domain. Otherwise if R is decomposable, $R = eR \oplus (1-e)R$ where $e^2 = e$. Since e is in the center, eR and $(1-e)R$ are p.p. rings with identities. By Noetherian condition, we can get only a finite number of summands and hence each direct summand is indecomposable and thus is an integral domain.

THEOREM 7. *Let R be a ring with identity. Then the following are equivalent:*

(i) *R is a finite direct sum of division rings.*

(ii) *R is a normal right Noetherian divisible* p.p. *ring.*

(iii) *R is a normal divisible* p.p. *ring in which every maximal right ideal is principal.*

PROOF. It suffices to prove that (ii) and (iii) separately $\Rightarrow$ (i). First we prove (ii) and (iii) separately imply that R is a semisimple ring (right Artinian ring with zero Jacobson radical).

Assume (ii). Since R is a normal p.p. ring, R has no nilpotents by proposition 4. Then the right Noetherian condition makes the right quotient ring of R, which is R itself, semi-simple.

Assume (iii). Let aR be a proper maximal right ideal of R. Since a is a non unit in the divisible ring R, $a^r=0$ or $a^l\neq 0$. But $a^r=0 \Leftrightarrow a^l=0$ since R is a normal right p.p. ring [1, Lemma 3]. So $a^r=eR$ where $e^2=e$ and $aR \subseteqq (1-e)R$. Since $(1-e)R \neq R$, $aR=(1-e)R$. Thus every maximal ideal is a direct summand. This implies that R is semisimple by a theorem in [4].

Now R is a semi-simple ring in both the cases (ii) and (iii) and so R is a finite direct sum of simple rings, each being isomorphic to a matrix ring over a division ring. These direct summands reduce to division rings since by hypothesis R has no nilpotents and hence the direct summands have no nilpotents.

REMARK 8. The hypotheses in (ii) and (iii) in the above theorem are minimum possible as can be seen in the counter examples of the ring of integers and the ring R in Remark 3.

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