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EXTREME POSITIVE LINEAR OPERATORS

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Introduction.

Suppose that A and B are the spaces of real valued continuous functions on compact Hausdorff spaces and that $K(A,B)$ is the set of all positive linear operators from A into B, then we are going to examine the extreme points of

$$K_0(A,B) = \{T \in K(A,B) : T(1) \leqq 1\} ,$$

the extreme points of

$$K_1(A,B) = \{T \in K(A,B) : T(1) = 1\} ,$$

and the indecomposable elements in $K(A,B)$. The multiplicative properties of these elements are examined in [2], so that, as F. F. Bonsall does for functionals in [1], we are particularly interested in characterizing these elements in terms of their order theoretic properties.

Since there are convex sets of positive linear operators mapping B' into A' which contain the adjoints of the elements in $K_0(A,B)$ and $K_1(A,B)$, we shall show that there is a connection between the extremal character of an operator and its adjoint. Then, since Ellis [5] has already characterized the extreme points of $K_1(A,B)$, we shall extend this characterization to $K_0(A,B)$. Before this, however, we shall characterize the indecomposable operators whenever they are in a setting such that the space spanned by the positive linear operators is a vector lattice (although this setting may not include $K(A,B)$, an argument similar to that in theorem 9 of [2] can be used to extend the characterization to include $K(A,B)$). This will lead to an extension of H. Gordon's theorem 5 in [7]. In particular, it will lead to the decomposition of the positive operators into diffuse and atomic components, and we shall show that the extremal character of the operators in $K_0(A,B)$ and $K_1(A,B)$ and their adjoints is usually preserved under the decomposition.

We refer the reader to [12] for the fundamental definitions and nota-

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tions. However, in order that the presentation be somewhat self-contained, we shall list some of the fundamentals and well known results which we shall need. Besides [12], some of this material can be found in [3], [4], and [11] as well as [9] and [13].

Our vector spaces will have the field R of real numbers for their scalar fields, and θ will denote the additive identity. We shall denote the cone of all positive elements in an ordered vector space E by E^+. If E is a vector lattice, then $a, b \in E$ are disjoint ($a \perp b$), if $|a| \wedge |b| = \theta$; moreover, $a \perp b$ if and only if $|a| \vee |b| = |a+b|$.

Let E be an order complete vector lattice. If $E = N \oplus M$ is an order direct sum of N and M, then N and M are bands and $N = M^\perp$. If P_n is the projection of E onto N which vanishes on $N^\perp$, then

$$P_N(x) = \sup\{y \in N : \theta \leqq y \leqq x^+\} - \sup\{y \in N : \theta \leqq y \leqq x^-\}$$

for all $x \in E$. For each $x \in E$, $(\{x\}^\perp)^\perp$ is the band generated by x, and the projection P_x of E onto this band which vanishes on $\{x\}^\perp$ satisfies

$$P_x(y) = \sup_n \{(n|x|)^\wedge y^+\} - \sup_n \{(n|x|)^\wedge y^-\}$$

for all $y \in E$. We shall call the projections P_n and P_x the *projections associated with* N and x, respectively.

If E, F are ordered vector spaces, the space of all order bounded linear operators from E into F will be denoted by $L^b(E, F)$; however, $L^b(E, \mathsf{R})$ will be denoted by E^b. If E is a vector lattice and if F is an order complete vector lattice, then $L^b(E, F)$ is an order complete vector lattice and for $T \in L^b(E, F)$ we have

$$T^+(x) = \sup\{T(z) : \theta \leqq z \leqq x\},$$
$$|T|(x) = \sup\{T(z) : -x \leqq z \leqq x\}$$

for all $\theta \leqq x \in E$. Moreover, $|T(y)| \leqq |T|(|y|)$ and T is a *lattice homomorphism* if $|T(y)| = T(|y|)$ for all $y \in E$. The projections P_n and P_x mentioned above are lattice homomorphisms.

Let E be an Archimedean ordered vector lattice with an order unit e, then the order topology $\mathscr{T}_0$ is a normable topology with the gage functional of $[-e, e]$ as the norm. We shall refer to this norm as the *order unit norm*. If E is also $\mathscr{T}_0$-complete, then $E(\mathscr{T}_0)$ is an (M)-space and, as such, is isomorphic to a space $C(X)$ of continuous real valued functions on a suitable compact Hausdorff space X. In fact, X may be identified with the subset of $E(\mathscr{T}_0)'$ consisting of the extreme points of $\{f \geqq \theta : f(e) = 1\}$, that is, the set of extreme points of the positive face in the dual unit ball. If $E(\mathscr{T}_0)$ is order complete, then $E(\mathscr{T}_0)$ is $\mathscr{T}_0$-complete.

