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NET CHARACTERIZATIONS OF EQUICONTINUITY

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Introduction.

Since the advent of equicontinuity, introduced by Ascoli, there have been numerous variations and generalizations of the original definition, cf. [1], [2], [3], [4], [7], [8]. In this paper, some characterizations of equicontinuity, simple equicontinuity, and even continuity in terms of universal nets are studied (Theorems 2, 8, 11). We also show by an example that the pointwise topology for a family of continuous functions having simple equicontinuity is not jointly continuous in general. Therefore, simple equicontinuity is not equivalent to even continuity.

In several cases we consider nets which have as their domains neighborhood bases. In all other cases the domains are arbitrary directed sets. The context will make clear what directed set we have in mind in each case, enabling us to conserve on notation by not bothering with a symbol for the domain of nets.

Throughout this paper, X will be a topological space, and Y a uniform space, unless otherwise specified. F will denote a family of continuous functions on X into Y. As a general reference for the basic definitions and notions occurring here, see [7].

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DEFINITION 1. A universal net[1] $\{f_n\}$ in F is said to be *eventually equicontinuous at* $x \in X$ if and only if, for each index V of Y, there exist a neighborhood N of x and an n_0 such that $(f_n(x), f_n(y)) \in V$ whenever $n \geqq n_0$ and $y \in N$. A universal net in F is said to be *eventually equicontinuous on* X if it is eventually equicontinuous at each point x of X.

If $A \subset X$, we denote by π_A the set of functions $\{\pi_x \mid x \in A\}$ on F into Y where $\pi_x(f)=f(x)$ for each $f \in F$. Note that every element of π_A is continuous if F is endowed with the pointwise topology.

The following is exercise 7 M (a) in [7].

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[1] Recall the remark at the end of the introduction which concerns our convention on domains of nets.

LEMMA 1. *Suppose Y is a topological space. A topology τ for F is jointly continuous, if and only if $\{f_n(x_n)\}$ converges to $f(x)$ whenever $\{x_n\}$ is a net in X converging to x and $\{f_n\}$ is a net in F, τ-converging to f.*

The following theorem is the first of the type promised in the opening remarks of this paper.

THEOREM 2. *F is equicontinuous at $x \in X$ if and only if every universal net in F is eventually equicontinuous at x.*

PROOF. The necessity is obvious.

Suppose that every universal net in F is eventually equicontinuous at x but F is not equicontinuous at x. Then there is an index V of Y such that, for each neighborhood N of x, there exist $f_N \in F$ and $x_N \in N$ such that $(f_N(x), f_N(x_N)) \notin V$. Now $\{x_N\}$ converges to x. Let $\{f_{Np}\}$ be a subnet of $\{f_N\}$ which is a universal net, then, since every universal net in F is eventually equicontinuous at x, there is a neighborhood U of x and a p_1 such that $(f_{N_p}(x), f_{N_p}(y)) \in V$ whenever $p \geqq p_1$ and $y \in U$. Since $\{x_{N_p}\}$ converges to x, there exists p_2 such that $x_{N_p} \in U$ for all $p \geqq p_2$. Choose $p_0 \geqq p_1, p_2$, then $x_{N_p} \in U$ for all $p > p_0$ and

$$(f_{N_p}(x), f_{N_p}(x_{N_p})) \in V .$$

The desired conclusion follows from this contradiction.

COROLLARY 2.1. *If every universal net in π_X is eventually equicontinuous on (F, τ), then τ is jointly continuous.*

PROOF. Let $(f,x) \in (F, \tau) \times X$ and V be an index of Y. Suppose W is a symmetric index of Y such that $W^2 \subset V$. Since f is continuous at x, there exists a neighborhood N of x such that $(f(x), f(z)) \in W$ whenever $z \in N$. By Theorem 2, π_X is equicontinuous, hence there exists a neighborhood U of f such that $(f(z), g(z)) \in W$ whenever $g \in U$ and $z \in X$. Hence, for all $g \in U$ and $z \in N$,

$$(f(x), g(z)) = (f(x), f(z)) \circ (f(z), g(z)) \in W^2 \subset V$$

and the corollary is proved.

