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EXTENSIONS TO REGRESSIVE ISOLS

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1. Introduction.

Let E denote the collection of all non-negative integers (*numbers*), Λ the collection of all isols, Λ^* the collection of all isolic integers, and Λ_R the collection of all regressive isols. In [13] and [14], A. Nerode associated with each recursive function $f\colon E \to E$ a function $D_f\colon \Lambda \to \Lambda^*$ and with each recursive set of numbers α a set α_Λ of isols. D_f is an extension of the function f from E to Λ, and $\alpha \subseteqq \alpha_\Lambda$. For α a recursive set, let $\alpha_R = \Lambda_R \cap \alpha_\Lambda$. Let f be a strictly increasing recursive function and let α be its range. Then by [1, Corollary 4] and [4, Proposition 3] the following properties are true, $D_f\colon \Lambda_R \to \Lambda_R$ and $\alpha_R = D_f(\Lambda_R)$.

In the present paper we wish to associate with each strictly increasing function $f\colon E \to E$, a function E_f from a subset of Λ_R into Λ_R, and with each set of numbers α a set α_* of regressive isols. Let δE_f denote the domain of the mapping E_f. Then it will be shown that $E \subseteqq \delta E_f \subseteqq \Lambda_R$ and that E_f is an extension of the function f from E to δE_f. Also, in the special case that f is a (strictly increasing) recursive function then $\delta E_f = \Lambda_R$, and $E_f(A) = D_f(A)$ for all regressive isols A. Regarding sets of numbers, the following properties will be shown,

(a) $\alpha \subseteqq \alpha_*$,
(b) if α is an infinite set of numbers, then α_* will have the cardinality of the continuum,
(c) if α is a recursive set, then $\alpha_* = \alpha_R$,
(d) if α is any set and β is any recursive set, then $\alpha \subseteqq \beta$ implies $\alpha_* \subseteqq \beta_R$.

Remark. We would like to mention that we were motivated toward the idea for the kind of extensions of functions and sets described above by our interest in two particular questions that are related to regressive isols and the Nerode extension for recursive sets. We call an infinite regressive isol T *torre*, if it has the property that for every recursive

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set α, either $T \in \alpha_R$ or $T \in (E-\alpha)_R$. The two questions that we were interested in are:

Q 1. Do torre regressive isols exist?

Q 2. If α is an infinite recursive set, does α_R contain a torre regressive isol?

It turns out that both of these questions have affirmative answers; this fact can be obtained almost directly from Theorem 4.1, Corollary 4.3 and Theorem 4.4 of Nerode [15]. Our approach in the present paper will also lead us to these answers. It is a bit different from the approach in [15], and may be of interest for its own sake.

2. Background.

Almost all of our notation will be as in the papers [1] through [8]. We will review here some of the principal definitions and properties which we need.

If f is a function from a subset of E into E, then δf will denote its domain and ϱf will denote its range. If F is a function from a subset of Λ into Λ, then δF will denote the domain of F. We let j denote the familiar recursive function of two arguments defined by

$$j(x,y) = x + \tfrac{1}{2}(x+y)(x+y+1)\,.$$

We recall that j maps E^2 onto E in a one-to-one manner. For any number n, we let $\nu(n) = \{x \mid x < n\}$. For any number n and set δ, we let $j(n,\delta) = \{j(n,y) \mid y \in \delta\}$. We define the function j of three arguments by

$$j(x,y,z) = j(j(x,y),z)\,.$$

Then $j(x,y,z)$ will be a recursive function and will map E^3 onto E in a one-to-one manner. We will sometimes write $j[x,y,z]$ for $j(x,y,z)$. We let c denote the cardinality of the continuum. If α is any set of numbers, then $\bar{\alpha}$ will denote the complement of α, that is, $\bar{\alpha} = E - \alpha$. If α is any set then $\bar{\alpha}_\Lambda$ and $\bar{\alpha}_R$ will stand for, $(\bar{\alpha})_\Lambda$ and $(\bar{\alpha})_R$ respectively. If α is an infinite set, then the strictly increasing function with domain E and range α is called the *principal function of* α.

Let a_n, b_n and c_n denote any functions from E into E. We write $a_n \leq^* b_n$, if there is a partial recursive function $p(x)$ such that

$$(1.1) \qquad \varrho a_n \subseteq \delta p \quad \text{and} \quad (\forall n)[p(a_n) = b_n]\,.$$

We write $a_n \simeq b_n$, if there is a one-to-one partial recursive function $p(x)$ such that (1.1) holds. It can be shown that,

$$(1.2) \qquad (a_n \leqq^* b_n \text{ and } b_n \leqq^* c_n) \Rightarrow a_n \leqq^* c_n ,$$

$$(1.3) \qquad (a_n \leqq^* b_n \text{ and } a_n \leqq^* c_n) \Rightarrow a_n \leqq^* b_n + c_n .$$

Also, in the special case that a_n and b_n are each one-to-one functions then, by [6, Proposition P1],

$$(1.4) \qquad (a_n \leqq^* b_n \text{ and } b_n \leqq^* a_n) \Leftrightarrow a_n \simeq b_n .$$

