MATH. SCAND. 25 (1969), 109–112

COMPLEMENTATION AND CONTINUITY IN SPACES OF ALMOST AUTOMORPHIC FUNCTIONS

WILLIAM A. VEECH

1. Introduction.

We prove two unrelated results:

THEOREM 1. *The space of almost periodic functions on the group of integers is uncomplemented in the space of almost automorphic functions.*

THEOREM 2. *Let f be an almost automorphic function on a locally compact group G. If f is measurable, then f is continuous.*

Recall that a complex-valued function on a group G is *almost automorphic* if every net $\alpha' = \{\alpha_{\lambda'}\}_{\lambda' \in \Lambda'}$ of group elements has a subnet $\alpha = \{\alpha_\lambda\}_{\lambda \in \Lambda}$ such that $T_\alpha f(t) = \lim_\lambda f(\alpha_\lambda t)$ exists for each $t \in G$, and also

$$T_{\alpha^{-1}} T_\alpha f(t) \;=\; \lim_\lambda T_\alpha f(\alpha_\lambda^{-1} t) \;=\; f(t) \tag{1}$$

holds for each t. Almost periodic functions are almost automorphic, but the reverse inclusion is generally false [3]. The theory of almost automorphic functions has been developed in [4].

2. Proof of Theorem 1.

We will use a construction from [3]. Let G be the group of integers, and let $G_1 \supset G_2 \supset \ldots$ be a descending sequence of proper subgroups of G. Let $1 < M_1 < M_2 < \ldots$ be a sequence of generators for these groups ($M_k \mid M_{k+1}$, $k = 1,2,\ldots$) and choose integers $a_1, a_2, \ldots$ in such a way that if $A_k = a_k + G_k$, then

(i) $A_k \cap A_l = \emptyset$, $k \neq l$,
(ii) $\bigcup_{k=1}^{\infty} A_k = G$.

Let m be the Banach space of bounded, complex-valued sequences $b = (b_1, b_2, \ldots)$. With each $b \in m$ we associate a function $f = \pi b$ on G by letting $f(x) = b_k$, $x \in A_k$. For each $b \in m$ πb is almost automorphic

Received May 25, 1968.
Research in part supported by NSF grant GP-5585.

on G [4]. If $\mathscr{A}_0$ is the space of almost automorphic functions on G with the supremum norm, then $\pi\colon m \to \mathscr{A}_0$ is an isometry into.

LEMMA. *If $b \in m$, a necessary and sufficient condition for πb to be almost periodic is that whenever $\lim_{j\to\infty}\{a_{k_j} \pmod{M_l}\}$ exists for $l=1,2,\ldots$, then also $\lim_{j\to\infty} b_{k_j}$ exists.*

PROOF. Let X be the space of infinite sequences of integers $x=(x_1,x_2,\ldots)$, where $0 \leqq x_l < M_l$, $l=1,2,\ldots$, and $x_{l+1} \equiv x_l \pmod{M_l}$. X is a compact abelian group if addition is defined coordinate-wise mod M_l. Let $\sigma\colon G \to X$ be the natural homomorphism which reduces $s \in G$ mod M_l for each l. If $f=\pi b$, $b \in m$, define $g(\sigma s)=f(s)$, $s \in G$. Evidently g is continuous on σG in the X-topology. Since f is constant on each set A_k, the condition of our lemma is necessary and sufficient for g to extend to be continuous on X. If g extends, then f is almost periodic. Conversely, since g is continuous on σG, if f is also almost periodic, g must extend to be continuous on X. For suppose $\alpha=\{\alpha_i\}$, $\beta=\{\beta_j\}$ are such that $T_\alpha f, T_\beta f$ exist and

$$\lim_{i\to\infty} \sigma\alpha_i = \lim_{j\to\infty} \sigma\beta_j \, .$$

By continuity on σG, $T_{\alpha-1} T_\beta f = f$. If f is almost periodic, then $T_\beta f$ is almost automorphic (even almost periodic) and so

$$T_\alpha f = T_\alpha T_{\alpha-1} T_\beta f = T_\beta f \, .$$

This implies that g extends to be continuous, and our lemma is proved.

PROOF OF THEOREM 1. Let $\mathscr{A}$ be the space of almost periodic functions on G, and let $\mathscr{A}_1 \subset \mathscr{A}$ be the subspace consisting of those f for which $g(\sigma s)=f(s)$ is relatively continuous. $\mathscr{A}_1$ is a closed, self-adjoint, translation invariant subalgebra of $\mathscr{A}$ containing the constants, and any such subalgebra is known to be complemented in $\mathscr{A}$. (The maximal ideal space of $\mathscr{A}$ is the Bohr compactification Y of G and that of $\mathscr{A}_1$ is Y/Y_1, where Y_1 is a closed subgroup. If ν is normalized Haar measure on Y_1, then $F \to \nu * F$ may be regarded as a projection of $C(Y)$ onto $C(Y/Y_1)$, and this projection induces a projection $P\colon \mathscr{A} \to \mathscr{A}_1$.)

