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A NOTE ON A PROBLEM OF BUSEMANN

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Let K and K' be two bodies in n-dimensional euclidean space E^n, $n \geqq 3$, which are convex and centrally symmetric about the origin. For any unit vector u in E^n, the $(n-1)$-dimensional measures (or areas) of the intersection of K or K' and the hyperplane $(x,u)=0$ are denoted by $A(u)$ and $A'(u)$, respectively. The volumes of K and K' are denoted by V and V'. The problem in question [1] is:

Does $A(u) \geqq A'(u)$ for all unit vectors u imply $V \geqq V'$?

Although easily stated, this important question seems remarkably difficult to answer. Much effort has been spent in trying to establish an affirmative answer, or to find a counterexample. The answer is, indeed, affirmative if K' is an ellipsoid, or if $A'(u)=cA(u)$ (see [1] for references), but no other special case seems to have been settled. The purpose of this note is to show that the answer is yes also if K and K' are 3-dimensional bodies of rotation with a common axis.

In a plane equipped with polar coordinates r, ψ we consider continuous closed curves represented by π-periodic expressions $r(\psi)$ of the following type: For $0 \leqq \psi \leqq \frac{1}{2}\pi$ we have $r(\pi-\psi)=r(\psi)>0$, and as ψ increases from 0 to $\frac{1}{2}\pi$, $r(\psi)\sin\psi$ is nondecreasing, i.e. any line parallel to the polar axis meets the curve in at most two points or segments. When such a curve is rotated around the polar axis we shall call the result a central symmetric monotonic body of rotation generated by $r(\psi)$. The set of all such bodies is denoted by M. Clearly, all central symmetric convex bodies of rotation belong to M.

Given a body B in M, let $A_B(\varphi)$ denote the area which B cuts out of a plane through the center, whose normal makes an angle φ, $0 \leqq \varphi \leqq \frac{1}{2}\pi$, with the axis of rotation. We then have the following

Theorem. *If for two bodies* 1 *and* 2 *in* M *it is true that* $A_1(\varphi) \geqq A_2(\varphi)$ *for all* φ *in* $[0, \frac{1}{2}\pi]$, *and* $A_1(\varphi) > A_2(\varphi)$ *for some* φ *in this interval, then the volume of* 1 *is greater than that of* 2.

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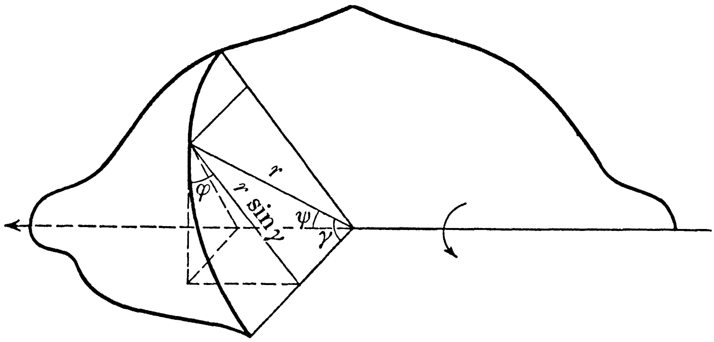

Proof. With the notation used in the accompanying figure, we have $\sin\gamma \sin\varphi = \cos\psi$. For a body B, generated by $r_B(\psi)$, this relation gives us

$$A_B(\varphi) = 2\int_0^{\frac{1}{2}\pi} r_B{}^2(\psi)\,d\gamma = \int_0^{\varphi} r_B{}^2(\psi) f(\varphi,\theta)\,d\theta ,$$

where $\psi = \frac{1}{2}\pi - \theta$, and

$$f(\varphi,\theta) = (\sin^2\varphi - \sin^2\theta)^{-\frac{1}{2}} \cos\theta, \quad 0 \leqq \theta < \varphi \leqq \tfrac{1}{2}\pi .$$

Let the two bodies be generated by $r_1(\psi) = r(\theta) + \varrho(\theta)$, and $r_2(\psi) = r(\theta)$. Then by assumption

$$A_1(\varphi) - A_2(\varphi) = 2\int_0^{\varphi} (2r\varrho + \varrho^2) f(\varphi,\theta)\,d\theta \geqq 0$$

for all φ in $(0, \frac{1}{2}\pi]$, and positive for some φ.

