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ON $(k-1)$-CONNECTED $(2k+1)$-MANIFOLDS

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1. Introduction.

This paper is concerned with the classification problem for closed, infinitely differentiable, $(k-1)$-connected $(2k+1)$-manifolds under the relation of orientation preserving diffeomorphism. It is assumed that k is even, $k>2$, and that the manifolds are either k-parallelizable or almost parallelizable. These assumptions permit the application of certain techniques of differential topology that yield a rather simple analysis of such manifolds. There is also a result for k odd (Theorem 2).

A preliminary result is the following: for k even there are at most a finite number of nondiffeomorphic $(k-1)$-connected almost parallelizable $(2k+1)$-manifolds with the k-dimensional homology group cyclic and not zero. In particular, such a manifold is almost diffeomorphic (and hence homeomorphic) to either the product of spheres $S^k \times S^{k+1}$ or the tangent k-sphere bundle $V_{k+2,2}$ to the $(k+1)$-sphere. An upper bound is obtained here on the number of such manifolds; the exact number has been obtained in most cases in [1]. More generally we have the following.

THEOREM 1. *Let $k>2$ be even and let M^{2k+1} be a $(k-1)$-connected, closed, almost parallelizable $(2k+1)$-manifold. Then M^{2k+1} is diffeomorphic to the connected sum*

$$S^k \times S^{k+1} + \ldots + S^k \times S^{k+1} + V_{k+2,2} + \ldots + V_{k+2,2} + M_T ,$$

where M_T is a $(k-1)$-connected π-manifold such that $H_k(M_T)$ is a finite group that is not cyclic and is isomorphic to the torsion part of $H_k(M)$.

REMARKS. If $H_k(M^{2k+1})$ is free, then M_T in the above decomposition must be a homotopy $(2k+1)$-sphere. If $k \equiv 6 \pmod 8$, then the assumption of almost parallelizability in this theorem may be removed.

The decomposition of this theorem has also been obtained by A. Vasquez [9] who also decomposes M_T into a connected sum of rather simple

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manifolds. I. Tamura [8] has results in the torsion-free case. The work of the present paper was done independently and without the knowledge of [8] and [9]. The approach presented in this paper is somewhat different, employing some of the techniques of [4], and results in what the author believes to be a simpler and more geometric exposition. The most general and complete study of $(k-1)$-connected $(2k+1)$-manifolds appears in the recent paper [11] of Wall.

ADDED IN PROOF. Since this paper was written in 1964 the paper [11] of Wall has appeared.

It was originally proved by Wall that a $(k-1)$-connected $(2k+1)$-manifold M^{2k+1} with $H_k(M)$ cyclic and not zero must have $H_k(M)$ infinite cyclic, provided that k is even. This is not the case when k is an odd integer. In fact, according to [5] there are, for a given prime p, exactly 1984 4-connected 11-manifolds M^{11} that bound π-manifolds such that $H_5(M^{11})$ is cyclic of order p.

We shall use the standard terminology. *Differentiable* or *smooth* will always mean of class C_∞, and all manifolds are understood to be smooth and oriented. The term diffeomorphism means an orientation preserving diffeomorphism, and all embeddings are assumed to be orientation preserving whenever this makes sense. A manifold is *almost parallelizable* if the removal of a point yields a parallelizable manifold, and a *π-manifold* is a manifold with a stably trivial tangent bundle. A manifold is *k-parallelizable* if the restriction of the tangent bundle to the k-skeleton is trivial. The unit k-sphere in euclidean $(k+1)$-space R^{k+1} is denoted by S^k and the unit $(k+1)$-disc in R^{k+1} is denoted by D^{k+1}. Finally, ∂M denotes the boundary of a manifold M; thus $S^k=\partial D^{k+1}$.

2. A decomposition.

The following proposition follows easily from results of S. Smale.