Suppose $Q\colon \mathscr{A}_0 \to \mathscr{A}$ is a projection. We define $\mu\colon m \to \mathscr{A}_1$ by

$$\mu b = \pi \circ i \circ P \circ Q \circ \pi \, .$$

Here $i\colon \mathscr{A}_1 \to m$ is given by $(if)_k = f(a_k)$. If $f \in \mathscr{A}_1$, then $\pi \circ if \in \mathscr{A}_1$ by our lemma. If $b \in m$, then $P \circ Q \circ \pi b \in \mathscr{A}_1$, and therefore $\mu b \in \mathscr{A}_1$ for $b \in m$. Clearly $\pi \circ i \circ \mu b = \mu b$, $b \in m$. Also, if $f \in \mathscr{A}_1$, and if $\pi \circ i \; f = f$, then $\mu \circ i\ f = f$. The range of μ is precisely the set of $f \in \mathscr{A}_1$ for which $\pi \circ i\ f = f$.

We specialize G_k. Let $M_k = 2^k$, $k = 1, 2, \ldots$, and set $a_k = \frac{1}{3}(1 - (-2)^{k-1})$. If $a_k + G_k = A_k$, an easy induction verifies (i) and (ii). Also

$$a_{k+j} - a_k = (-2)^{k-1} a_{j+1} \equiv 0 \pmod{2^{k-1}} .$$

Thus, $\lim_{k\to\infty} \sigma a_k$ exists in X. Now by our lemma $\pi b \in \mathscr{A}$ if and only if $\lim_{k\to\infty} b_k$ exists. Letting c be the space of convergent sequences we see that $i \circ \mu$ maps m into c. Since $i \circ \mu \circ i \circ \mu\ c_1 = c_1$, $c_1 \in c$, it follows that $i \circ \mu$ is a projection of m onto c. This contradicts the classical result of Phillips [2] that c is uncomplemented in m. Theorem 1 is proved.

Remark. As would be expected Theorem 1 holds in greater generality, but we have chosen to present it on the integers where the constructions are more explicit.

3. Proof of Theorem 2.

A set $E \subset G$ is *relatively dense* if there exist elements $s_1, \ldots, s_p \in G$ with

$$\bigcup_{i=1}^{p} s_i E = G = \bigcup_{i=1}^{p} E s_i .$$

In a locally compact group a relatively dense set has positive outer (right or left) Haar measure.

Let f be almost automorphic on a group G. Given $\varepsilon > 0$ and a finite set $M \subset G$, there exists $\delta > 0$ and a finite set $N \subset G$ such that if

$$\max_{x,y\in N} |f(x\sigma_i y) - f(xy)| < \delta, \qquad i = 1, 2 ,$$

then

$$\max_{x,y\in M} |f(x\sigma_1\sigma_2^{-1}y) - f(xy)| < \varepsilon .$$

Also, for such δ, N

$$E(\delta, N) = \{\sigma \in G \mid \max_{x,y\in N} |f(x\sigma y) - f(xy)| < \delta\}$$

is relatively dense. (See respectively Lemmas 2.1.2 and 2.1.1 of [4].)

Now let f be measurable almost automorphic, and let G be locally compact. Given $x \in G$ and $\varepsilon > 0$, we set $M = \{x, e\}$, e = identity, and then find $E(\delta, N)$ for this M and ε. Being measurable and relatively dense, E has positive measure, and therefore EE^{-1} contains a neighborhood U of e. If $\sigma \in U$, we have $\sigma = \sigma_1 \sigma_2^{-1}$, $\sigma_i \in E$, $i = 1, 2$, and therefore $|f(x\sigma) - f(x)| < \varepsilon$. Thus f is continuous at x.

REMARK. For almost periodic functions Theorem 2 is a classical result of von Neumann. See also [1] for an "elementary" proof of the latter.

REMARK. It is easy to show, even on the line, that uniform continuity need not hold in Theorem 2.

REMARK. In Theorem 2 we have used only that f is almost automorphic and that $E(\delta, N)$ has positive inner measure for all $\delta > 0$ and finite N.

REFERENCES

1. H. Davis, *An elementary proof that Haar measurable almost periodic functions are continuous*, Pacific J. Math., 21 (1967), 241–248.
2. R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), 516–541.
3. W. A. Veech, *Almost automorphic functions*, Proc. Nat. Acad. Sci. U.S.A. 49 (1963), 462–464.
4. W. A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. 87 (1965), 719–751.
5. B. B. Wells, *Uncomplemented subspace of continuous functions and weak compactness of measures*, Thesis, University of California, Berkeley, 1967.

UNIVERSITY OF CALIFORNIA, BERKELEY, CALIF., U.S.A.