MATH. SCAND. 25 (1969), 133–144

A SHARPER FORM OF THE DOEBLIN - LÉVY - KOLMOGOROV - ROGOZIN INEQUALITY FOR CONCENTRATION FUNCTIONS

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1. Introduction.

Let $X_1, X_2, \ldots$ be independent random variables and

$$S_n = \sum_{i=1}^n X_i .$$

It is well known (see [4, Theorem III.2.9] or [9, Sec. 42–44]) that the distribution of S_n is "spread out" more and more over the real line as $n \to \infty$, unless $S_n - \mathrm{median}(S_n)$ converges with probability 1. Doeblin and Lévy [3], [9, Sec. 48], [1], were the first to give quantitative estimates for the spreading out of the distribution of S_n by means of concentration functions. For any random variable Y, its concentration function is defined by

$$(1.1) \qquad Q(Y;\lambda) = \sup_x P\{x \leqq Y \leqq x+\lambda\}, \qquad \lambda \geqq 0 .$$

Kolmogorov [8] improved the results of [3] and [Sec. 48 in 9] and all estimates were put together in one inequality by Rogozin [11]. Rogozin's result is the following:

THEOREM 1. *There exists a universal constant C such that for any independent random variables $X_1, \ldots, X_n$, and real numbers $0 < \lambda_1, \ldots, \lambda_n \leqq 2L$, one has*

$$(1.2) \qquad Q(S_n; L) \leqq CL\{\textstyle\sum_{i=1}^n \lambda_i^2 [1-Q(X_i;\lambda_i)]\}^{-\frac{1}{2}} .$$

The aim of this note is to prove the following sharper form of this inequality:

THEOREM 2. *For the constant C of Theorem 1 and any independent random variables $X_1, \ldots, X_n$, and real numbers $0 < \lambda_1, \ldots, \lambda_n \leqq 2L$, one has*

$$(1.3) \qquad Q(S_n; L) \leqq 4 \cdot 2^{\frac{1}{2}}(1+9C)L \frac{\sum_{i=1}^n \lambda_i^2 [1-Q(X_i;\lambda_i)]\, Q(X_i; L)}{\{\sum_{i=1}^n \lambda_i^2 [1-Q(X_i;\lambda_i)]\}^{3/2}}$$

Received May 4, 1968.

Research supported by the U. S. National Science Foundation under grant GP 7128.

$$\leq 4\cdot 2^{\frac{1}{2}}(1+9C)L \max_{k\leq n} Q(X_k;\, L)\{\textstyle\sum_{i=1}^n \lambda_i^2[1-Q(X_i;\, \lambda_i)]\}^{-\frac{1}{2}}\,.$$

An application of this theorem will be given in Section 3.

NOTE. Shortly after the completion of this manuscript Professor Esseen kindly informed the author of an improvement of Theorem 1. In Theorem 3.1 of *On the concentration function of a sum of independent random variables*, Z. Wahrscheinlichkeitstheorie Verw. Geb. 9 (1968), 290–308, Esseen proves that (1.2) may be replaced by

$$Q(S_n;\, L) \leq CL[\textstyle\sum_{i=1}^n \lambda_i^2 D^2(X_i^s;\, \lambda_i)]^{-\frac{1}{2}}\,, \tag{1.4}$$

where

$$D^2(X_i^s;\, \lambda_i) = \lambda_i^{-2}\int_{|x|<\lambda_i} x^2 P\{|X_i^s|\in dx\} + P\{|X_i^s|\geq \lambda_i\}\,,$$

and X_i^s has the distribution of $X_i - X_i'$ for some X_i' independent of X_i but with the same distribution as X_i. Without essential changes in Sec. 2 below, this improvement can be used to sharpen (1.3) to

$$Q(S_n;\, L) \leq 4\cdot 2^{\frac{1}{2}}(1+9C)L\frac{\sum_{i=1}^n \lambda_i^2 D^2(X_i^2;\, \lambda_i)\, Q(X_j;\, L)}{[\sum_{i=1}^n \lambda_i^2 D^2(X_i^s;\, \lambda_i)]^{3/2}} \tag{1.5}$$

$$\leq 4\cdot 2^{\frac{1}{2}}(1+9C)L\frac{\max_{k\leq n} Q(X_k;\, L)}{[\sum_{i=1}^n \lambda_i^2 D^2(X_i^s;\, \lambda_i)]^{\frac{1}{2}}}\,.$$

However, the interest of Theorem 2 lies in the cases with $\max_{k\leq n} Q(X_k;L)$ small, and one easily sees that in these cases (1.5) is no better than (1.3).