Let t_n denote any (one-to-one) regressive function. In view of the definition of the relation $\leqq^*$, it is easy to see that there are exactly $\aleph_0$ functions a_n such that $t_n \leqq^* a_n$. Because t_n is a regressive function we also see that $t_n \leqq^* n$. It follows from this property that if a_n is any recursive function, then $t_n \leqq^* a_n$. Let T denote an infinite regressive isol. We write $T \leqq^* a_n$, if there is a regressive function t_n that ranges over a set in T and such that $t_n \leqq^* a_n$. It is well-known that if t_n and $\tilde{t}_n$ are two regressive functions that range over sets that belong to the same isol then $t_n \simeq \tilde{t}_n$. Combining this fact with (1.2) and (1.4), it follows that if $T \leqq^* a_n$, then $t_n \leqq^* a_n$ for every regressive function t_n that ranges over a set in T. In view of some of our remarks above, we also have the following properties,

$$(1.5) \qquad (T \leqq^* a_n \text{ and } T \leqq^* b_n) \Rightarrow T \leqq^* (a_n + b_n) ,$$

$$(1.6) \qquad a_n \text{ a recursive function} \Rightarrow T \leqq^* a_n .$$

With every regressive isol T and function a_n from E into E, J. C. E. Dekker in [5] associated a particular isol, denoted by $\sum_T a_n$ and called an *infinite series of isols*. If T is a finite number, suppose $T = t$, then

$$(1.7) \qquad \sum\nolimits_T a_n = \begin{cases} 0, & \text{if } t = 0 , \\ a_0 + \ldots + a_{t-1}, & \text{if } t \geqq 1 . \end{cases}$$

In the special case that T is an infinite regressive isol, then

$$(1.8) \qquad \sum\nolimits_T a_n = \operatorname{Req} \sum_0^\infty j(t_n, \nu(a_n)) ,$$

where t_n can be chosen to be any regressive function that ranges over a set in T. By [5, Theorem 1], $\sum_T a_n$ is an isol, and its value depends on the regressive isol T but not on the particular choice of a regressive function whose range is in T. In the event that a_n is a recursive function, then $\sum_T a_n$ is a regressive isol [1, Theorem 1].

Let $f: E \to E$ be a strictly increasing function. The function $e(x)$ defined by,

$$e(0) = f(0), \qquad e(n+1) = f(n+1) - f(n) ,$$

is called the *e-difference function of f*; we will also write e_n for $e(n)$. It is easy to see that $e_x \geqq 0$ for all numbers x, and that

$$f(x) = e_0 + \ldots + e_x . \tag{1.9}$$

In addition, if f is a (strictly increasing) recursive function, then e_x is also a recursive function. The following result is a special case of [1, Proposition 2].

THEOREM A. *Let $f: E \to E$ be a strictly increasing recursive function and let e_x be its e-difference function. Then for all regressive isols T,*

$$D_f(T) = \Sigma_{T+1} e_n .$$

3. Six Lemmas.

LEMMA 3.1. *Let a_n be any function from E into E. Then there are c (infinite) regressive isols T such that $T \leqq^* a_n$.*

PROOF. It is well known that retraceable functions range over sets that are either recursive or immune. Also, every infinite isol contains exactly $\aleph_0$ sets. By combining these two properties we see that to prove the lemma, it suffices to show that there are c retraceable functions $\tilde{t}_n$ such that $\tilde{t}_n \leqq^* a_n$. This will be our approach here.

Let the function d_n be defined by

$$d_n = 2^{1+a_0} 3^{1+a_1} \ldots p_n{}^{1+a_n} ,$$

where p_n denotes the $(n+1)^{st}$ prime number. It is easy to see that d_n is a retraceable function and $d_n \leqq^* a_n$. For any retraceable function t_n set

$$\tilde{t}_n = j(t_n, d_n) .$$

Then $\tilde{t}_n$ is also a retraceable function and $\tilde{t}_n \leqq^* a_n$. It is easy to see that different functions t_n give rise to different functions $\tilde{t}_n$. Because there are c retraceable functions, and therefore c possible choices for the function t_n, it follows that there are c retraceable functions $\tilde{t}_n$ such that $\tilde{t}_n \leqq^* a_n$. As we noted above, this property gives the desired result of the lemma.

LEMMA 3.2. *Let $f: E \to E$ be a strictly increasing function, and let $e(x)$ be its e-difference function. Let T be any infinite regressive isol. Then*

$$T \leqq^* f(n) \Leftrightarrow T \leqq^* e(n) . \tag{1}$$

Proof. Let t_n be a regressive function that ranges over a set in T. We note that statement (1) is equivalent to

$$(2) \qquad t_n \leqq^* f(n) \Leftrightarrow t_n \leqq^* e(n) ,$$

and we will prove statement (2).

Firstly $\Rightarrow$: Assume that $t_n \leqq^* f(n)$. We wish to prove that $t_n \leqq^* e(n)$, that is, that the mapping $t_n \to e(n)$ has a partial recursive extension. For this purpose, let the value of t_n be given. We wish to find the value of $e(n)$. From the number t_n we can compute the number n (since t_n is a regressive function) and the number $f(n)$ (since $t_n \leqq^* f(n)$). If $n=0$, then $e(0)=f(0)$, and we are done. If $n \geqq 1$, then using the regressive property of t_n, we can find the number t_{n-1}. From t_{n-1} we can then find the value of $f(n-1)$. Since $e(n)=f(n)-f(n-1)$, it follows that the number $e(n)$ can then be computed. In view of these remarks we can conclude that the mapping $t_n \to e(n)$ will have a partial recursive extension, and therefore $t_n \leqq^* e(n)$.