Assuming first that the number of intervals in which ϱ is positive (or negative) is finite, we let (θ_k), $0 = \theta_0 < \theta_1 < \ldots < \theta_N = \frac{1}{2}\pi$, be a sequence dividing $[0, \frac{1}{2}\pi]$ into intervals where ϱ has constant sign. Clearly ϱ can not be negative in $[\theta_0, \theta_1]$, so ϱ is nonnegative in $[\theta_{2k}, \theta_{2k+1}]$ and nonpositive in $[\theta_{2k-1}, \theta_{2k}]$. Thus

$$A_1(\theta_n) - A_2(\theta_n) = 2\sum_{k=0}^{n-1} (-1)^k \int_{\theta_k}^{\theta_{k+1}} |2r\varrho + \varrho^2| f(\theta_n, \theta)\,d\theta, \quad n = 1, 2, \ldots, N .$$

Putting

$$\sigma_k = \int_{\theta_k}^{\theta_{k+1}} |2r\varrho + \varrho^2|\,d\theta$$

$= $ twice the area enclosed by $r + \varrho$ and r in the interval $[\theta_k, \theta_{k+1}]$,

we have

$$\int_{\theta_k}^{\theta_{k+1}} |2r\varrho+\varrho^2|\, f(\theta_n,\theta)\, d\theta = \alpha_{k,n}\,\sigma_k\,,$$

where

$$\alpha_{k,n} = f(\theta_n,\xi_{k,n}) \quad \text{for some } \xi_{k,n} \text{ in } [\theta_k,\theta_{k+1}]\,.$$

For any φ, $0<\varphi<\frac{1}{2}\pi$, the function f is strictly increasing in θ, so

$$0 < \alpha_{0,n} < \alpha_{1,n} < \alpha_{2,n} < \dots .$$

But our assumption

$$\sum_{k=0}^{n-1} (-1)^k \alpha_{k,n}\sigma_k \geqq 0, \quad n=1,2,\dots,N\,,$$

then implies (by Abel's formula for partial summation)

$$\sum_{k=0}^{n-1} (-1)^k \beta_k\sigma_k > 0\,,$$

for any decreasing sequence (β_k). This proves that $V_1>V_2$ since V_1-V_2 can be written as a sum of this type, with $\beta_k=\pi y_k$, where y_k is the distance from the axis of rotation to the center of gravity of the region with area σ_k.

If $r+\varrho$ and r intersect at infinitely many points, we still must have $\varrho(\theta)>0$ in some interval $\theta'<\theta<\theta''<\frac{1}{2}\pi$. Denote the area enclosed by $r+\varrho$ and r in this interval by σ. Let p be any outer polygonal approximation to $r+\varrho$, and q any inner polygonal approximation to r. Let θ_k, σ_k and β_k be defined as above for p and q. For some even number n the interval $[\theta_n,\theta_{n+1}]$ contains $[\theta',\theta'']$, and $\sigma_n\geqq\sigma$. There must also exist a positive δ, independent of p and q, such that $\beta_n-\beta_{n+1}>\delta$. Since (again using partial summation)

$$V_p-V_q = \sum_{k=0}^{N-1} (-1)^k\beta_k\sigma_k > (\beta_n-\beta_{n+1})\sigma_n > \delta\sigma\,,$$

we conclude that

$$V_1-V_2 \geqq \delta\sigma > 0\,.$$

This completes the proof.

It was recently brought to the author's attention by J. Schaer (Calgary) that H. Hadwiger (Bern) has obtained by different methods a very similar result, as a special case of a theorem appearing in volume 23 of this journal [2]. Neither result contains the other. The author also wishes to express his gratitude to Professor Hadwiger for pointing out that a certain condition assumed in the first draft of this note was redundant.

REFERENCES

1. H. Busemann and C. M. Petty, *Problems on convex bodies*, Math. Scand. 4 (1956), 88–94.
2. H. Hadwiger, *Radialpotenzintegrale zentralsymmetrischer Rotationskörper und Ungleichheitsaussagen Busemannscher Art*, Math. Scand. 23 (1968), 193–200.

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