Before we give the proof of Theorem 2 we would like to point out a few things.

Firstly, the last member of (1.3) is an improvement over the right hand side of (1.2) only if

$$\max_{k\leq n} Q(X_k;\, L) \leq C[4\cdot 2^{\frac{1}{2}}(1+9C)]^{-1}\,,$$

so that the main interest of Theorem 2 lies in the cases where $\max_{k\leq n} Q(X_k;\, L)$ becomes small. On the other hand the loss incurred by replacing C by $4\cdot 2^{\frac{1}{2}}(1+9C)$ seems unimportant because for applications the precise value of C does not seem to play a role.

Secondly, we would like to write out the special case obtained by taking $\lambda_1 = \ldots = \lambda_n$ and $X_1,\ldots,X_n$ identically distributed.

COROLLARY 1. *If $X_1,\ldots,X_n$ are independent, identically distributed random variables and $0<\lambda\leq 2L$, then*

$$(1.6) \qquad Q(S_n; L) \leqq 4\cdot 2^{\frac{1}{2}}(1+9C)\,\frac{1}{n^{\frac{1}{2}}}\,\frac{L}{\lambda}\,\frac{Q(X_1; L)}{[1-Q(X_1; \lambda)]^{\frac{1}{2}}}.$$

Especially for discrete variables this should be compared with Theorem 2 in [10].

Thirdly, we want to draw the reader's attention to Esseen's elegant and short proof [5] of Theorem 1, using only characteristic functions. Our proof of Theorem 2 uses Theorem 1 and combinatorial methods. It would be interesting to have a proof by means of characteristic functions for Theorem 2 as well.

Finally we point out that Rogozin announced a sharpening of Theorem 1 in another direction than Theorem 2 in footnote 1 of [11].

2. Proof of Theorem 2.

We begin with a few simple reductions. For the situation given in the theorem, introduce

$$a_i = \lambda_i^2[1-Q(X_i; \lambda_i)],$$

$$\mu = \frac{\sum_{i=1}^n a_i Q(X_i; L)}{\sum_{i=1}^n a_i},$$

and

$$A = \{i:\ 1 \leqq i \leqq n \text{ and } Q(X_i; L) \leqq 2\mu\}.$$

The second member of (1.3) can then be written as

$$(2.1) \qquad 4\cdot 2^{\frac{1}{2}}(1+9C)L\left(\textstyle\sum_{i=1}^n a_i\right)^{-\frac{1}{2}}\mu .$$

But it follows from

$$\sum_{\substack{i=1\\ i\notin A}}^n a_i \leqq \frac{1}{2\mu}\sum_{\substack{i=1\\ i\notin A}}^n a_i Q(X_i; L) \leqq \tfrac{1}{2}\sum_{i=1}^n a_i$$

that

$$\sum_{\substack{i=1\\ i\in A}}^n a_i \geqq \tfrac{1}{2}\sum_{i=1}^n a_i$$

and hence

$$(2.2) \qquad 4\cdot 2^{\frac{1}{2}}(1+9C)L\left(\sum_{i=1}^n a_i\right)^{-\frac{1}{2}}\mu \geqq 2(1+9C)L\left(\sum_{\substack{i=1\\ i\in A}}^n a_i\right)^{-\frac{1}{2}}\max_{k\in A} Q(X_k; L).$$

It is almost immediate from the definition of Q that for any pair of independent random variables Y, Z