PROPOSITION 3. *If a universal net $\{f_n\}$ in F is eventually equicontinuous at each point x of a compact space X, then it is eventually uniformly equicontinuous on X.*

PROOF. Let V be an index of Y and W a symmetric index of Y such that $W^2 \subset V$. For each $x \in X$, the net $\{f_n\}$ is eventually equicontinuous at x, there exist a neighborhood N_x of x and an n_x such that $(f_n(x), f_n(y)) \in W$ whenever $n \geqq n_x$ and $y \in N_x$. There is a finite subset $\{x_1, x_2, \ldots, x_k\}$ of X such that $X = \bigcup_{i=1}^{k} N_{x_i}$. Let $n_0 \geqq n_{x_1}, \ldots, n_{x_k}$, and let y, z be an arbitrary pair of elements of N_{x_i}, $1 \leqq i \leqq k$, then, for each $n \geqq n_0$, we have $(f_n(y), f_n(x_i)) \in W$ and $(f_n(x_i), f_n(z)) \in W$. It follows that $(f_n(y), f_n(z)) \in V$ whenever $n \geqq n_0$. Let $\mathscr{V}$ be the uniformity of X compatible with the compact topology for X. Then, by the uniform cover property, there exists an index $U \in \mathscr{V}$ such that, for each $x \in X$, $U[x] \subset N_{x_i}$ for some i, $1 \leqq i \leqq k$. Now for $n \geqq n_0$ and $(x, y) \in U$ we have $(f_n(x), f_n(y)) \in V$. This shows that $\{f_n\}$ is eventually uniformly equicontinuous on X and the proof is complete.

Proposition 4. *If $\{\pi_{x_n}\}$ is a universal net in π_X such that $\{x_n\}$ converges to $x \in X$, and a topology τ for F is jointly continuous, then $\{\pi_{x_n}\}$ is eventually equicontinuous on (F, τ).*

Proof. Let $f \in F$ and V be an index of Y. Let W be a symmetric index of Y such that $W^2 \subset V$. There exist neighborhoods N_f and N_x of f and x respectively such that $\pi(N_f \times N_x) \subset W[f(x)]$. Since $\{x_n\}$ converges to x, $\{f(x_n)\}$ converges to $f(x)$. There exists n_1 such that $n \geqq n_1$ implies that $(f(x_n), f(x)) \in W$. There exists an n_2 such that $x_n \in N_x$ for all $n \geqq n_2$. Choose $n_0 \geqq n_1, n_2$. Then, for every $g \in N_f$ and $n \geqq n_0$, $(f(x), g(x_n)) \in W$ and $(f(x_n), f(x)) \in W$. It follows that $(f(x_n), g(x_n)) \in V$ for every $n \geqq n_0$ and $g \in N_f$. This completes the proof.

Corollary 4.1. *If X is compact and a topology τ for F is jointly continuous, then π_X is equicontinuous on (F, τ).*

Proposition 4 has the following dual.

Proposition 5. *If $\{f_n\}$ is a universal net in F, τ-converging to $f \in F$, and τ is jointly continuous, then $\{f_n\}$ is eventually equicontinuous on X.*

We note that Theorem 7.16 of [7] may be regarded as a corollary to Proposition 5.

The following theorem is a filter characterization of equicontinuity.

Theorem 6. *F is equicontinuous at $x \in X$ if and only if every ultrafilter $\mathscr{H}$ in F satisfies: If V is an index of Y, there exist $H \in \mathscr{H}$ and a neighborhood N of x such that $(f(x), f(y)) \in V$ for all $f \in H$ and $y \in N$.*

Proof. Suppose every universal net in F is eventually equicontinuous at $x \in X$, but not every ultrafilter in F has the stated property. Then, for some ultrafilter $\mathscr{H}$ in F, there is an index V of Y such that for some

$H \in \mathscr{H}$ and each neighborhood N of x there exist $x_N \in N$ and $f_N \in H$ such that $(f_N(x), f_N(x_N)) \notin V$. There is a subnet $\{f_{N_p}\}$ of the net $\{f_N\}$ which is a universal net in F. The corresponding net $\{x_{N_p}\}$ is a subnet of $\{x_N\}$. There exist a neighborhood N_x of x and a p_1 such that $(f_{N_p}(x), f_{N_p}(y)) \in V$ for all $y \in N_x$ and $p \geqq p_1$. Since $\{x_{N_p}\}$ converges to x, there exists p_2 such that $x_{N_p} \in N_x$ for all $p \geqq p_2$. Choose $p_0 \geqq p_1, p_2$. It follows that $x_{N_p} \in N_x$ for all $p \geqq p_0$, hence $(f_{N_p}(x), f_{N_p}(x_{N_p})) \in V$ for all $p \geqq p_0$, which is a contradiction.