PROPOSITION 1. *Let M^{2k+1} be a closed, $(k-1)$-connected, k-parallelizable $(2k+1)$-manifold, where $k>2$, and suppose that $\sigma_1,\ldots,\sigma_p,\tau_1,\ldots,\tau_q$ is a set of generators for a direct sum decomposition of $H_k(M)$ such that σ_i is free and τ_j is of finite order with q minimal for such a decomposition of $H_k(M)$. Then there is a diffeomorphism h of the connected sum of $p+q$ copies of $S^k\times S^k$ such that M^{2k+1} is diffeomorphic to the disjoint union of two copies of $\sum_{i=1}^{p+q}(S^k\times D^{k+1}, S^k\times S^k)_i$ with points identified along the boundary $\sum_{i=1}^{p+q}(S^k\times S^k)_i$ under the diffeomorphism h. If $k\equiv 3,5,6,7 \pmod 8$, then the above decomposition is valid for any $(k-1)$-connected $(2k+1)$-manifold.*

For the definition of the connected sum along the boundary $\sum_{i=1}^{p+q}(S^k \times D^{k+1}, S^k \times S^k)$ of $p+q$ copies of $S^k \times D^{k+1}$ the reader is referred to [4].

PROOF. According to Smale [7, Theorem 6.1] there is a real valued C_∞ function f on M^{2k+1} with only nondegenerate critical points such that at each critical point $\beta \in M^{2k+1}$, $f(\beta)$ is equal to the index of f at β. Furthermore, f has the property that the number m_i of critical points of index i is such that $m_0 = m_{2k+1} = 1$, $m_i = 0$ for $0 < i < k$ and $k+1 < i < 2k+1$, and $m_k = m_{k+1} = p+q$. It follows that M^{2k+1} may be constructed in the following manner. There is an embedding

$$g_1\colon \bigcup_{i=1}^{p+q} S_i^{k-1} \times D^{k+1} \to S^{2k}$$

of $p+q$ copies of $S^{k-1} \times D^{k+1}$ disjointly in $S^{2k} = \partial D^{2k+1}$, and an embedding

$$g_2\colon \bigcup_{j=1}^{p+q} S_j^{k} \times D^{k} \to \partial H^{2k+1}$$

of $p+q$ copies of $S^k \times D^k$ disjointly in the boundary of the handlebody $H^{2k+1} = D^{2k+1} \cup_{g_1} (\bigcup_{i=1}^{p+q} D_i^k \times D^{k+1})$. Finally, there is a diffeomorphism

$$g_3\colon S^{2k} \to \partial[H^{2k+1} \cup_{g_2} (\textstyle\bigcup_{j=1}^{p+q} D_j^{k+1} \times D^k)]$$

of S^{2k} onto the boundary of $H^{2k+1} \cup_{g_2} (\bigcup_{j=1}^{p+q} D_j^{k+1} \times D^k)$ such that M^{2k+1} is diffeomorphic to the latter manifold with D^{2k+1} attached along the boundary S^{2k} under the diffeomorphism g_3.

We shall show that H^{2k+1} is diffeomorphic to $\sum_{i=1}^{p+q}(S^k \times D^{k+1}, S^k \times S^k)_i$, which is also a handlebody determined by some embedding