(2.3) $$Q(Y+Z;L) \leqq \max\{Q(Y;L), Q(Z;L)\},$$

(see also Section 29 of [9]). Therefore

(2.4) $$Q(S_n;L) \leqq Q(\textstyle\sum_{i\in A} X_i; L).$$

In view of (2.2) and (2.4), the first inequality in (1.3) will follow (by renumbering of the X_i with $i \in A$) once we have proved the inequality

(2.5) $$Q(S_m;L) \leqq 2(1+9C)L\frac{\max_{1\leqq i\leqq m} Q(X_i;L)}{(\sum_{i=1}^m a_i)^{\frac{1}{2}}}.$$

for any independent random variables $X_1,\ldots,X_m$, $S_m = \sum_{i=1}^m X_i$, and any real numbers $0<\lambda_1,\ldots,\lambda_m \leqq L$.

The second inequality in (1.3) is obvious so that we may restrict ourselves to (2.5). We see at once that (2.5) is implied by Theorem 1 if $\max_{1\leqq i\leqq m} Q(X_i;L) \geqq \frac{1}{16}$, so that without loss of generality we can take

(2.6) $$\max_{1\leqq i\leqq m} Q(X_i;L) \leqq \tfrac{1}{16}.$$

Moreover,

$$\textstyle\sum_{i=1}^m a_i \leqq \sum_{i=1}^m \lambda_i^2 \leqq mL^2,$$

so that it suffices to show

(2.7) $$Q(S_m;L) \leqq 2(1+9C)\frac{\max_{1\leqq i\leqq m} Q(X_i;L)}{m^{\frac{1}{2}}}$$

under the side condition (2.6). In addition we may assume $m \geqq 4$ since by (2.3)

$$Q(S_m;L) \leqq Q(X_i;L), \quad 1 \leqq i \leqq m.$$

For our last reduction we define, for $0<a\leqq 1$, $m \geqq 1$,

$$R(m,a) = \sup Q(S_m;L)$$

where the sup is over all sequences $X_1,\ldots,X_m$ of independent random variables with $\max_{1\leqq i\leqq m} Q(X_i;L) \leqq a$ (of course, $S_m = \sum_{i=1}^m X_i$). (By changing scale we see that $R(m,a)$ is independent of $L>0$, but, since we think of L as fixed, this point is of no importance.) We claim that for all $0<a<\frac{1}{2}$

(2.8) $$R(m,a) \leqq 8Cam^{-\frac{1}{2}} + 2^{-[\frac{1}{4}m]}R(m,2a).$$

Before actually proving (2.8) we show that it implies (2.7), provided (2.6) holds and $m \geqq 4$. In fact let $X_1,\ldots,X_m$ be independent and let

(2.9) $$a = \max_{1\leqq i\leqq m} Q(X_i;L) \leqq \tfrac{1}{16}.$$

Let $r \geqq 1$ be the unique integer with

(2.10) $$2^r a < \tfrac{1}{4} \leqq 2^{r+1} a \leqq \tfrac{1}{2}.$$

Then (2.8) will imply

(2.11)
$$\begin{aligned} Q(S_m; L) &\leqq 8Cam^{-\frac{1}{2}} + 2^{-[\frac{1}{2}m]} R(m, 2a) \\ &\leqq 8Cam^{-\frac{1}{2}} + 2^{-[\frac{1}{2}m]} 8C2am^{-\frac{1}{2}} + 2^{-2[\frac{1}{2}m]} R(m, 2^2 a) \\ \cdots &\leqq 8Cam^{-\frac{1}{2}} \textstyle\sum_{j=0}^{r} 2^{-j[\frac{1}{2}m]+j} + 2^{-(r+1)[\frac{1}{2}m]} R(m, 2^{r+1} a) \\ &\leqq 16Cam^{-\frac{1}{2}} + 2^{-2(r+1)} R(m, 2^{r+1} a) \quad \text{for } m \geqq 4. \end{aligned}$$