For the reverse implication, assume that every ultrafilter in F has the stated property, and let $\{f_n\}$ be a universal net in F. Let $F_n = \{f_m \mid m \geqq n\}$ and let $\mathscr{H}$ be the ultrafilter in F such that $\mathscr{H} \supset \{F_n\}$. Then, for each index V of Y, there is $H \in \mathscr{H}$ and a neighborhood N of x such that $(g(x), g(y)) \in V$ whenever $g \in H$ and $y \in N$. There exists an n_0 such that $H \supset F_{n_0}$. Then, for each $g \in F_{n_0}$ and $y \in N$, we have $(g(x), g(y)) \in V$. Therefore, for each $n \geqq n_0$ and each $y \in N$, we have $f_n \in F_{n_0}$ and $(f_n(x), f_n(y)) \in V$. This shows that the net $\{f_n\}$ is eventually equicontinuous at $x \in X$, and the proof is completed.

Proposition 7. *If X is a uniform space, then a family F is uniformly equicontinuous on X if and only if every universal net in F is eventually uniformly equicontinuous on X.*

Proof. Suppose F is uniformly equicontinuous and $\{f_n\}$ is a universal net in F. For each index V of Y there is an index U of X such that $(f(x), f(y)) \in V$ whenever $f \in F$ and $(x, y) \in U$. It follows specially that $\{f_n\}$ is eventually uniformly equicontinuous on X.

Conversely, suppose every universal net in F is eventually uniformly equicontinuous on X, but F is not uniformly equicontinuous. Then there is an index V of Y so that, for each index U of X, there are x_U, z_U in X and $f_U \in F$ such that $(x_U, z_U) \in U$ but $(f_U(x_U), f_U(z_U)) \notin V$. If we introduce an ordering in the uniformity of X by defining $U_1 \leqq U_2$ if and only if $U_1 \supset U_2$ for every pair of indices U_1, U_2 of X. Then the uniformity along with such an ordering is a directed set, and $\{f_U\}$ is a net in F. Suppose $\{f_{U_n}\}$ is a subnet of $\{f_U\}$ which is a universal net. Then there exist an index W of X and an n_1 such that $(f_{U_n}(x), f_{U_n}(z)) \in V$ whenever $n \geqq n_1$ and $(x, z) \in W$. Corresponding to W there exists n_0 such that, if $n \geqq n_0 \geqq n_1$, then $U_n \geqq W$. For $n \geqq n_0$, $(x, z) \in U_n$ implies $(x, z) \in W$; hence $(f_{U_n}(x), f_{U_n}(z)) \in V$. Specially, we have that $((f_{U_n}(x_{U_n}), f_{U_n}(z_{U_n})) \in V$, which is a contradiction.

Using Propositions 3 and 7 we have the following well-known result as our corollary.

COROLLARY 7.1. *If F is equicontinuous on a compact space X, then it is uniformly equicontinous on X.*

DEFINITION 2. F is said to have *simple equicontinuity at* $x \in X$ if, for every index V of Y and every ultrafilter $\mathscr{H}$ in F there exists a neighborhood N of x such that, for each $y \in N$, there exists $H_y \in \mathscr{H}$ such that $(f(x), f(y)) \in V$ for all $f \in H_y$.

The concept of simple equicontinuity was first studied by Brace [2]. Brace obtained a strengthened version of the Alaoglu–Bourbaki Theorem, replacing equicontinuity by simple equicontinuity. Comparing Definition 2 with Theorem 6 we can see that simple equicontinuity is clearly weaker than equicontinuity. The following examples show that simple equicontinuity is not equivalent to equicontinuity, even if X is compact. Furthermore, the examples also show that the pointwise topology for a family having simple equicontinuity is not jointly continuous in general. Therefore simple equicontinuity is not equivalent to even continuity.

EXAMPLE 1. Let X be the space of all real sequences with all but finitely many terms equal to zero. Let X have the norm topology with $||x|| = \sum\{|x_n| \mid n \in \omega\}$. If $f_n(x) = nx_n$, then the sequence $\{f_n\}$ converges to zero relative to the pointwise topology. Let $F = \{f_n\} \cup \{0\}$. Then F is compact in the pointwise topology, hence F has simple equicontinuity by Theorem 4.7 of [2]. However, F is not equicontinuous.

EXAMPLE 2. Let f be a continuous real-valued function on the closed unit interval such that $f(0) = f(1) = 0$ and $f \not\equiv 0$. Let $g_n(x) = f(x^n)$ for each non-negative integer n. Then $\{g_n\}$ converges pointwise (but not uniformly) to the function h which is identically zero. The set $F = \{g_n\} \cup \{h\}$ is compact relative to the pointwise topology, but is not compact relative to the uniform topology. Since X is compact, the uniform topology coincides with the compact-open topology. Therefore the pointwise topology is smaller than the compact-open topology. Consequently the pointwise topology is not jointly continuous and F is not equicontinuous. However F has simple equicontinuity by Theorem 4.7 of [2].

