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MEASURE THEORY FOR C^* ALGEBRAS IV

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This paper deals with the problem of finding for a self-adjoint bounded functional on a C^* algebra A, or more generally for a self-adjoint C^* integral on A, a unique decomposition into mutually singular positive parts. For bounded functionals this is of course not new, but our proof has the merit of avoiding von Neumann algebra techniques, working within A itself. However, for the unbounded case we shall need the enveloping von Neumann algebra A''. We assume first that A has an identity element.

PROPOSITION 1. *If f and g are bounded positive functionals on A, the following conditions are equivalent*:

(1) $\|f-g\| = \|f\| + \|g\|$.

(2) *For any $\varepsilon > 0$ there exists $z \in A$, $0 \leqq z \leqq 1$, such that $f(z) < \varepsilon$ and $g(1-z) < \varepsilon$.*

If f and g are extended to A'', these conditions are again equivalent to:

(3) *There exists $z \in A''$, $0 \leqq z \leqq 1$, such that $\tilde{f}(z) = \tilde{g}(1-z) = 0$.*

(4) *f and g have orthogonal supports in A''.*

PROOF. (1) $\Rightarrow$ (2). For $\varepsilon > 0$ there exists $x \in A$, $-1 \leqq x \leqq 1$, such that

$$f(1) + g(1) = \|f-g\| < \varepsilon + (f-g)(x) .$$

We put $z = \frac{1}{2}(1-x)$, $1-z = \frac{1}{2}(1+x)$ and have

$$f(z) + g(1-z) < \tfrac{1}{2}\varepsilon .$$

(2) $\Rightarrow$ (1). Since $-1 \leqq 1-2z \leqq 1$, we have

$$\|f-g\| \geqq (f-g)(1-2z) = f(1-2z) + g(1-2(1-z)) \geqq \|f\| + \|g\| - 4\varepsilon .$$

(2) $\Rightarrow$ (3). Let $\{\varepsilon_n\}$ be a sequence tending to zero. Since the positive part of the unit ball of A'' is compact, we can choose a weak limit point z of the corresponding sequence $\{z_n\}$. As $\tilde{f}$ and $\tilde{g}$ are normal on A'', we have $\tilde{f}(z) = \tilde{g}(1-z) = 0$.

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(3) $\Rightarrow$ (4). If p and q are the supports of f and g respectively, we have $z \leqq 1-p$ and $1-z \leqq 1-q$ hence $p+q \leqq 1$ and $p \perp q$.

(4) $\Rightarrow$ (1). With p and q as above we have

$$\|f-g\| \geqq (\tilde{f}-\tilde{g})(p-q) = \|f\|+\|g\| .$$

When f and g satisfy the conditions of the proposition we call them mutually *singular* and write $f \perp g$.

Proposition 2. *For any self-adjoint bounded functional h there exists a unique decomposition*

$$h = f-g, \qquad f \geqq 0,\ g \geqq 0,\ f \perp g .$$

Proof. The existence part of the proposition is [3, Corollaire 2.6.4]. We prove the uniqueness assertion [3, Corollaire 12.3.4]. Suppose therefore

$$f-g = f'-g', \quad f \perp g, \quad f' \perp g' .$$

Since obviously $\|f\|+\|g'\| = \|f'\|+\|g\|$, we get by condition (1) in proposition 1 that $\|f\| = \|f'\|$ and $\|g\| = \|g'\|$. For $\varepsilon > 0$ we choose z to f and g satisfying condition (2) in proposition 1. Then

$$\begin{aligned} \|f'\| &\geqq f'(1-z) \geqq (f'-g')(1-z) \\ &= (f-g)(1-z) \geqq f(1)-2\varepsilon = \|f'\|-2\varepsilon . \end{aligned}$$

Hence $f'(z) < 2\varepsilon$ and similarly $g'(1-z) < 2\varepsilon$. By Cauchy–Schwarz' inequality we get for any $x \in A^+$

$$|f(zx(1-z))|^2 \leqq f(zxz)\, f((1-z)x(1-z)) \leqq \varepsilon \|f\|\, \|x\|^2$$

and similar expressions for g, f' and g'. Moreover

$$|(f-f')(zxz)| \leqq 3\varepsilon \|x\|, \qquad |(g-g')((1-z)x(1-z))| \leqq 3\varepsilon \|x\| .$$

Since by assumption $f-f' = g-g'$, we get

$$|(f-f')x| = |(f-f')((z+1-z)x(z+1-z))| \leqq \delta \|x\| ,$$

where δ can be chosen arbitrary small. Hence $f=f'$ and $g=g'$.