Secondly $\Leftarrow$: Assume that $t_n \leqq^* e(n)$. Then, in view of the regressive property of the function t_n, it is easily seen that

$$t_n \leqq^* (e_0 + \dots + e_n) ,$$

and therefore by (1.9), this means that also $t_n \leqq^* f(n)$. This completes the proof.

Lemma 3.3. *Let T be an infinite regressive isol. Let a_n by any function such that $T \leqq^* a_n$. Then*

(a) $T+1 \in \Lambda_R$,
(b) $\sum_{T+1} a_n \in \Lambda_R$, *and*
(c) *if $\tilde{t}_n$ is any regressive function that ranges over a set in $T+1$,*

$$j(\tilde{t}_0, 0), \dots, j(\tilde{t}_0, a_0 \dot{-} 1), j(\tilde{t}_1, 0), \dots, j(\tilde{t}_1, a_1 \dot{-} 1), j(\tilde{t}_2, 0), \dots$$

(*here no terms of the form $j(\tilde{t}_x, y)$ would appear if $a_x = 0$*) *represents a regressive enumeration of a set belonging to $\sum_{T+1} a_n$.*

Proof. Part (a) is a well-known result and parts (b) and (c) follow from the proof of [3, Proposition 5].

Remark. The next lemma is [3, Lemma 4], and we will state it without proof.

Lemma 3.4. *Let T be an infinite regressive isol. Let a_n and b_n be functions such that $T \leqq^* a_n$ and $T \leqq^* b_n$. Then*

(a) $$T \leqq^* (a_n + b_n),$$

and

(b) $$\Sigma_{T+1}(a_n + b_n) = \Sigma_{T+1} a_n + \Sigma_{T+1} b_n .$$

LEMMA 3.5. *Let a_n be any function from E into E. Let p be a partial recursive function such that $\rho a \subseteq \delta p$. Let T be any infinite regressive isol. Then*

$$T \leqq^* a_n \Rightarrow T \leqq^* p(a_n) .$$

PROOF. Left to the reader.

LEMMA 3.6. *Let f and g be strictly increasing functions, with g a recursive function. Let h denote the composition function $g \circ f$. Let T be any infinite regressive isol. Then*

$$T \leqq^* h(n) \Leftrightarrow T \leqq^* f(n) .$$

PROOF. We first note that the direction from left to right follows from Lemma 3.5 and the fact that the function g^{-1} [the inverse function of g] will be partial recursive. Finally the direction from right to left follows directly from Lemma 3.5.

4. Extensions of strictly increasing functions to regressive isols.

We wish to introduce in this section an extension procedure for strictly increasing functions f (from E into E) to functions E_f (from particular subsets of Λ_R into Λ_R). We will use the following notations: we write $T \leqq^* f$ to mean that $T \leqq^* f(n)$, and we write $f\uparrow$ to mean that f is a strictly increasing function from E into E.

DEFINITION D1. Let $f\uparrow$ and let $e(x)$ be the e-difference function of f. Then we define,

$$\delta E_f = E \cup \{T \mid T \in \Lambda_R - E \text{ and } T \leqq^* f\};$$

and for $T \in \delta E_f$, we let

$$E_f(T) = \Sigma_{T+1} e_n .$$

REMARK. In view of D1 and (1.9) it is easy to see that E_f is an extension of the function f from E to δE_f.

THEOREM 4.1. *Let $f\uparrow$ be a recursive function. Then*

(a) $\delta E_f = \Lambda_R$, *and*

(b) $E_f(T) = D_f(T)$, *for* $T \in \Lambda_R$.

PROOF. Because f is a recursive function, we know that for any infinite regressive isol T, one has $T \leqq^* f$. In light of D1, it follows then that $\delta E_f = \Lambda_R$; and this verifies (a). Part (b) follows directly from D 1 and Theorem A.

THEOREM 4.2. *Let* $f\uparrow$. *Then*

(a) *the cardinality of* δE_f *is* c, *and*
(b) $E_f \colon \delta E_f \to \Lambda_R$.

PROOF. Apply Lemmas 3.1, 3.2 and 3.3.

THEOREM 4.3. *Let* $f\uparrow$. *Let* S *and* T *be any regressive isols with* $S,T \in \delta E_f$. *Then*

$$E_f(S) = E_f(T) \Rightarrow S = T\,.$$

PROOF. Let $e(x)$ denote the e-difference function of f. Because f is a strictly increasing function it follows that $e(x) \geqq 1$ whenever $x \geqq 1$. Combining this property with the definition of an infinite series of isols, we see that if X is a regressive isol, then $\sum_X e_n$ is an infinite isol if and only if X is an infinite isol. Assume that $E_f(S) = E_f(T)$. If $E_f(S)$ is a finite isol, then each of the isols S and T will also be finite. In this event

$$f(S) = E_f(S) = E_f(T) = f(T)\,,$$

and the desired result follows from the one-to-one property of the function f; that is, in this event $S = T$.