Finally, by Theorem 1 with $\lambda_i \equiv L$, for any sequence $Z_1, \ldots, Z_m$ of independent random variables with

$$\max_{1 \leqq i \leqq m} Q(Z_i; L) \leqq 2^{r+1} a \quad \text{and} \quad \tfrac{1}{4} \leqq 2^{r+1} a \leqq \tfrac{1}{2},$$

we have

$$\begin{aligned} Q(\textstyle\sum_{i=1}^m Z_i; L) &\leqq \frac{C}{[\sum_{i=1}^m (1 - Q(Z_i; L))]^{\frac{1}{2}}} \\ &\leqq \frac{C}{m^{\frac{1}{2}}} \frac{1}{[1 - 2^{r+1} a]^{\frac{1}{2}}} \leqq C \frac{2^{\frac{1}{2}}}{m^{\frac{1}{2}}} \leqq \frac{4 \cdot 2^{\frac{1}{2}} C}{m^{\frac{1}{2}}} 2^{r+1} a, \end{aligned}$$

so that

(2.12) $$R(m, 2^{r+1} a) \leqq \frac{4 \cdot 2^{\frac{1}{2}} C}{m^{\frac{1}{2}}} 2^{r+1} a$$

whenever (2.10) holds. Since $r \geqq 1$, (2.11) together with (2.12) implies (2.7). Now we give the

Proof of (2.8). Let $X_1, \ldots, X_m$ be independent random variables satisfying

(2.13) $$\max_{1 \leqq i \leqq m} Q(X_i; L) \leqq a < \tfrac{1}{2};$$

let $I_i = [b_i, b_i + c_i]$ be closed intervals for which $P\{X_i \in I_i\} \geqq \frac{1}{2}$, but $P\{X_i \in I\} < \frac{1}{2}$ for any interval I of length less than c_i. The existence of such I_i is easily shown and by extending our probability space we may even assume that there exist events E_i with

$$\{X_i \in [b_i, b_i + c_i]\} \subset E_i \subset \{X_i \in [b_i, b_i + c_i]\},$$
$$P\{E_i\} = \tfrac{1}{2},$$

and such that $X_{j_1}, \ldots, X_{j_r}$, $E_{j_1}, \ldots, E_{j_r}$ are independent of $X_{i_1}, \ldots, X_{i_s}$, $E_{i_1}, \ldots, E_{i_s}$ whenever $\{j_1, \ldots, j_r\} \cap \{i_1, \ldots, i_s\} = \emptyset$. We denote the complement of E_i by E_i', and assume without loss of generality that the X_i are so ordered that

(2.14) $$c_1 \leqq c_2 \leqq \cdots \leqq c_m.$$

Then, for any closed interval $J=[u,u+L]$ of length L,

(2.15) $$P\{S_m \in J\} = \sum_{i=1}^{[\frac{1}{2}m]} P\{E_i \cap \bigcap_{j=1}^{i-1} E_j'\} \cdot$$

$$P\{S_m \in J \mid E_i \cap \bigcap_{j=1}^{i-1} E_j'\} + P\{\bigcap_{j=1}^{[\frac{1}{2}m]} E_j'\} P\{S_m \in J \mid \bigcap_{j=1}^{[\frac{1}{2}m]} E_j'\}$$
$$= \sum_{i=1}^{[\frac{1}{2}m]} 2^{-i} P\{S_m \in J \mid E_i \cap \bigcap_{j=1}^{i-1} E_j'\} + 2^{-[\frac{1}{2}m]} P\{S_m \in J \mid \bigcap_{j=1}^{[\frac{1}{2}m]} E_j'\} .$$

Now let $Y_i(Y_i')$ be random variables with the conditional distribution of X_i given that E_i (resp. E_i') occurs. Also assume that $X_1,\ldots,X_m$, $Y_1,\ldots,Y_m$, $Y_1',\ldots,Y_m'$ are independent. Then

(2.16)
$$P\{S_m \in J \mid E_i \cap \bigcap_{j=1}^{i-1} E_j'\} = P\{Y_1' + \ldots + Y_{i-1}' + Y_i + \sum_{j=i+1}^{m} X_j \in J\}$$

and also, when $x \in J - I_i$,

(2.17) $$P\{Y_i \in J-x\} = P\{X_i \in J-x \mid E_i\} \leqq 2P\{X_i \in (J-x) \cap I_i\} \leqq 2Q(X_i; L) .$$