$$g\colon \bigcup_{i=1}^{p+q} S_i^{k-1} \times D^{k+1} \to S^{2k}\,.$$

In view of dimension considerations it is clear that the restricted embeddings $g \mid \bigcup_{i=1}^{p+q} S_i^{k-1} \times 0$ and $g_1 \mid \bigcup_{i=1}^{p+q} S^{k-1} \times 0$ are diffeotopic (deform the embeddings $g_1 \mid S_i^{k-1} \times 0$ into $g \mid S_i^{k-1} \times 0$ respectively, one at a time), and hence by diffeotopy extension we may assume (cf. [6]) that the embeddings g and g_1 coincide on $\bigcup_{i=1}^{p+q} S_i^{k-1} \times 0$. At this point it is clear that H^{2k+1} is determined merely by the product structure chosen on the normal bundle of $g \mid \bigcup_{i=1}^{p+q} S_i^{k-1} \times 0$ in S^{2k}; that is to say, by the extension of $g \mid \bigcup_{i=1}^{p+q} S_i^{k-1} \times 0$ to $\bigcup_{i=1}^{p+q} S_i^{k-1} \times D^{k+1}$, and this will be the embedding g_1. This product structure must be chosen so that H^{2k+1} is parallelizable and hence is unique, as follows. The embedded sphere $g(S_i^{k-1} \times 0)$ bounds a smooth k-disc B_i^k in D^{2k+1} such that $B_i^k \cap S^{2k} = \partial B_i^k$ and B_i^k is unique up to diffeotopy. There is a unique (up to orientation) product structure on the normal bundle of $g(S_i^{k-1}\times 0)$ in S^{2k} that extends to a product structure on the normal bundle of B_i^k in D^{2k+1}; letting

$$\psi_i\colon\ S_i^{k-1}\times D^{k+1}\to S^{2k}$$

denote this unique product structure we have $\psi_i=g\,|\,S_i^{k-1}\times D^{k+1}$. Now given any other product structure φ_i on the normal bundle of $g(S_i^{k-1}\times 0)$ in S^{2k}, then, by application of the tubular neighborhood theorem, $\varphi_i(S_i^{k-1}\times D^{k+1})$ may be deformed onto $\psi_i(S_i^{k-1}\times D^{k+1})$ under a diffeotopy leaving $\varphi_i(S_i^{k-1}\times 0)$ pointwise fixed such that after the diffeotopy

$$\varphi_i(u,v) = \psi_i(u,v\cdot\alpha(u)),$$

where $\alpha\colon S^{k-1}\to SO_{k+1}$ is a smooth map and $v\cdot\alpha(u)$ denotes the usual action of SO_{k+1} on D^{k+1}. The homotopy class of α in $\pi_{k-1}(SO_{k+1})$ classifies the normal bundle of the k-sphere $B_i^k\cup(D_i^k\times 0)$ (which may be assumed to be smoothly embedded) in the handlebody $D^{2k+1}\cup_{\varphi_1}(D_i^k\times D^{k+1})$. If this handlebody is to be parallelizable, then this normal bundle must be trivial (cf. [4, Lemma 5.3]). It follows that α may be taken to be the constant map: $\alpha(u)=1$ for each $u\in S^{k-1}$, and hence $g_1=g$ as desired.

It is easy to see that the closure $\mathrm{cl}(M^{2k+1}-H^{2k+1})$ is diffeomorphic to H^{2k+1}. In fact, if in the preceding argument the function f is replaced by $-f$, then the manifold M^{2k+1} is turned "upside down" by applying handlebody theory to $-f$. It is sufficient to observe that the type numbers m_i' of $-f$ are same as those of f since $m_i'=m_{2k+1-i}$.

The final statement of the proposition follows from the above proof by noting that $\pi_{k-1}(SO_{k+1})=0$ for $k\equiv 3,5,6,7 \pmod 8$.

3. Classification.

Let M^{2k+1} be a $(k-1)$-connected, closed, almost parallelizable $(2k+1)$-manifold such that $k>2$ is even. We shall first assume that $H_k(M)$ is cyclic and not zero. Specifically we shall show the following.