Let us assume now that $E_f(S) = E_f(T)$, and that each of the regressive isols S and T is infinite. Let $\tilde{s}_n$ and $\tilde{t}_n$ be regressive functions that range over sets belonging to $S+1$ and $T+1$ respectively. In view of D1 and Lemma 3.3, it follows that

$$j(\tilde{s}_0, 0), \ldots, j(\tilde{s}_0, e_0 \dot{-} 1), j(\tilde{s}_1, 0), \ldots, j(\tilde{s}_1, e_1 \dot{-} 1), j(\tilde{s}_2, 0), \ldots\,,$$

and

$$j(\tilde{t}_0, 0), \ldots, j(\tilde{t}_0, e_0 \dot{-} 1), j(\tilde{t}_1, 0), \ldots, j(\tilde{t}_1, e_1 \dot{-} 1), j(\tilde{t}_2, 0), \ldots\,,$$

represent regressive enumerations of sets belonging to $E_f(S)$ and $E_f(T)$ respectively. Because $E_f(S) = E_f(T)$, it follows that the (regressive) functions determined by these two enumerations will be recursively equivalent. Therefore, there exists a one-to-one partial recursive function $p(x)$ such that, for each number n and number $0 \leqq y < e_n$,

$$j(\tilde{s}_n, y) \in \delta p \qquad \text{and} \qquad pj(\tilde{s}_n, y) = j(\tilde{t}_n, y)\,.$$

In particular, since $e_x \geqq 1$ for $x \geqq 1$, we have that for each number $n \in E$,

$$(1) \qquad j(\tilde{s}_{n+1}, 0) \in \delta p \quad \text{and} \quad pj(\tilde{s}_{n+1}, 0) = j(\tilde{t}_{n+1}, 0) .$$

It follows from (1), that $j(\tilde{s}_{n+1}, 0) \simeq j(\tilde{t}_{n+1}, 0)$, and hence also that

$$(2) \qquad \tilde{s}_{n+1} \simeq \tilde{t}_{n+1} .$$

The function $\tilde{s}_{n+1}$ ranges over a set in S, since the function $\tilde{s}_n$ ranges over a set in $S+1$; and similarly the function $\tilde{t}_{n+1}$ ranges over a set in T. We can therefore conclude from (2) that $S=T$, and this completes the proof.

COROLLARY 4.1. *Let* $f\uparrow$. *Then the cardinality of the collection* $E_f(\delta E_f)$ *is* c.

PROOF. Combine Theorems 4.2(a) and 4.3.

THEOREM 4.4. *Let* $f\uparrow$ *and* $g\uparrow$. *Then*

(a) $(f+g)\uparrow$,

(b) $\delta E_f \cap \delta E_g \subseteqq \delta E_{f+g}$,

(c) $E_{f+g}(T) = E_f(T)+E_g(T)$, *for* $T \in \delta E_f \cap \delta E_g$.

PROOF. Part (a) of the theorem is clear. Regarding part (b), let $T \in \delta E_f \cap \delta E_g$. If T is finite, then $T \in \delta E_{f+g}$ by the definition of the set δE_{f+g}. Assume now that T is an infinite regressive isol. Then $T \leqq^* f$ and $T \leqq^* g$. By (1.5), this property implies that $T \leqq^* f+g$; and therefore also that $T \in \delta E_{f+g}$. This verifies property (b).

To prove part (c), let $T \in \delta E_f \cap \delta E_g$. If T is finite then the desired result is readily seen to be true. Let us assume now that T is an infinite regressive isol. Let e_f, e_g and e_{f+g} denote the e-difference functions of the functions f, g and $(f+g)$ respectively. It is an easy computation to show that, for $n \in E$

$$(1) \qquad e_{f+g}(n) = e_f(n) + e_g(n) .$$

Because $T \in \delta E_f \cap \delta E_g$, we know that both $T \leqq^* f$ and $T \leqq^* g$. By Lemma 3.2, it then also follows that

$$(2) \qquad T \leqq^* e_f \quad \text{and} \quad T \leqq^* e_g .$$

Combining (1), (2) and Lemma 3.4 gives

$$\sum_{T+1} e_{f+g}(n) = \sum_{T+1} e_f(n) + \sum_{T+1} e_g(n);$$

and therefore also,

$$E_{f+g}(T) = E_f(T) + E_g(T) .$$

This is the desired result and completes the proof.

THEOREM 4.5. *Let $f\uparrow$ and $g\uparrow$, and with g a recursive function. Let h denote the composition function $g \circ f$. Then*

(a) $h\uparrow$,
(b) $\delta E_h = \delta E_f$,
(c) $E_h(T) = E_g[E_f(T)]$, *for* $T \in \delta E_h$.

REMARK. Regarding the proofs of the three parts of the theorem, parts (a) and (b) have short proofs and part (c) has a long proof. In order to avoid some additional comments that would be needed in the course of proving part (c), we will use the following notation: we let the two symbols 000 and 00 play the role of numbers and imagine that the natural enumeration of E would look like $000, 00, 0, 1, 2, \ldots$.

PROOF. We observe that part (a) of the theorem is clear. Regarding part (b), we first note that, by D1,

$$E \subseteqq \delta E_h \cap \delta E_f .$$

Hence to prove (b), it suffices to show that if T is an infinite regressive isol, then

$$T \in \delta E_h \Leftrightarrow T \in \delta E_f ,$$

or equivalently,

$$T \leqq^* h \Leftrightarrow T \leqq^* f;$$

and this particular relation follows directly from Lemma 3.6.