By a similar chain of inequalities,

(2.18) $$P\{Y_i \in J-x\} = 0 \qquad \text{when } x \notin J - I_i .$$

From (2.16)–(2.18) one deduces

(2.19) $$P\{S_m \in J \mid E_i \cap \bigcap_{j=1}^{i-1} E_j'\}$$
$$= \int P\{Y_1' + \ldots + Y_{i-1}' + \sum_{j=i+1}^{m} X_j \in dx\} P\{Y_i \in J-x\}$$
$$\leqq P\{Y_1' + \ldots + Y_{i-1}' + \sum_{j=i+1}^{m} X_j \in J - I_i\} 2Q(X_i; L) .$$

Since the length of $J-I_i$ is at most $L+c_i \leqq 2c_i$ (because (2.13) implies $0<L<c_i$), another application of (2.3) and Theorem 1 shows for $i \leqq [\frac{1}{2}m]$

(2.20) $$P\{\sum_{j=1}^{i-1} Y_j' + \sum_{j=i+1}^{m} X_i \in J - I_i\} \leqq Q(\sum_{j=[\frac{1}{2}m]+1}^{m} X_j; 2c_i)$$
$$\leqq C\, 2c_i \limsup_{\varepsilon \downarrow 0} \{\sum_{j=[\frac{1}{2}m]+1}^{m} (c_i - \varepsilon)^2 [1 - Q(X_j; c_i - \varepsilon)]\}^{-\frac{1}{2}} \leqq 2C(\tfrac{1}{2} \cdot \tfrac{1}{2}m)^{-\frac{1}{2}} .$$

For the last inequality we used the estimate

$$Q(X_j; c_i - \varepsilon) \leqq Q(X_j; c_j - \varepsilon) < \tfrac{1}{2}, \qquad i \leqq j,\ 0<\varepsilon<c_i ,$$

which is a consequence of (2.14), and the definition of c_j. The inequality (2.19) together with (2.20) shows

(2.21) $$P\{S_m \in J \mid E_i \cap \bigcap_{j=1}^{i-1} E_j'\} \leqq 8Cm^{-\frac{1}{2}} Q(X_i; L) \leqq 8Cam^{-\frac{1}{2}} ,$$

whereas

(2.22)

$$P\{S_m \in J \mid \bigcap_{j=1}^{[\frac{1}{2}m]} E_j'\} = P\{Y_1' + \ldots + Y'_{[\frac{1}{2}m]} + X_{[\frac{1}{2}m]+1} + \ldots + X_m \in J\} \leqq R(m, 2a) ,$$

because $Y_1', \ldots, Y'_{[\frac{1}{2}m]}, X_{[\frac{1}{2}m]+1}, \ldots, X_m$ are independent, (2.13) holds, and

$$Q(Y_i'; L) = \sup_v P\{X_i \in [v, v+L] \mid E_i'\} \leqq \frac{1}{P\{E_i'\}} Q(X_i; L) \leqq 2a .$$

(2.15), (2.21) and (2.22) imply (2.8), and the proof is therefore complete.

3. Application.

For any random variable Y, the dispersion function $D(Y; q)$ is defined as the inverse function of the concentration function, that is,

$$(3.1) \qquad \begin{aligned} D(Y; q) &= \inf \{L \mid Q(Y; L) \geqq q\} \\ &= \inf \{L \mid \exists\, v \text{ such that } P\{Y \in [v, v+L]\} \geqq q\} , \\ &\qquad\qquad 0 \leqq q \leqq 1 , \end{aligned}$$

where we take the inf over the empty set as ∞.