PROPOSITION 2. *Assume that $k>2$ is even and let M^{2k+1} be a $(k-1)$-connected, closed, almost parallelizable $(2k+1)$-manifold such that $H_k(M)$ is cyclic and not zero. Then M^{2k+1} is homeomorphic to either $S^k\times S^{k+1}$ or $V_{k+2,2}$. In particular, there is a homotopy $(2k+1)$-sphere Σ^{2k+1} such that M^{2k+1} is diffeomorphic to either $(S^k\times S^{k+1})+\Sigma^{2k+1}$ or $V_{k+2,2}+\Sigma^{2k+1}$.*

REMARKS. Since there are only a finite number of homotopy spheres (in dimension $\neq 3$) there are only a finite number of such manifolds

M^{2k+1}. It is a result due to I. M. James and J. H. C. Whitehead that $S^k \times S^{k+1}$ and $V_{k+2,2}$ have different homotopy types when $k \neq 0, 2, 6$.

PROOF. According to the preceding section M^{2k+1} may be constructed by taking the manifold $S^k \times D^{k+1}$ (an orientation is chosen on $S^k \times D^{k+1}$ and held fixed throughout) and first attaching the handle $D^{k+1} \times D^k$ by means of an appropriate embedding

$$\varphi\colon\ S^k \times D^k \to S^k \times S^k;$$

φ determines an embedding of S^k in $S^k \times S^k$ with a trivial normal bundle and a product structure on the normal bundle. Pick a fixed $p_0 \in S^k$. Then $H_k(S^k \times S^k)$ is the free abelian group generated by the homology classes λ and μ represented by the respective embedded spheres $S^k \times p_0$ and $p_0 \times S^k$. Thus $\varphi(S^k \times 0)$ represents a homology class $a\lambda + b\mu$, a and b integers, with normal bundle classified by an element $\alpha(a\lambda + b\mu) \in \pi_{k-1}(SO_k)$ which must be zero. If $\partial\colon \pi_k(S^k) \to \pi_{k-1}(SO_k)$ is the boundary homomorphism of the fibration $S^k = SO_{k+1}/SO_k$, then

$$\alpha(a\lambda + b\mu) \;=\; ab\;\partial\iota_k$$

(cf. [10, Lemma 2]), where $\iota_k \in \pi_k(S^k)$ is the class of the identity map. Hence we must have

$$ab\;\partial\iota_k \;=\; 0\;.$$

Since k is even, $\partial\iota_k$ has infinite order and it follows that either $a = 0$ or $b = 0$.

Case 1. The case $b = 0$ will be considered first. Then $\varphi(S^k \times 0)$ represents the homology class $a\lambda$. Now the embedding φ determines a spherical modification of $S^k \times S^k$ and the result of this modification must be S^{2k} (in order to attach the last cell, the $(2k+1)$-disc D^{2k+1}). That is, the spherical modification determined by φ must have the effect of killing the entire homotopy group $\pi_k(S^k \times S^k)$ and this is the case if and only if $a\lambda$ is indivisible (i.e. $a = \pm 1$), for otherwise the modified manifold would have an element of finite order a in its homology in dimension k (the verification of this is left to the reader). It follows that $(S^k \times D^{k+1}) \cup_\varphi (D^{k+1} \times D^k)$ is contractible and hence the manifold is a homotopy sphere.

Case 2. If $a = 0$, then by exactly the same argument as in Case 1 we have $b = \pm 1$. We shall assume that $b = 1$ since the case where $b = -1$ reduces to this. By application of the embedding theorems of Haefliger [2] it may be assumed that $\varphi(u, 0) = (p_0, u)$ for each $(u, 0) \in S^k \times 0$. It remains to determine the product structures that are possible on the embedding $\varphi \mid S^{k-1} \times 0$. In any case it is clear at this point that $H_k(M^{2k+1})$ is infinite cyclic.