We recall that a *Baire* algebra* is a monotone σ-closed C^* algebra [5], and a *Σ^* algebra* is a weakly σ-closed C^* algebra [2] (probably the notions coincide). A functional on a Baire*algebra is *σ-normal* if it preserves limits of monotone sequences, and a functional on a Σ^* algebra is *σ-continuous* if it preserves limits of weakly convergent sequences. With these notions in mind we can state

PROPOSITION 3. *For any self-adjoint σ-normal (respectively σ-continuous) bounded functional h on a Baire*algebra (respectively Σ^* algebra) there exists a unique decomposition*

$$h = f - g, \qquad f \geqq 0,\ g \geqq 0,\ f \perp g\ ,$$

where f and g are σ-normal (respectively σ-continuous).

PROOF. Let $h=f-g$ be the unique decomposition given by proposition 2, and for $\varepsilon > 0$ choose z as in condition (2) of proposition 1. By the polarization identity it follows that if h is σ-normal, so is the functional $h(z\cdot)$. Applying the Cauchy–Schwarz inequality we obtain

$$\|g + h(z\cdot)\|^2 = \|g - g(z\cdot) + f(z\cdot)\|^2 \leqq 2\varepsilon\|h\|\ .$$

Since the set of σ-normal functionals is a norm-closed subset of the dual space, we have g, hence also f, σ-normal.

The case of σ-continuous functionals is proved similarly.

The above proposition is the non-commutative version of the *Jordan decomposition* of signed measures. The next proposition shows the possibility of a *Hahn decomposition.*

PROPOSITION 4. *If f and g are singular positive σ-normal (respectively σ-continuous) functionals on a Baire*algebra (respectively Σ^* algebra) A, there exists a projection $p \in A$ such that $f(p) = g(1-p) = 0$.*

PROOF. It suffices to consider the case where A is represented on a Hilbert space H and f and g are vector functionals associated with $\xi, \eta \in H$. We can then use the results contained in the proof of [4, Lemma 1] of which we give the following sketch: Since we can find $z \in A$ such that $\|z\xi\|$ and $\|(1-z)\eta\|$ are arbitrarily small, we can find a sequence $\{z_n\} \subset A$ such that both series

$$\sum |z_{n+1} - z_n|\xi, \qquad \sum |z_{n+1} - z_n|\eta$$

are convergent in H. This shows that if x^2 is the limit of the decreasing sequence $(1 + \sum |z_{n+1} - z_n|)^{-1}$, then $x \in A$ and contains both ξ and η in its range. Now the sequence $x(\sum |z_{n+1} - z_n|)x$ is convergent in A, hence also the sequence $x(\sum (z_{n+1} - z_n))x$ converges in A. It follows that if z is a weak limit point of $\{z_n\}$ in the weak closure of A, then

$$z\xi = (1-z)\eta = 0\ ,$$
$$xzx = \lim_k x\left(\textstyle\sum_1^k (z_{n+1} - z_n)\right)x - xz_1x \in A\ .$$

An ingenious calculation shows that $xzx \in A$ implies $[x]z[x] \in A$, where

$$[x]z[x]\xi = 0, \qquad (1-[x]z[x])\eta = [x](1-z)[x]\eta = 0 .$$

As our projection p we can now take the range projection of $[x]z[x]$.

Clearly p is not unique, in fact for any projection q with "total measure zero", that is, $(f+g)(q)=0$, we can use $p \vee q$ and $p \wedge q$ as well.

Now assume that A has no identity element. Then by [6, Theorem 1.3] A has a minimal dense hereditary ideal K. (A *subalgebra B of A is *hereditary* if B^+ is an order ideal of A^+ and B is the linear span of B^+. The terms *order-related* and *facial* are also used.) We give K the topology τ defined in [7, Theorem 2.1] (see also [9, Theorem 2.4]) and notice that if A were commutative, that is, $A = C_0(X)$, X locally compact Hausdorff, then (K,τ) would be the set of continuous functions on X with compact supports equipped with the inductive limit topology arising from uniform convergence on compact subsets of X. Hence the C^* integrals of A, defined as the elements of the dual space of (K,τ), are the non-commutative analogues of the Radon measures on a locally compact Hausdorff space.

The positive C^* integrals can be characterized as the positive functionals f on K such that for any $x \in K^+$

$$\sup \{f(y^*xy) \mid y \in A,\ \|y\| \leqq 1\} < \infty .$$

If J denotes the smallest norm-closed ideal of A'' containing A, then by [1, Proposition 4.4] $K(J)$ is the smallest ideal of A'' containing K, and every positive C^* integral of A has a unique extension as a C^* integral of J. Moreover, every positive C^* integral has a unique extension to A''^+ as a weakly lower semi-continuous extended functional. For a positive C^* integral f on A we shall denote the extensions to $K(J)$ and A''^+ by $\tilde{f}$. Following [1, p. 95] we define the *support* of f as the smallest projection p in A'' such that $\tilde{f}(1-p)=0$.