THEOREM 8. *F has simple equicontinuity at $x \in X$ if and only if, for every universal net $\{f_n\}$ in F and every index V of Y, there exists a neighborhood N of x so that, for each $y \in N$, there exists n_y such that $(f_n(x), f_n(y)) \in V$ whenever $n \geqq n_y$.*

PROOF. Suppose F has simple equicontinuity at $x \in X$, and let $\{f_n\}$ be a universal net in F. Let $F_n = \{f_m \mid m \geqq n\}$ and let $\mathscr{H}$ be the ultrafilter in F such that $\mathscr{H} \supset \{F_n\}$. Then for each index V of Y, there exists a neighborhood N of x and for each $y \in N$, there exists $H_y \in \mathscr{H}$ such that $(f(x), f(y)) \in V$ for all $f \in H_y$. There is an n_y such that $H \supset F_{n_y}$. It follows that $(f_n(x), f_n(y)) \in V$ for all $n \geqq n_y$.

Suppose, conversely, that F has the stated property but does not have simple equicontinuity at $x \in X$. Then, there exist an index V of Y and an ultrafilter $\mathscr{H}$ in F such that, for each neighborhood N of x, there exists $x_N \in N$ and for each $H \in \mathscr{H}$, $(f_{N,H}(x), f_{N,H}(x_N)) \notin V$ for some $f_{N,H} \in H$. Define an ordering in $\mathscr{N} \times \mathscr{H}$ such that $(N_1, H_1) \geqq (N_2, H_2)$ if and only if $H_1 \subset H_2$, where $\mathscr{N}$ is the neighborhood filter for x. Now $\{f_{N,H}\}$ is a net on $\mathscr{N} \times \mathscr{H}$. Let $\{f_p\}$ be a universal subnet of $\{f_{N,H}\}$. Then there exists a neighborhood N_0 of x such that, for each $z \in N_0$, there exists p_z such that $(f_p(x), f_p(z)) \in V$ whenever $p \geqq p_z$. Hence there is $H \in \mathscr{H}$ such that $(f_{N_0,H}(x), f_{N_0,H}(x_{N_0})) \in V$, which is a contradiction.

We note that the property stated in Theorem 8 is the ε-related condition defined in [8] and was essentially given by E. W. Hobson [5, p. 409] as a necessary and sufficient condition for interchange of order in repeated limits.

LEMMA 9. (Cf. [7, p. 241].) *A family F of continuous functions on X to Y is evenly continuous if and only if, for each net $\{(f_n, x_n)\}$ in $F \times X$ such that $\{x_n\}$ converges to x and $\{f_n(x)\}$ converges to y, it is true that $\{f_n(x_n)\}$ converges to y.*

DEFINITION 3. A net $\{f_n\}$ in F is said to be *uniformly Cauchy at $x \in X$* if, for every index V of Y, there exist a neighborhood N of x and an n_0 such that $(f_n(y), f_m(y)) \in V$ whenever $m, n \geqq n_0$ and $y \in N$.

PROPOSITION 10. *If F is equicontinuous at $x \in X$, then every universal net in F which is Cauchy at x is uniformly Cauchy at x. The converse also holds if $F(x)$ is a totally bounded subset of Y.*

PROOF. Suppose F is equicontinuous at x and $\{f_n\}$ is a universal net in F such that $\{f_n(x)\}$ is Cauchy in Y. Let V be an index of Y and W a symmetric index of Y such that $W^3 \subset V$. Since F is equicontinuous at x, there exists a neighborhood N of x such that $(f(x), f(z)) \in W$ for all $z \in W$ and $f \in F$. There exists n_0 such that $(f_p(x), f_q(x)) \in W$ for $p, q \geqq n_0$. Hence, for every $z \in N$ and $p, q \geqq n_0$, we have

$$(f_p(z), f_q(z)) = (f_p(z), f_p(x)) \circ (f_p(x), f_q(x)) \circ (f_q(x), f_q(z)) \in W^3 \subset V ,$$

and the first statement is proved.