As an application of Theorem 2 we shall now prove

THEOREM 3. *Let $X_1, X_2, \ldots$ be independent, identically distributed random variables and $S_n = \sum_{i=1}^n X_i$. Assume that for some fixed $L > 0$,*

$$(3.2) \qquad \lim_{m\to\infty} \limsup_{n\to\infty} \frac{Q(S_n; L)}{mQ(S_{nm}; L)} = 0 .$$

Then, for all $0 < q_1 \leqq q_2 < 1$,

$$(3.3) \qquad \limsup_{n\to\infty} \frac{D(S_n; q_2)}{D(S_n; q_1)} < \infty .$$

REMARK. Let

$$m_n = m(S_n) = \text{median of } S_n .$$

Since any interval which contains S_n with probability greater than $\frac{1}{2}$ must contain m_n, one has for $1 - \varepsilon > \frac{1}{2}$

$$(3.4) \qquad P\{|S_n - m_n| > D(S_n; 1-\varepsilon)\} \leqq \varepsilon .$$

It is an easy consequence of (3.3) and (3.4) that for each fixed q, $0 < q < 1$, there exists a function $\varepsilon(x; q)$ such that $\varepsilon(x; q) \to 0$ as $x \to \infty$, and

$$\sup_{n\geqq 1} P\left\{\frac{|S_n - m_n|}{D(S_n; q)} > x\right\} \leqq \varepsilon(x; q) . \tag{3.5}$$

In other words, $D(S_n; q)$ is "the right normalization factor for S_n". It is this fact which makes Theorem 3 the main tool for the proof of the following Tauberian theorem for random walks which will be proved in [7]:

THEOREM 4. *Let $X_1, X_2, \ldots$ be independent random variables, all with the same distribution function, symmetric about the origin. If for some fixed interval I, for fixed $1 < \alpha \leqq 2$, and for a slowly varying function M,*

$$\sum_{k=1}^{n} P\{S_k \in I\} \sim \frac{n^{1-1/\alpha}}{M(n)} \quad \text{as } n \to \infty ,$$

then

$$\lim_{n\to\infty} P\left\{\frac{CS_n}{n^{1/\alpha} M(n)} \leqq x\right\} = F_\alpha(x) ,$$

where $F_\alpha(\cdot)$ is the symmetric stable distribution function with characteristic function $\exp(-|t|^\alpha)$ and C is a constant depending on I and the support of $P\{X_1 \in dx\}$. If X_1 does not have a lattice distribution, then

$$C = \pi(\alpha - 1)\{\Gamma(\alpha^{-1})|I|\}^{-1} .$$

We would also like to point out that Theorem 3 is related to Doeblin's results in [2]. In fact, in the terminology of [2], (3.5) implies that the collection of powers of the distribution of X_1 is strongly compact. Thus (3.5) implies the condition in Theorem 10 of [2].

PROOF OF THEOREM 3. Assume that (3.3) fails for some fixed $0 < q_1 \leqq q_2 < 1$. Let $n_i \to \infty$ be such that

$$\lim_{i\to\infty} \frac{D(S_{n_i}; q_2)}{D(S_{n_i}; q_1)} = \infty . \tag{3.6}$$

We prove the theorem by showing that (3.6) contradicts (3.2). Put

$$L_i = D(S_{n_i}; q_1) .$$

Since $Q(S_n; \lambda) = O(n^{-\frac{1}{2}})$ for each fixed λ (by Theorem 1) one has necessarily

$$L_i \to \infty \quad \text{as } i \to \infty , \tag{3.7}$$

and by the definition of L_i we can choose intervals $J_i = [x_i, x_i + 2L_i]$ such that

$$b_i = P\{S_{n_i} \in J_i\} \geqq q_1 . \tag{3.8}$$

By (3.6), $L_i^{-1}D(S_{n_i}; q_2) \to \infty$ so that we even have

$$q_1 \leqq b_i \leqq q_2 . \tag{3.9}$$

Next we introduce the conditional distribution functions of S_{n_i} given $S_{n_i} \in J_i$ resp. $S_{n_i} \notin J_i$. These are given by

$$F_i(x) = \frac{1}{b_i} P\{S_{n_i} \leqq x,\ S_{n_i} \in J_i\}$$

and

$$G_i(x) = \frac{1}{(1-b_i)} P\{S_{n_i} \leqq x,\ S_{n_i} \notin J_i\} .$$

Of course

$$P\{S_{n_i} \leqq x\} = b_i F_i(x) + (1-b_i) G_i(x) \tag{3.10}$$

and thus, if $Y_1^{(i)}, Y_2^{(i)}, \ldots, Z_1^{(i)}, Z_2^{(i)}, \ldots$ are independent random variables, all $Y_j^{(i)}$ with distribution function F_i and all $Z_j^{(i)}$ with distribution function G_i, then for any Borel set A