PROOF OF (c): Let $T \in \delta E_h$. If T is finite then the identity

(1) $$E_h(T) = E_g[E_f(T)]$$

is easily seen to hold. Let us assume how that T is an infinite regressive isol. Let u, e and r denote the e-difference functions of the functions f, g and h respectively. We wish to prove the identity of (1), and this is equivalent to proving that

(2) $$\sum_{T+1} r_n = \sum_{1+\Sigma_{T+1} u_n} e_n .$$

We note that e_n is a recursive function since g is a recursive function.

REMARK. We wish to give an expression for r_n in terms of values of u_m and e_p. It can be readily shown using the definition of the e-difference function and some easy computation, that

$$(3)\qquad \begin{aligned} r_0 &= e_0 + \dots + e_{u_0}\,, \\ r_k &= e_{u_0+\dots+u_{k-1}+1} + \dots + e_{u_0+\dots+u_k}\,, \qquad \text{for } k \geqq 1\,. \end{aligned}$$

We will omit the details that verify the identities of (3).

Let $t_0, t_1, t_2, \dots,$ and $t_{00}, t_0, t_1, t_2, \dots$ denote regressive enumerations of sets belonging to T and $T+1$ respectively. Then

$$(4)\qquad j(t_{00}, \nu(r_0)) + \sum_0^\infty j(t_i, \nu(r_{i+1})) \in \sum_{T+1} r_n\,.$$

Because $T \in \delta E_h$, we know by Lemma 3.2 that

$$(5)\qquad t_n \leqq^* r_n\,.$$

Consider the set appearing in (4); its member are listed in the following array.

Array I.

Row 00	$j(t_{00}, 0), \dots, j(t_{00}, r_0 \dot{-} 1)$,
Row 0	$j(t_0, 0), \dots, j(t_0, r_1 \dot{-} 1)$,
Row 1	$j(t_1, 0), \dots, j(t_1, r_2 \dot{-} 1)$,
...	....

In view of (4) and (5), it follows that the natural enumeration of the numbers appearing in Array I (by this we mean the enumeration obtained by listing the elements in the array first by rows from top toward bottom and then in each row from left to right) will be a regressive enumeration of a set belonging to $\sum_{T+1} r_n$. We now wish to construct a set that belongs to

$$(*)\qquad \sum_{1+\Sigma_{T+1} u_n} e_n\,.$$

We first observe that

$$(6)\qquad j(t_{00}, \nu(u_0)) + \sum_0^\infty j(t_i, \nu(u_{i+1})) \in \sum_{T+1} u_n\,.$$

We now list in the following array, the members of the set that appears in (6).

Array II.

$$\begin{aligned} &j(t_{00}, 0), \dots, j(t_{00}, u_0 \dot{-} 1)\,, \\ &j(t_0, 0), \dots, j(t_0, u_1 \dot{-} 1)\,, \\ &j(t_1, 0), \dots, j(t_1, u_2 \dot{-} 1)\,, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

We know that $T \in \delta E_f$ and hence also that

(7) $$t_n \leqq^* u_n \, .$$

It follows from (6) and (7) that the natural enumeration of the numbers appearing in Array II will be a regressive enumeration of a set belonging to $\sum_{T+1} u_n$. Let t_{000} be a number that does not appear in this set (there will be such a number because the set is isolated). We adjoin t_{000} at the beginning of our enumeration of Array II; it is then easy to see that this new enumeration will be a regressive enumeration of a set belonging to $1 + \sum_{T+1} u_n$. Let us now consider the following array of numbers.

Array III.

$$\textit{Group } 00 \left\{ \begin{array}{c} j(t_{000}, 0), \ldots, j(t_{000}, e_0 \dot{-} 1)\, , \\ j[t_{00}, 0, 0], \ldots, j[t_{00}, 0, e_1 \dot{-} 1]\, , \\ j[t_{00}, 1, 0], \ldots, j[t_{00}, 1, e_2 \dot{-} 1]\, , \\ \vdots \qquad\qquad \vdots \\ j[t_{00}, u_0 \dot{-} 1, 0], \ldots, j[t_{00}, u_1 \dot{-} 1, e_{u_0} \dot{-} 1]\, , \end{array} \right.$$

$$\textit{Group } 0 \left\{ \begin{array}{c} j[t_0, 0, 0], \ldots, j[t_0, 0, e_{u_0+1} \dot{-} 1]\, , \\ j[t_0, 1, 0], \ldots, j[t_0, 1, e_{u_0+2} \dot{-} 1]\, , \\ \vdots \qquad\qquad \vdots \\ j[t_0, u_1 \dot{-} 1, 0], \ldots, j[t_0, u_1 \dot{-} 1, e_{u_0+u_1} \dot{-} 1]\, , \end{array} \right.$$

$$\textit{Group } 1 \left\{ \begin{array}{c} j[t_1, 0, 0], \ldots, j[t_1, 0, e_{u_0+u_1+1} \dot{-} 1]\, , \\ j[t_1, 1, 0], \ldots, j[t_1, 1, e_{u_0+u_1+2} \dot{-} 1]\, , \\ \vdots \qquad\qquad \vdots \\ j[t_1, u_2 \dot{-} 1, 0], \ldots, j[t_1, u_2 \dot{-} 1, e_{u_0+u_1+u_2} \dot{-} 1]\, , \end{array} \right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

It follows from our previous remarks and the definition of an infinite series of isols, that the set whose elements are listed in Array III, belongs to the (regressive) isol given in (*).