Let

$$\psi\colon\ S^k \times D^k \to S^k \times S^k\;,$$

such that $\psi(u,0)=(p_0,u)$ for all $(u,0)\in S^k\times 0$, be the unique product structure on $p_0\times S^k$ in $S^k\times S^k$ that extends to a product structure

$$\bar{\psi}\colon D^{k+1}\times D^k \to S^k\times D^{k+1}$$

on the $(k+1)$-disc $p_0\times D^{k+1}$ in $S^k\times D^{k+1}$. By the tubular neighborhood theorem and diffeotopy extension we can assume that

$$\varphi(u,v) = \psi(u,v\cdot\alpha(u))$$

for all $(u,v)\in S^k\times D^k$, where $\alpha\colon S^k\to SO_k$ is a smooth map. We can also take the $(k+1)$-sphere $p_0\times D^{k+1})\cup_\varphi(D^{k+1}\times 0)$ to be smoothly embedded in the manifold $(S^k\times D^{k+1})\cup_\varphi(D^{k+1}\times D^k)$; then this $(k+1)$-sphere has a normal bundle classified by the homotopy class $[\alpha]$ of α in the group $\pi_k(SO_k)$. But the manifold $(S^k\times D^{k+1})\cup_\varphi(D^{k+1}\times D^k)$ is parallelizable and hence the normal bundle of any embedded sphere is stably trivial. It follows that the element $[\alpha]$ lies in the kernel of the homomorphism $\pi_k(SO_k)\to\pi_k(SO)$ induced by the natural injection $SO_k\subset SO$, where SO is the infinite special orthogonal group. Now this kernel is of order four when k is even (cf. [3]) but it turns out that the four elements of this kernel determine exactly two distinct manifolds of the form $(S^k\times D^{k+1})\cup_\varphi(D^{k+1}\times D^k)$. These two manifolds are $S^k\times S^{k+1}$ and $V_{k+2,2}$, each with the interior of a $(2k+1)$-disc removed. In fact, let $\alpha\colon S^k\to SO_k$ be a smooth map representing an element in the kernel of $\pi_k(SO_k)\to\pi_k(SO)$ and let $E(\alpha)$ denote the total space of the k-disc bundle over S^{k+1} that is classified by α. Let $S_k\colon\pi_k(SO_k)\to\pi_k(SO_{k+1})$ denote the homomorphism that is induced by the natural inclusion $SO_k\subset SO_{k+1}$. Then the total space of the k-sphere bundle that is classified by $S_k[\alpha]\in\pi_k(SO_{k+1})$ is exactly the double $D(E(\alpha))$ of $E(\alpha)$. (Observe that $D(E(\alpha))$ is two copies of $E(\alpha)$ pasted together along the boundaries under the identity map.) If the interior of a $(2k+1)$-disc is removed from $D(E(\alpha))$, then the result is clearly the parallelizable manifold $(S^k\times D^{k+1})\cup_\varphi(D^{k+1}\times D^k)$, where $\varphi\colon S^k\times D^k\to S^k\times S^k$ is defined by $\varphi(u,v)=\psi(u,v\cdot\alpha(u))$ (recall how ψ was defined above). It remains to show that the bundle $D(E(\alpha))$ which is classified by $S_k[\alpha]$ is either trivial or is the tangent k-sphere bundle to S^{k+1}. Now if $S_{k+1}\colon\pi_k(SO_{k+1})\to\pi_k(SO)$ denotes the homomorphism that is induced by the natural injection $SO_{k+1}\subset SO$, then of course the composition $S_{k+1}\circ S_k$ is the homomorphism $\pi_k(SO_k)\to\pi_k(SO)$ that is induced by the injection $SO_k\subset SO$ and hence $S_k[\alpha]$ lies in the kernel of S_{k+1}. But, as is well known, the kernel of S_{k+1} is the subgroup of order two (when k is even) that is generated by the element $\partial\iota_{k+1}$ which classifies the tangent bundle of S^{k+1}, and the proof of Proposition 2 is complete.