$$\begin{aligned} &P\{S_{mn_i} \in A\} \\ &= \textstyle\sum_{k=0}^{m} \binom{m}{k} b_i^k (1-b_i)^{m-k} P\{Y_1^{(i)} + \ldots + Y_k^{(i)} + Z_1^{(i)} + \ldots + Z_{m-k}^{(i)} \in A\} \end{aligned} \tag{3.11}$$

(recall that the distribution of S_{mn_i} is the m-fold convolution of the distribution of S_{n_i} and use (3.10)).

Before we can apply (3.11) we need some estimates for $Q(Y_1^{(i)}; \cdot)$ and $Q(Z_1^{(i)}; \cdot)$. First, for any Borel set A

$$\begin{aligned} P\{Y_j^{(i)} \in A\} &= P\{S_{n_i} \in J_i\}^{-1} P\{S_{n_i} \in A \cap J_i\} \\ &\leqq b_i^{-1} P\{S_{n_i} \in A\} \\ &\leqq q_1^{-1} P\{S_{n_i} \in A\} , \end{aligned}$$

so that

$$Q(Y_j^{(i)}; L) \leqq q_1^{-1} Q(S_{n_i}; L) = o(1) \text{ as } i \to \infty,\ L \text{ fixed}, \tag{3.12}$$

and, similarly,

$$Q(Z_j^{(i)}; L) \leqq (1-q_2)^{-1} Q(S_{n_i}; L) . \tag{3.13}$$

Also

$$Q(Z_j^{(i)}; mL_i) = \sup_y P\{S_{n_i} \notin J_i\}^{-1} P\{S_{n_i} \in [y, y+mL_i],\ S_{n_i} \notin J_i\} . \tag{3.14}$$

We claim that there exists a $\lambda < 1$ such that for each m

$$\limsup_{i\to\infty} Q(Z_j^{(i)}; mL_i) \leqq \lambda < 1 . \tag{3.15}$$

For if no such λ exists then we can find a subsequence $B=\{i_1,i_2,\ldots\}$ and $m_i \geqq 1$ such that

$$Q(Z_j^{(i)}; m_iL_i) \to 1 \quad \text{as} \quad i \to \infty,\ i \in B\ . \tag{3.16}$$

By (3.6) and the definition of L_i we can even choose B and m_i such that

$$\frac{m_iL_i}{D(S_{n_i};\, q_2)} \to 0 \quad \text{as} \quad i \to \infty,\ i \in B\ . \tag{3.17}$$

On the other hand, by (3.14), (3.16) means that there exist y_i such that

$$P\{S_{n_i} \in [y_i, y_i + m_iL_i] \cup [x_i, x_i + 2L_i]\} \to 1 \quad \text{as } i \to \infty,\ i \in B\ . \tag{3.18}$$

Since by (3.17) no interval with length $O(m_iL_i)$ can contain S_{n_i} with probability q_2, we must then have

$$\frac{m_iL_i}{|y_i - x_i|} \to 0 \quad \text{as} \quad i \to \infty,\ i \in B$$

and, by (3.18) this means that the distribution of $(S_{n_i} - x_i)|y_i - x_i|^{-1}$ converges weakly to a distribution concentrated on 0 and 1 or 0 and -1 as $i \to \infty$ along a suitable subsequence of B. The limit distribution must be a genuine two point distribution, since by (3.9)

$$0 < q_1 \leqq P\{S_{n_i} \in [x_i, x_i + 2L_i]\} \leqq q_2 < 1\ .$$

This, however, is not possible by [6, Theorem 24.2], because a two point distribution is not the square of any distribution, and a fortiori it is not infinitely divisible (if F is supported on one point, so is $F*F$ and if the support of F contains more than 1 point, then the support of $F*F$ contains at least 3 points).