We have up to this point obtained two particular sets that we are interested in; the first is given in Array I and belongs to $\sum_{T+1} r_n$, we let ω denote this set; and the second is given in Array III and belongs to

$$\sum_{1+\Sigma_{T+1} u_n} e_n \, ,$$

and we let λ denote this set. We wish to prove the identity of (2), and therefore it suffices to prove that

(8) $$\omega \simeq \lambda \, .$$

Consider the Array I, and let the set of numbers of ω that appear in the nth row be denoted by ω_n. Consider the Array III, and let the set of numbers of λ that are listed in the nth group be denoted by λ_n. We see that each of $\{\omega_n\}$ and $\{\lambda_n\}$ is a sequence of mutually disjoint sets, and

$$\omega = \sum_{00}^{\infty} \omega_n \qquad \text{and} \qquad \lambda = \sum_{00}^{\infty} \lambda_n .$$

In addition, in view of (3), it also follows that, for each number $n=00,0,1,\ldots,$

$$\text{cardinality } \omega_n = \text{cardinality } \lambda_n .$$

Consider the enumerations of the sets ω and λ that are obtained from the natural enumerations of the Arrays I and III, respectively. Let q denote the mapping of ω onto λ that is order preserving with respect to these enumerations. Then

$$q : \omega \to \lambda \quad \text{onto},$$

in a one-to-one manner, and for each number $n=00,0,1,\ldots,$

$$q : \omega_n \to \lambda_n \quad \text{onto},$$

in a one-to-one manner. Let q^{-1} denote the inverse function of q; then

$$q^{-1} : \lambda \to \omega \quad \text{onto},$$

in a one-to-one manner. In view of (1.4), we see that to prove (8), it suffices to show that both

(A) q has a partial recursive extension, and
(B) q^{-1} has a partial recursive extension.

This will be our approach here. We will verify (A); and leave the details for (B) to the reader because these are similar to those for (A).

Proof of (A). We may assume without any loss of generality that the value of t_{000} is known to us. Let the number $z=j(t_n,y) \in \omega$ be given. We wish to find the value of $q(z)$. Since j is a one-to-one recursive function and t_x is a regressive function, we can find the numbers, n, t_n and y. Then $z \in \omega_n$; and hence also $q(z) \in \lambda_n$. We consider separately two cases.

Case 1. $n=00$. Then $0 \leqq y \leqq r_0$ [we note that at this point we have not computed the value of r_0]. In view of (3), we see that

$$r_0 = e_0 + \ldots + e_{u_0},$$

[also at this point we have not computed the value of u_0]. Because e_x is a recursive function, we can find the value of e_x for any number x. If

$$0 \leqq y < e_0, \quad \text{then} \quad q(z) = j(t_{000}, y),$$

and we are done (recall that the value of t_{000} is known to us). Otherwise, we can determine the numbers k and m such that

$$e_0 + \ldots + e_k \leqq y < e_0 + \ldots + e_k + e_{k+1},$$

and

$$y = e_0 + \ldots + e_k + m .$$

In this event, then

$$q(z) = j[t_{00}, k, m] = j[t_n, k, m],$$

and the value of $q(z)$ can be found.

Case 2. $n \geqq 0$. Then $0 \leqq y < r_{n+1}$, and by (3),

$$r_{n+1} = e_{u_0+\ldots+u_n+1} + \ldots + e_{u_0+\ldots+u_n+u_{n+1}} .$$

Since the value of t_n is known, we can compute the numbers $t_0, t_1, \ldots, t_n$ and therefore also, in view of (7), the numbers $u_0, u_1, \ldots, u_n$. We then can find the numbers

$$u_0 + \ldots + u_n + 1 \quad \text{and} \quad e_{u_0+\ldots+u_n+1} .$$

If $y < e_{u_0+\ldots+u_n+1}$, then $q(z) = j[t_n, 0, y]$, and we are done. Otherwise, we can compute the numbers k and m such that

$$e_{u_0+\ldots+u_n+1} + \ldots + e_{u_0+\ldots+u_n+k} \leqq y < e_{u_0+\ldots+u_n+1} + \ldots + e_{u_0+\ldots+u_n+k+1},$$

and

$$y = e_{u_0+\ldots+u_n+1} + \ldots + e_{u_0+\ldots+u_n+k} + m .$$

Also, in this event,

$$q(z) = j[t_n, k, m];$$

and therefore the value of $q(z)$ can be found. In view of our remarks above we can conclude that the mapping

$$q : \omega \to \lambda ,$$

will have a partial recursive extension; and this verifies (A).

As we remarked earlier, we will omit the details for (B). Combining (A) and (B) implies the desired result of (8). This verifies part (c) of the theorem and completes the proof.

5. Extensions of sets to regressive isols.

We wish to introduce in this section an extension procedure for any set of numbers α to a particular set α_* of regressive isols.