PROPOSITION 5. *If f and g are positive C^* integrals on A, the following conditions are equivalent:*

(1) *There exist two sets $\{f_i\}$ and $\{g_j\}$ of bounded positive functionals such that*

$$f = \sum f_i, \quad g = \sum g_j, \qquad f_i \perp g_j \text{ for all } i,j .$$

(2) *For any two bounded positive functionals f' and g', $f' \leqq f$ and $g' \leqq g$ imply $f' \perp g'$.*

(3) *There exists $z \in A''$, $0 \leqq z \leqq 1$, such that*

$$\tilde{f}(z) = \tilde{g}(1-z) = 0 .$$

(4) *f and g have orthogonal supports in A''.*

PROOF. (1) $\Rightarrow$ (4). Clearly the supports of f and g are the suprema of the supports of the f_i's and the g_j's respectively. Since these are mutually orthogonal by assumption, so are the supports of f and g.

(4) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (2). Follows from condition (3) in proposition 1.

(2) $\Rightarrow$ (1). By [6, Theorem 3.1] there exist sets $\{f_i\}$ and $\{g_j\}$ such that $f = \sum f_i$, $g = \sum g_j$. By assumption $f_i \perp g_j$ for all i,j.

When f and g satisfy the conditions of the proposition, we call them mutually *singular* and write $f \perp g$. The following lemma is the key to the proof that a decomposition in singular parts is unique.

LEMMA 6. *If f and g are positive C^* integrals on A and u,v are unitary operators in A'' such that for all $x \in K$*

$$\tilde{f}(ux) = \tilde{f}(xu) = \tilde{g}(vx) = \tilde{g}(xv) ,$$

then

$$f = g .$$

PROOF. For any $x \in K(J)^+$ define

$$\varrho(x) = \sup \{\tilde{f}(y^*xy) \mid y \in A'', \|y\| \leqq 1\} < \infty .$$

Using the relation $\tilde{f}(x) = \tilde{g}(u^*xv)$ which is valid for $x \in K(J)^+$ as well, together with the Cauchy–Schwarz inequality we get

$$(1) \qquad \tilde{f}(x)^2 = \tilde{g}(u^*x^{\frac{1}{2}}x^{\frac{1}{2}}v)^2 \leqq \tilde{g}(u^*xu)\,\tilde{g}(v^*xv) = \tilde{g}(u^*xu)\,\tilde{g}(x)$$

and similarly

$$(2) \qquad \tilde{g}(x)^2 \leqq \tilde{f}(v^*xv)\,\tilde{f}(x) .$$

It follows that we have the inequalities

$$\tilde{f}(x) \leqq \varrho(x), \qquad \tilde{g}(x) \leqq \varrho(x) .$$

Suppose we have proved for all $x \in K(J)^+$ that

$$(3) \qquad \tilde{f}(x)^n \leqq \varrho(x)\,\tilde{g}(x)^{n-1}, \qquad \tilde{g}(x)^n \leqq \varrho(x)\,\tilde{f}(x)^{n-1} .$$

Then, applying (1) and (3) in succession, we get

$$\begin{aligned} \tilde{f}(x)^{2n} &\leqq \tilde{g}(u^*xu)^n\,\tilde{g}(x)^n \\ &\leqq \varrho(u^*xu)\,\tilde{f}(u^*xu)^{n-1}\,\tilde{g}(x)^n = \varrho(x)\,\tilde{f}(x)^{n-1}\,\tilde{g}(x)^n \end{aligned}$$

and a similar inequality for g applying (2) and (3); hence

$$\tilde{f}(x)^{n+1} \leqq \varrho(x)\,\tilde{g}(x)^n, \qquad \tilde{g}(x)^{n+1} \leqq \varrho(x)\,\tilde{f}(x)^n .$$

We now prove (3) by induction, and extracting the nth roots we get in the limit $\tilde{f}(x) = \tilde{g}(x)$.

THEOREM 7. *For any self-adjoint C^* integral h there exists a unique decomposition*

$$h = f - g, \qquad f \geqq 0,\ g \geqq 0,\ f \perp g ,$$

where f and g are C^* integrals.