Now that (3.15) is established the proof is easily completed. Fix $m \geqq 1$ for the moment. For any $0<k<m$ and any closed interval $I = [u, u+L]$ of length L,

$$\begin{aligned} &P\{Y_1^{(i)} + \ldots + Y_k^{(i)} + Z_1^{(i)} + \ldots + Z_{m-k}^{(i)} \in I\} \\ &\quad = \int P\{Z_1^{(i)} + \ldots + Z_{m-k}^{(i)} \in dx\}\, P\{Y_1^{(i)} + \ldots + Y_k^{(i)} \in I - x\}\ . \end{aligned} \tag{3.19}$$

Since the distribution of $Y_j^{(i)}$ is concentrated on J_i,

$$P\{Y_1^{(i)} + \ldots + Y_k^{(i)} \in I - x\} = 0, \text{ when } (I-x) \cap [kx_i, kx_i + 2kL_i] = \emptyset\ ,$$

and by Theorem 2, for any x,

$$\begin{aligned} P\{Y_1^{(i)} + \ldots + Y_k^{(i)} \in I - x\} &\leqq Q(\textstyle\sum_{j=1}^k Y_j^{(i)}; L) \\ &\leqq \{k[1 - Q(Y_1^{(i)}; L)]\}^{-\frac{1}{2}}\, 4 \cdot 2^{\frac{1}{2}}(1+9C)\, Q(Y_1^{(i)}; L) \\ &\leqq 8 \cdot 2^{\frac{1}{2}}(1+9C)(q_1 k^{\frac{1}{2}})^{-1} Q(S_{n_i}; L) \qquad \text{(see (3.12))}\ . \end{aligned}$$

If we write, for short,

$$D_1 = 8\cdot 2^{\frac{1}{2}}(1+9C)q_1^{-1},$$

then we can bound (3.19) above by

$$D_1 k^{-\frac{1}{2}} Q(S_{n_i}; L)\, P\{Z_1^{(i)}+\ldots+Z_{m-k}^{(i)} \in u-kx_i-2kL_i,\, u+L-kx_i\}.$$

Since $L_i \geqq L$ for sufficiently large i (see (3.7)), and $k \leqq m$, the last factor is bounded eventually by

$$\begin{aligned} Q(\textstyle\sum_{j=1}^{m-k} Z_j^{(i)};\, 3mL_i) &\leqq C\,3mL_i\big((m-k)m^2L_i^2[1-Q(Z_1^{(i)};\, mL_i)]\big)^{-\frac{1}{2}} \\ &\leqq 3C\,(1-\lambda)^{-\frac{1}{2}}\,(m-k)^{-\frac{1}{2}}; \end{aligned}$$

here Theorem 1 and (3.15) have been used. In total this gives the bound

$$\frac{3CD_1}{(1-\lambda)^{\frac{1}{2}}}\,\frac{Q(S_{n_i};L)}{(k(m-k))^{\frac{1}{2}}}$$

for (3.19). In view of (3.11), (2.3), (3.12), (3.13), and (3.9), this shows that for all sufficiently large i and all u

$$\begin{aligned} P\{S_{mn_i} \in [u,u+L]\} &\leqq Q(Z_1^{(i)};\, L)(1-b_i)^m + \textstyle\sum_{k=1}^{m-1}\binom{m}{k}\, b_i^k(1-b_i)^{m-k}\cdot \\ &\quad \cdot\frac{3CD_1}{(1-\lambda)^{\frac{1}{2}}}\,\frac{Q(S_{n_i};L)}{(k(m-k))^{\frac{1}{2}}} + Q(Y_1^{(i)};L)b_i^m \\ &\leqq D_2 m^{-1} Q(S_{n_i};\, L) \end{aligned}$$

for some D_2 independent of $m \geqq 1$. In other words, for each fixed m,

$$\liminf_{i\to\infty} \frac{Q(S_{n_i};\, L)}{Q(S_{mn_i};\, L)} \geqq mD_2^{-1},$$

which contradicts (3.2). This completes the proof.

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