DEFINITION D2. Let $\alpha \subseteq E$ be any set of numbers. If α is a finite set, then we let $\alpha_* = \alpha$. If α is an infinite set, then we let

$$\alpha_* = E_f(\delta E_f) ,$$

where f denotes the principal function of the set α.

THEOREM 5.1. *Let α be a recursive set of numbers. Then $\alpha_* = \alpha_R$.*

PROOF. If α is a finite set then the desired result follows from the well-known property that $\alpha = \alpha_\Lambda = \alpha_R$.

Assume now that α is an infinite set. Let f denote the principal function of α. Then f is a recursive and strictly increasing function. By [4, Proposition 3], it follows that

$$\alpha_R = D_f(\Lambda_R) . \tag{1}$$

In addition, by Theorem 4.1, it also follows that $\delta E_f = \Lambda_R$ and

$$E_f(\Lambda_R) = D_f(\Lambda_R) . \tag{2}$$

Combining (1), (2), and D2 gives the desired result; and this completes the proof.

THEOREM 5.2. *Let α be an infinite set of numbers. Then*

(a) $\alpha \subseteq \alpha_* \subseteq \Lambda_R$, *and*
(b) *cardinality of* $\alpha_* = c$.

PROOF. The first inclusion in (a) follows from the Remark that appears after D1 in § 4, and the second inclusion follows from Theorem 4.2(b). Part (b) of the theorem follows from D2 and Corollary 4.1.

THEOREM 5.3. *Let α be any set and β a recursive set. Then*

$$\alpha \subseteq \beta \Rightarrow \alpha_* \subseteq \beta_R .$$

PROOF. Let us assume that $\alpha \subseteq \beta$. If α is a finite set then it is easy to see that $\alpha_* \subseteq \beta_R$.

Let us assume now that α, and hence also β, are infinite sets. Let f and g denote the principal functions of the sets α and β, respectively.

Then g is a recursive function. Let k denote the strictly increasing function such that

(1) $$f(n) = g(k(n)), \qquad \text{for } n \in E .$$

By Theorem 4.5, we then have from (1) that

$$\delta E_f = \delta E_k ,$$

and

(2) $$E_f(T) = E_g[E_k(T)], \qquad \text{for } T \in \delta E_f .$$

Let $S \in \alpha_*$. Then $S = E_f(T)$, for some regressive isol $T \in \delta E_f$. Set $Q = E_k(T)$; then Q is a regressive isol since $T \in \delta E_k$. In view of (2) and the fact that g is a recursive function, it follows that

(3) $$S = E_f(T) = E_g(Q) = D_g(Q) .$$

Because $\beta_R = D_g(\Lambda_R)$, we can conclude from (3), that $S \in \beta_R$. This completes the proof.

6. Torre regressive isols.

We call an infinite isol T *torre*, if for each recursive set α,

$$T \in \alpha_\Lambda \cup \bar{\alpha}_\Lambda .$$

It follows from the definition of α_R, that an infinite regressive isol T is torre, if for every recursive set α,

$$T \in \alpha_R \cup \bar{\alpha}_R .$$

It is an easy consequence of Theorem 4.1, Corollary 4.3 and Theorem 4.4 of [15], that torre regressive isols exist. We wish to give another proof of this fact. We recall from [16, p. 231], that a set δ is *cohesive* if

(i) δ is infinite, and
(ii) for each recursively enumerable set β, either $\delta \cap \beta$ is finite or $\delta \cap \bar{\beta}$ is finite.

The existence of cohesive sets is proved in [16]. In addition, the following properties are also known [16, p. 231],

(6.1) cohesive sets are immune, and
(6.2) every infinite set possesses a cohesive subset.

We state next two lemmas; the first follows in an easy manner from the definition of a cohesive set and we will omit its proof.

Lemma 6.1. *Let δ be a cohesive set and let β be any recursive set. Then either*

$$\delta \subseteq \beta + \lambda, \quad \textit{for some finite set } \lambda\,,$$

or

$$\delta \subseteq \bar{\beta} + \lambda, \quad \textit{for some finite set } \lambda\,.$$

Lemma 6.2. *Let α and β be two infinite recursive sets and let λ be a finite set such that,*

$$(1) \qquad \alpha = \beta + \lambda\,.$$

Let T be an infinite regressive isol. Then

$$(2) \qquad T \in \alpha_R \Leftrightarrow T \in \beta_R\,.$$

Proof. Let f and g denote the principal functions of α and β respectively. Then f and g will be strictly increasing recursive functions and

$$(3) \qquad \alpha_R = D_f(\Lambda_R) \quad \text{and} \quad \beta_R = D_g(\Lambda_R)\,.$$

Combining (1) and the fact that λ is a finite set, we see that there will be numbers p and q such that

$$(4) \qquad f(x+p) = g(x+q), \qquad \text{for } x \in E\,.$$

By a well-known theorem of Nerode, it follows from (4) that,

$$(5) \qquad D_f(X+p) = D_g(X+q), \qquad \text{for } X \in \Lambda\,.$$

To prove (2), let us assume first that $T \in \alpha_R$. By (3), we then have that

$$T = D_f(S), \qquad \text{for some } S \in \Lambda_R\,.$$

Because T is an infinite regressive isol, S will also be an infinite regressive isol. Let $\tilde{S} = S - p$. Then $\tilde{S} \in \Lambda_R$ and from (5) we have that

$$(6) \qquad T = D_f(\tilde{S}+p) = D_g(\tilde{S}+q)\,.$$

Since $\tilde{S}+q \in \Lambda_R$, we can conclude from (3) and (6) that

$$T \in D_g(\Lambda_R) = \beta_R\,.$$

We have therefore proved that

$$T \in \alpha_R \Rightarrow T \in \beta_R\,.$$

The implication in the other direction can be also proved in a similar manner and we will omit the details. Together they give the desired result of (2), and this completes the proof.