PROOF. By [9, Theorem 2.5] there exists on K^+ an invariant convex functional ϱ majorizing h and $-h$. For any C^* subalgebra B of K the restriction $\varrho|B$ is necessarily continuous by [9, Lemma 2.1]. Hence $h|B$ is continuous. (Since the topology τ on K is generated by semi-norms arising from the invariant convex functionals on K, this argument in fact shows that the restriction of τ to B gives the norm topology.) It follows from proposition 2 that there exist bounded positive functionals f', g' on B such that $h|B = f' - g'$, $f' \perp g'$. Hence there exists an element $z \in B''$, $0 \leqq z \leqq 1$, such that

$$(h|B)^{\tilde{}}\,(x(1-z)) \geqq 0, \qquad (h|B)^{\tilde{}}\,(xz) \leqq 0$$

for all $x \in B^+$. Since B'' has a canonical injection as a von Neumann subalgebra of A'', we may assume $z \in A''$.

Now by [8, Proposition 4] the set $\{B_i\}$ of finitely generated C^* subalgebras of K forms a net under inclusion which converges to K. To each B_i we choose z_i as above and since the positive part of the unit ball of A'' is compact, we may assume that the net $\{z_i\}$ converges weakly to some z.

For any $x \in K^+$ and $u, v \in A$, $\|u\| \leqq 1$, $\|v\| \leqq 1$, we have

$$4u^*xv = \sum_{n=0}^{3} i^n (u + i^n v)^* x (u + i^n v) ,$$

hence

$$4|h(u^*xv)| \leqq \sum_{n=0}^{3} \varrho\big((u + i^n v)^* x (u + i^n v)\big) \leqq 16\,\varrho(x) .$$

It follows that the bilinear functional $(u, v) \to h(uxv)$ is bounded on $A \times A$ and hence extends to a weakly continuous bilinear functional Φ_x on $A'' \times A''$. If $\sum u_n x_n v_n = 0$ and $\{u_\lambda\} \subset K^+$ is an approximative unit for A, then

$$\begin{aligned} 0 = \Phi_{u_\lambda}(\textstyle\sum u_n x_n v_n,\, 1) &= \textstyle\sum \Phi_{u_\lambda}(u_n x_n v_n,\, 1) \\ &= \textstyle\sum \Phi_{x_n}(u_n,\, v_n u_\lambda) \to \sum \Phi_{x_n}(u_n,\, v_n) . \end{aligned}$$

We conclude that for any $y = \sum u_n x_n v_n \in K(J)$ we may define $\tilde{h}(y) = \sum \Phi_{x_n}(u_n, v_n)$ and that $\tilde{h}$ is a linear functional which extends h.

For any $x \in K^+$ we have

$$\tilde{h}(xz_i) = (h|B_i)^\sim (xz_i), \qquad \tilde{h}(x(1-z_i)) = (h|B_i)^\sim (x(1-z_i))$$

when $x \in B_i$. Hence in the weak limit we get

$$\tilde{h}(x(1-z)) \geqq 0, \qquad \tilde{h}(xz) \leqq 0 .$$

We define positive linear functionals f and g on K by

$$f(x) = \tilde{h}(x(1-z)), \qquad g(x) = -\tilde{h}(xz) ,$$

and since for $x \geqq 0$ we have

$$f(x) \leqq 4\varrho(x), \qquad g(x) \leqq 4\varrho(x) ,$$

we conclude that f and g are C^* integrals. Finally

$$\begin{aligned} 0 \leqq \tilde{f}(z^{\frac{1}{2}}xz^{\frac{1}{2}}) &= \tilde{h}(z^{\frac{1}{2}}xz^{\frac{1}{2}}(1-z)) \\ &= \tilde{h}((1-z)^{\frac{1}{2}}x(1-z)^{\frac{1}{2}}z) = -\tilde{g}((1-z)^{\frac{1}{2}}x(1-z)^{\frac{1}{2}}) \leqq 0 \end{aligned}$$

for any $x \in K^+$. We conclude that $\tilde{f}(z)=\tilde{g}(1-z)=0$, hence $f \perp g$ by condition (3) in proposition 5.

To prove uniqueness of the decomposition let f,g,f',g' be positive C^* integrals with

$$f \perp g, \quad f' \perp g', \qquad f-g = f'-g' .$$

If p and q are the supports of g and g', we observe that $u=1-2p$ and $v=1-2q$ are unitary operators in A'' such that

$$\begin{aligned} (\tilde{f}+\tilde{g})(ux) = (\tilde{f}+\tilde{g})(xu) &= (f-g)(x) \\ &= (f'-g')(x) = (\tilde{f}'+\tilde{g}')(vx) = (\tilde{f}'+\tilde{g}')(xv) \end{aligned}$$

for all $x \in K$. By lemma 6 this implies $f+g=f'+g'$, hence $f=f'$ and $g=g'$.

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