Conversely, suppose $F(x)$ is totally bounded and every universal net in F which is Cauchy at x is uniformly Cauchy at x. If F is not equicontinuous at x, there exists an index V of Y such that, for each neighborhood N of x, there are $x_N \in N$ and $f_N \in F$ such that $(f_N(x_N), f_N(x)) \notin V$. Let W be a symmetric index of Y such that $W^3 \subset V$, and let $\{f_{N_p}\}$ be a universal subnet of $\{f_N\}$. Then $\{f_{N_p}(x)\}$ is a universal net in $F(x)$, hence a Cauchy net in $F(x)$. There exist a neighborhood N_1 of x and a p_1 such that, whenever $z \in N_1$ and $p, q \geqq p_1$, we have $(f_{N_p}(z), f_{N_q}(z)) \in W$. Since $\{x_{N_p}\}$ converges to x, there exists p_2 such that $x_{N_p} \in N_1$ for all $p \geqq p_2$. Choose $p \geqq p_1, p_2$. Then there exists a neighborhood N_2 of x such that $(f_{N_p}(z), f_{N_p}(x)) \in W$ whenever $z \in N_2$. Let $N = N_1 \cap N_2$. There is $p_0 > p_1$ such that $x_{N_{p_0}} \in N$. Now,

$$\begin{aligned}&(f_{N_{p_0}}(x_{N_{p_0}}), f_{N_{p_0}}(x)) \\ &\quad = (f_{N_{p_0}}(x_{N_{p_0}}), f_{N_p}(x_{N_{p_0}})) \circ (f_{N_p}(x_{N_{p_0}}), f_{N_p}(x)) \circ (f_{N_p}(x), f_{N_{p_0}}(x)) \in W^3 \subset V.\end{aligned}$$

This contradicts our assumption that F is not equicontinuous.

The following theorem characterizes even continuity in terms of nets when Y is a uniform space.

THEOREM 11. *If Y is a uniform space, then F is evenly continuous if and only if, for every $x \in X$, every universal net in F which converges at x is uniformly Cauchy at x.*

PROOF. Suppose every universal net in F which converges at x is uniformly Cauchy at x for each $x \in X$, and suppose F is not evenly continuous. Then there exists a net $\{(f_n, x_n)\}$ in $F \times X$ such that $\{x_n\}$ converges to x and $\{f_n(x)\}$ converges to y, but $\{f_n(x_n)\}$ does not converge to y. There is an index V of Y such that, for each n, there exists $p \geqq n$ such that $(f_p(x_p), y) \notin V$. Let W be a symmetric index of Y such that $W^3 \subset V$, and let $\{(f_{p_q}, x_{p_q})\}$ be a subnet of $\{(f_p, x_p)\}$ which is universal net. Then $\{f_{p_q}\}$ is a universal net in F and $\{f_{p_q}(x)\}$ converges to y. Thus $\{f_{p_q}\}$ is uniformly Cauchy at x and there exist a neighborhood N_1 of x and a k such that $q, r \geqq k$ and $z \in N_1$ imply that $(f_{p_q}(z), f_{p_r}(z)) \in W$ and $(f_{p_q}(x), y) \in W$. Then there exists a neighborhood N_2 of x such that $(f_{p_k}(z), f_{p_k}(x)) \in W$ whenever $z \in N_2$. Choose q_1 such that $x_{p_q} \in N = N_1 \cap N_2$ for each $q \geqq q_1$. Now if, $q \geqq k.q_1$, then

$$(f_{p_q}(x_{p_q}), y) \\ = (f_{p_q}(x_{p_q}), f_{p_k}(x_{p_q})) \circ (f_{p_k}(x_{p_q}), f_{p_k}(x)) \circ (f_{p_k}(x), y) \in W^3 \subset V ,$$

which is a contradiction and the sufficiency is then proved.

For the necessity, let F be evenly continuous, and suppose $\{f_n\}$ is a universal net in F which converges at x to $y \in Y$. Let V be an index of Y and W a symmetric index of Y such that $W^2 \subset V$. There exists a neighborhood N of x and an index U of Y such that, whenever $z \in N$ and $(f(x), y) \in U$, we have $(y, f(z)) \in W$. Since $\{f_n(x)\}$ converges to y, there exists an n_0 such that $(f_n(x), y) \in U$ for all $n \geqq n_0$; hence, for all $z \in N$ and $m, n \geqq n_0$, we have $(f_m(z), y) \in W$ and $(f_n(z), y) \in W$. It follows that $(f_m(z), f_n(z)) \in W^2 \subset V$ whenever $z \in N$ and $m, n \geqq n_0$. This completes the proof.

COROLLARY 11.1. *If F is evenly continuous and $\overline{F(x)}$ is a complete subset of Y, then every universal net in F which is Cauchy at x is uniformly Cauchy at x.*

COROLLARY 11.2. *If every universal net in F which is Cauchy at x is uniformly at x for each $x \in X$, then F is evenly continuous.*

As a consequence of Proposition 10 and Corollary 11.1, we have the following Proposition, which is Theorem 7.23 of [7].

PROPOSITION 12. *If F is evenly continuous and $F(x)$ is relatively compact, then F is equicontinuous at x.*

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