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ON POINT REALIZATIONS OF L^∞-ENDOMORPHISMS

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Let Z be a locally compact space and μ a positive Radon measure on Z. $M^\infty(Z,\mu)$ denotes the set of bounded complex-measurable functions, $N^\infty(Z,\mu)$ the set of functions in $M^\infty(Z,\mu)$ which are null locally almost everywhere (l.a.e.). $L_\mu^\infty(Z)$ is the quotient $M^\infty(Z,\mu)/N^\infty(Z,\mu)$ and is a commutative C^*-algebra with a complete ordering as the dual of the ordered space $L^1(Z,\mu)$.

A $*$-homomorphism $\Phi\colon L^\infty(Z,\mu) \to L^\infty(Z_1,\mu_1)$ is called normal if $\Phi(\sup\mathscr{F}) = \sup\varphi(\mathscr{F})$ for any upwards directed and bounded set $\mathscr{F} \subseteqq L^\infty(Z,\mu)$. J. von Neumann proved in [4] that such a Φ is implemented by a point map $\eta\colon Z_1 \to Z$ if Z and Z_1 are metrizable. Applying liftings and disintegration for measures, C. Ionesco Tulcea obtained the result for compact spaces [2]. The purpose of this note is to prove the result in the general case. This will be done only by use of liftings.

A lifting on (Z,μ) is a map $\varrho\colon L^\infty(Z,\mu) \to M^\infty(Z,\mu)$ such that ϱ is linear, positive, multiplicative, $\varrho(I)=1$ and $\varrho(\tilde{f})=f$ l.a.e. where $\tilde{f}$ is the canonical image in $L^\infty(Z,\mu)$ of $f \in M^\infty(Z,\mu)$. There exists always a lifting for any (Z,μ) [3].

For the sake of completeness, we state a reformulation of (P), Appendix I in [1].

PROPOSITION. Let $\mathscr{F} \subseteqq L^\infty(Z,\mu)$ be an upwards directed and bounded set and let ϱ be a lifting on (Z,μ). Then the function $f_\infty = \sup\{\varrho(f) \mid f \in \mathscr{F}\}$ is in $M^\infty(Z,\mu)$ and $\varrho(\sup\mathscr{F}) \geqq f_\infty$ and $\sup\mathscr{F} = \tilde{f}_\infty$.

THEOREM. *Let $\Phi\colon L^\infty(Z,\mu) \to L^\infty(Z_1,\mu_1)$ be a normal $*$-homomorphism with $\Phi(1)=1$. Then there exists a map $\eta\colon Z_1 \to Z$ with the following properties:*

1) *If $f \in M^\infty(Z,\mu)$, then $f\circ\eta \in M^\infty(Z_1,\mu_1)$.*
2) *If $A \subset Z$ is a null set, then $\eta^{-1}(A)$ is a null set.*
3) *If $f \in M^\infty(Z,\mu)$, then $\Phi(\tilde{f}) = \widetilde{f\circ\eta}$.*

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Proof. Without loss of generality we may assume that μ has support Z. Then, denoting the one-point compactification of Z by Z_{∞}, $C(Z_{\infty})$ is imbedded in $L^{\infty}(Z,\mu)$. Let ϱ be a lifting on (Z_1,μ_1). For $z_1 \in Z_1$, the functional $f \in C(Z_{\infty}) \to \varrho(\Phi(f))(z_1)$ is a character on $C(Z_{\infty})$. Hence there is a unique $z=\eta(z_1) \in Z_{\infty}$ such that

$$\varrho(\Phi(f))(z_1) = f(\eta(z_1)) \quad \text{for } f \in C(Z_{\infty}) . \tag{1}$$

In other words, η is a map $Z_1 \to Z_{\infty}$ such that $\varrho(\Phi(f))=f\circ\eta$ for $f \in C(Z_{\infty})$. Let now g be lower semicontinuous (l.s.c.), positive, bounded, and real on Z. Then $g=\sup\{f \in C_0(Z) \mid 0 \leqq f \leqq g\}$ and $\breve{g}=\sup\breve{f}$. Hence, by normality,

$$\begin{aligned}\Phi(\breve{g}) &= \sup\Phi(\breve{f}) = \sup \overparen{f\circ\eta}\\ &= (\sup \varrho(\overparen{f\circ\eta}))^{\smile} = (\sup f\circ\eta)^{\smile} = \overparen{g\circ\eta}\end{aligned}$$

where we agree that any function g on Z is extended to Z_{∞} by $g(\infty)=0$. Here we used the proposition; we get also $g\circ\eta \leqq \varrho(\Phi(g))$. In particular, we have

$$\overparen{1_{Z_1}} = \Phi(1) = (1_Z\circ\eta)^{\smile} = (1_{\eta^{-1}(Z)})^{\smile} .$$

Hence $Z_1 \setminus C$ is a local null set, where $C=\eta^{-1}(Z)$. Adding constants, we remove the condition that g is positive to the effect that the following holds:

$$\varrho(\Phi(\breve{g})) \geqq g\circ\eta \quad \text{on } C , \tag{2}$$

$$\Phi(\breve{g}) = \overparen{g\circ\eta} . \tag{3}$$

Let now $h \in M^{\infty}(Z,\mu)$ be real and

$$\mathscr{F}_u = \{g \mid g \text{ l.s.c.}, g \text{ real}, g \geqq h\} .$$

Then $\breve{h}=\inf\overparen{\mathscr{F}_u}$, hence $\Phi(\breve{h})=\inf\Phi(\overparen{\mathscr{F}_u})$. Let

$$\varphi_u = \inf\varrho(\Phi(\breve{g})) .$$

Then, by the proposition, φ_u is measurable and by (2)

$$\varphi_u \geqq \inf g\circ\eta \quad \text{on } C; \tag{4}$$

therefore $\varphi_u \geqq h\circ\eta$ on C and, by the proposition,

$$\breve{\varphi}_u = \big(\inf\varrho(\Phi(\breve{g}))\big)^{\smile} = \inf\Phi(\breve{g}) = \Phi(\breve{h}) . \tag{5}$$

In the same way, replacing h by $-h$, we find a $\varphi_1 \in M^{\infty}(Z_1,\mu_1)$ such that

$$\varphi_1 \leqq h\circ\eta \quad \text{on } C , \tag{6}$$

$$\breve{\varphi}_1 = \Phi(\breve{h}) . \tag{7}$$

Consequently we have $\overbrace{\varphi_1} = \Phi(\tilde{h}) = \overbrace{\varphi_u}$ and $\varphi_1 \leqq h \circ \eta \leqq \varphi_u$ on C. Hence $h \circ \eta$ is measurable and $\overbrace{h \circ \eta} = \Phi(\tilde{h})$.

Finally, we modify η on $\eta^{-1}(\infty)$ to take some fixed value z_0 in Z. The relation $\Phi(\tilde{h}) = \overbrace{h \circ \eta}$ is still true. From this we easily infer the three statements of the Theorem.

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