THEOREM 6.1. *Let δ be a cohesive set. Let T be an infinite regressive isol. Then*

$$T \in \delta_* \Rightarrow T \quad \textit{is a torre regressive isol.}$$

PROOF. Assume that $T \in \delta_*$. Let β be any recursive set. We wish to prove that either $T \in \beta_R$ or $T \in \bar{\beta}_R$. By Lemma 6.1, we know that either

(1) $\delta \subseteqq \beta+\lambda,$ for some finite set λ ,

or

(2) $\delta \subseteqq \bar{\beta}+\lambda,$ for some finite set λ .

We consider separately the two possibilities:

CASE 1. (1) is true. Then β will be an infinite set and $\beta+\lambda$ will be a recursive set. In this case, we have

(3) $$\delta_* \subseteqq (\beta+\lambda)_R ,$$

by Theorem 5.3. Also, because T is an infinite regressive isol and $T \in \delta_*$, it follows from (3) and Lemma 6.2 that $T \in \beta_R$.

CASE 2. (2) is true. Here one can proceed as in the previous case, to show that in this event $T \in \bar{\beta}_R$; we will omit the details.

In view of the previous remarks, we see that if (1) holds then $T \in \beta_R$, and if (2) holds then $T \in \bar{\beta}_R$; in any event it follows that

$$T \in \beta_R \cup \bar{\beta}_R .$$

We can conclude therefore that T is a torre regressive isol. This completes the proof.

COROLLARY 6.1. *There exist c torre regressive isols.*

PROOF. Combine Theorems 5.2 and 6.1.

COROLLARY 6.2. *Let α be any infinite recursive set. Then α_R contains c torre regressive isols.*

PROOF. By (6.2), α contains a cohesive subset. Let δ be a cohesive subset of α. Then $\delta_* \subseteqq \alpha_R$, by Theorem 5.3. The desired result now follows from this property and Theorems 5.2 and 6.1.

REMARK. We wish to close the paper with some observations related to torre isols that arise naturally. Let Λ_T denote the collection of all isols that are either finite or are infinite and torre. For any recursive set α, let

$$(*)\qquad \alpha_T = \alpha_\Lambda \cap \Lambda_T .$$

We call α_T the *torre extension of* α. Regarding the Nerode extension for recursive sets, the following properties are well-known:

(1) $\varphi_\Lambda = \varphi$ [where φ denotes the empty set],
(2) $\alpha \subseteq \alpha_\Lambda$,
(3) $\alpha \subseteq \beta \Rightarrow \alpha_\Lambda \subseteq \beta_\Lambda$,
(4) $(\alpha \cap \beta)_\Lambda = \alpha_\Lambda \cap \beta_\Lambda$,
(5) $(\alpha \cup \beta)_\Lambda$ need not equal $\alpha_\Lambda \cup \beta_\Lambda$.

In view of (*), it is easy to see that each of properties (1), (2), (3) and (4) will have analogues involving the torre extension of recursive sets; these are

(6) $\varphi_T = \varphi$,
(7) $\alpha \subseteq \alpha_T$,
(8) $\alpha \subseteq \beta \Rightarrow \alpha_T \subseteq \beta_T$,
(9) $(\alpha \cap \beta)_T = \alpha_T \cap \beta_T$.

In addition, the following property will also be true,

(10) $(\alpha \cup \beta)_T = \alpha_T \cup \beta_T$.

To verify (10), let α and β denote recursive sets. The inclusion

$$\alpha_T \cup \beta_T \subseteq (\alpha \cup \beta)_T ,$$

follows readily from (8). To verify the inclusion in the other direction, let $X \in (\alpha \cup \beta)_T$. If X is a finite isol, then $X \in \alpha \cup \beta$. This fact follows from the well-known property that, if $n \in E$, then $n \in \alpha_\Lambda \Leftrightarrow n \in \alpha$. In view of (8), it follows in this event that $X \in \alpha_T \cup \beta_T$. Let us assume now that X is an infinite torre isol. Consider the following implications,

$$\begin{aligned} X \in (\alpha \cup \beta)_T &\Rightarrow X \notin (\overline{\alpha \cup \beta})_T \Rightarrow X \notin (\bar{\alpha} \cap \bar{\beta})_T \\ &\Rightarrow X \notin \bar{\alpha}_T \cap \bar{\beta}_T \Rightarrow X \in \alpha_T \cup \beta_T . \end{aligned}$$

The first implication follows from (6) and (9). The second one is clear, and the third follows from (9). Finally, the last one follows from the torre property of X. Together they imply,

$$X \in (\alpha \cup \beta)_T \Rightarrow X \in \alpha_T \cup \beta_T .$$

Combining our previous remarks, we can conclude that the desired relation of (10) holds.

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