If $k \equiv 6 \pmod 8$, then the condition of almost parallelizability in Proposition 2 may be removed. In fact, recall the final statement of Proposition 1 and then apply the argument given in Case 2 of the proof of Proposition 2 above, recalling that $\pi_k(SO) = 0$ for $k \equiv 6 \pmod 8$. More generally, one may prove the following, noting that

$$\pi_k(SO) = 0 \quad \text{for} \quad k \equiv 5, 6 \pmod 8 .$$

PROPOSITION 3. *If $k \equiv 5, 6 \pmod 8$, then a $(k-1)$-connected $(2k+1)$-manifold is a π-manifold.*

PROOF. The only obstructions to making M^{2k+1} a π-manifold lie in the groups $\pi_k(BSO)$, $\pi_{k+1}(BSO)$, and $\pi_{2k+1}(BSO)$, which are all zero if $k \equiv 5, 6 \pmod 8$. Since M^{2k+1} has the homotopy type of

$$(\vee_j S_j^k) \cup (\cup_i D_i^{k+1}) \cup D^{2k+1} ,$$

the classifying map $M^{2k+1} \to BSO$ of the stable tangent bundle is homotopically trivial.

We are now prepared to prove Theorem 1 stated in the introduction.

PROOF OF THEOREM 1. It is well known that M^{2k+1} may be reduced to a homotopy sphere by a sequence of framed spherical modifications of type $(k+1, k+1)$. (M^{2k+1} is a π-manifold since it is almost parallelizable; see the remark following Lemma 2 of [1].) For convenience we shall be specific about those modifications that we shall begin with. By Proposition 1 M^{2k+1} is the disjoint union of two copies of

$$\Sigma_{i=1}^{p+q} (S^k \times D^{k+1}, S^k \times S^k)_i \tag{1}$$

identified along the boundaries; choose one copy and hold it fixed throughout. Here p and q are those integers defined in Proposition 1 and hence we can suppose that the first p embedded spheres $(S^k \times 0)_j$, $j = 1, \ldots, p$, in (1) represent a free set of generators for the free part of $H_k(M)$. Then in order to reduce M^{2k+1} to a homotopy sphere we may begin the modifications with the spheres $(S^k \times 0)_j$, $j = 1, \ldots, p$. The product structures that are to be chosen on each of these k-spheres may differ from the obvious one (given by

$$(S^k \times D^{k+1})_j \subset \Sigma_{i=1}^{p+q} (S^k \times D^{k+1}, S^k \times S^k)_i)$$

by actions of SO_{k+1} on the fibers D^{k+1} for each $j = 1, \ldots, p$. Then according to [4, Lemma 5.7], by performing these p modifications we shall obtain a manifold M_T' that is $(k-1)$-connected and with $H_k(M_T')$ isomorphic to the torsion part of $H_k(M)$. Now $H_k(M_T')$ is generated by the q embedded k-spheres $(S^k \times 0)_j$, $j=p+1,\ldots,p+q$, and according to [4, proof of Theorem 5.1 for k even], M_T' may be reduced to a homotopy sphere U^{2k+1} by a finite sequence of modifications (in fact, by exactly $2q$ modifications). It follows that the original manifold M^{2k+1} may be obtained from the homotopy sphere U^{2k+1} by performing $p+2q$ modifications on U^{2k+1}. In fact, the first p modifications to be performed on U^{2k+1} may be taken to be the ones that remove the k-spheres $(p_0 \times S^k)_j$, $j=1,\ldots,p$, where $p_0 \in S^k$ is a base point.

Assertion. The first p k-spheres $(p_0 \times S^k)_j$, $j=1,\ldots,p$, that are to be removed from U^{2k+1} are not pairwise linked in U^{2k+1}, nor is any one of them linked with the remaining $2q$ k-spheres that are to be removed from U^{2k+1}.

To prove this assertion let the homology class of $(S^k \times 0)_i$ be denoted by α_i for each $i=1,\ldots,p+q$. Choose a fixed integer j such that $1 \leqq j \leqq p$. Then α_j is free and hence by the Poincaré duality theorem there is a class $\beta_j \in H_{k+1}(M)$ such that the intersection number $\alpha_i \cdot \beta_j = \delta_{ij}$, the Kronecker delta. It follows from the theorems of Hurewicz and Haefliger that β_j may be represented by an embedded sphere $S_j^{k+1} \subset M^{2k+1}$ such that S_j^{k+1} intersects $(S^k \times 0)_j$ in exactly one point $(p_0, 0) \in (S^k \times 0)_j$ and is disjoint from $(S^k \times 0)_i$ for $i \neq j$. (Proof: Since M^{2k+1} is $(k-1)$-connected, the Hurewicz homomorphism $\pi_{k+1}(M) \to H_{k+1}(M)$ is surjective and hence β_j may be represented by a map $g\colon S^{k+1} \to M^{2k+1}$. Then by an embedding theorem of Haefliger [2] the map g is homotopic to an embedding $\varphi\colon S^{k+1} \to M^{2k+1}$ (since $k \geqq 4$). But since $\alpha_i \cdot \beta_j = \delta_{ij}$, we can arrange matters so that $\varphi(S^{k+1}) = S_j^{k+1}$ intersects $(S^k \times 0)_j$ in exactly one point and is disjoint from $(S^k \times 0)_i$ for $i \neq j$. This is done essentially by Whitney's method as described in [12].) Furthermore, we can assume that

$$S_j^{k+1} \cap (S^k \times D^{k+1})_j = (p_0 \times D^{k+1})_j$$

and hence $(p_0 \times S^k)_j$ bounds a disc in the complement of $(S^k \times D^{k+1})_j$ in M^{2k+1} that is disjoint from $(S^k \times 0)_i$ for $i \neq j$. We can also assume that this disc is disjoint from $(p_0 \times S^k)_i$ for each $i \neq j$. Thus the k-sphere $(p_0 \times S^k)_j$ bounds a disc in U^{2k+1} that is disjoint from the sphere $(p_0 \times S^k)_i$ for each $i \neq j$ since we can perform all of the required modifications (on M^{2k+1} in order to obtain U^{2k+1}) away from this disc. That is, the k-sphere $(p_0 \times S^k)_j$ is not linked with $(p_0 \times S^k)_i$ in U^{2k+1} for each $i \neq j$ and thus the assertion follows.

It now follows easily from the above assertion that M^{2k+1} is a connected sum of $p+1$ manifolds that are $(k-1)$-connected, such that each of the first p of these manifolds has an infinite cyclic kth homology group and the last manifold is M_T' with $H_k(M_T')$ a torsion group that is not finite cyclic. Up until now the assumption of k-parallelizability (in place of almost parallelizability) was actually sufficient in all of the arguments. But since M^{2k+1} is assumed to be almost parallelizable, it follows from Proposition 2 that each of the first p manifolds in the decomposition obtained is either $(S^k \times S^{k+1}) + Z$ or $V_{k+2,2} + Z$, where Z is a homotopy $(2k+1)$-sphere. The theorem now follows easily.

The above proof actually establishes the following more general result for any $k \geqq 4$ (the application of [4, Lemma 5.7] in the above proof is valid for any k-parallelizable manifold and any integer $k > 1$).

THEOREM 2. *Let* $k \geqq 4$ *and let* M^{2k+1} *be a* $(k-1)$*-connected, closed,* k*-parallelizable manifold. Then* M^{2k+1} *is diffeomorphic to the connected sum of* $p+1$ $(k-1)$*-connected manifolds*

$$M^{2k+1} \approx M_1 + \ldots M_p + M_{p+1},$$

such that $H_k(M_i)$ *is infinite cyclic for* $i = 1, \ldots, p$ *and* $H_k(M_{p+1})$ *is a finite group. If* k *is even, then* $H_k(M_{p+1})$ *is not cyclic.*

It is possible to apply the methods of [4] in order to decompose the manifold M_T of Theorem 1 into simpler manifolds.

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