If F is an ordered vector space, then a convex set $B \subset F^+$ is a *base* for F if $\theta \notin B$ and if every element in $F^+ \setminus \{\theta\}$ has a unique representation as a scalar multiple of some element in B. Every B has the property that whenever $\sum_{i=1}^n \lambda_i b_i = \sum_{j=1}^m \gamma_j b_j$ with $b_i, b_j \in B$ and $\lambda_i, \gamma_j \in \mathsf{R}$ $(i=1,\ldots,n;$ $j=1,\ldots,m)$, then $\sum_{i=1}^n \lambda_i = \sum_{j=1}^m \gamma_j$. Let F be a vector lattice with a base B. Then the gage functional p of the convex hull of $B \cup (-B)$ is a norm on F, which is called a *base norm.* The unit ball for the base norm is solid and the norm itself is additive on F^+. If F is complete for this norm, then F is an (L)-space and $F(\|\cdot\|p)' = F^b$ is an (M)-space with an order unit e which satisfies $e(b) = 1$ for all $b \in B$. We shall refer to this order unit e as the *dual order unit.*

If E is an Archimedean ordered vector lattice with an order unit e, then $E(\mathscr{T}_0)' = E^b$ has a base

$$B = \{f \in E^b : \theta \leqq f \text{ and } f(e) = 1\}$$

which we shall call the *dual base* of E^b. The base is $\sigma(E^b, E)$-compact and convex, so it is the $\sigma(E^b, E)$-closed convex hull of its extreme points. If b is an extreme point of B, then $\theta \leqq x \leqq b$ implies that $x = \lambda b$ for some $\lambda \in [0,1]$, that is, b is indecomposable. Conversely if $b \in B$ is indecomposable, then b is an extreme point of B. Thus, the indecomposable elements belonging to B are total over E.

Finally, we remark that an (L)-space E is a band in $(E^b)^b = E''$ and if $y_\alpha \downarrow \theta$ in E, then $\{y_\alpha\}$ converges to θ in norm. Moreover every positive linear operator from E into an ordered locally convex space with a normal cone (every Banach lattice is such a space) is both topologically and order continuous.

We shall adopt the notations of rng for range, dmn for domain, and LH$\{\cdot\}$ for the linear hull spanned by the elements within the braces. Moreover, φ will always denote the canonical embedding of an ordered vector space into its second order dual.

1. Indecomposable operators.

If E is an ordered vector space, then $\theta < a \in E$ is *indecomposable* if $b \in [\theta, a]$ implies that $b = \lambda a$ for some $\lambda \geqq 0$. If E is an order complete vector lattice, then a is indecomposable if and only if $a = b + c$ with $b \geqq \theta$, $c \geqq \theta$, and $b \perp c$ implies that $b = \theta$ or $c = \theta$; moreover, if a is indecomposable in E, then it is easily deduced from the form of the projection P_a associated with a that $\operatorname{rng} P_a = \mathrm{LH}\{a\}$. The following is a well known result (see theorem 3 in [7]).

(1.1) LEMMA. *Let E be a vector lattice and let $\theta \leqq f \in E^b$. Then the following statements are equivalent.*

(a) *f is indecomposable in E^b.*

(b) *The null space of f is a maximal lattice ideal in E.*

(c) *f is a lattice homomorphism.*

(d) *If $x, y \in E$ with $x \perp y$, then either $f(x) = 0$ or $f(y) = 0$.*

We now give the characterization of the indecomposable linear operators which we are interested in. We note that, as with the functionals, the indecomposable linear operators are lattice homomorphisms.

(1.2) PROPOSITION. *Let E and F be vector lattices and let F be order complete. Then $T \in L^b(E,F)$ is indecomposable if and only if* $\operatorname{rng} T = \operatorname{LH}\{a\}$ *for some indecomposable $a \in F$ and T is a lattice homomorphism.*

PROOF. Let T be indecomposable. Let $\theta < x_0 \in E$ satisfy $T(x_0) > \theta$, and define $a = T(x_0)$. Suppose that $a = b + c$ where $\theta \leqq b$, $\theta \leqq c$ and $b \perp c$. Since T is indecomposable in $L^b(E,F)$, there exist $\beta, \gamma \in [0,1]$ such that $P_b \circ T = \beta T$ and $T_c \circ T = \gamma T$ where P_b and P_c are the projections associated with b and c, respectively. Hence, $b = \beta T(x_0)$ and $c = \gamma T(x_0)$, but since $b \perp c$, we conclude that either b or c is θ. It follows that the only disjoint decomposition of a is the trivial one where one of the components is θ, and this implies that a is indecomposable. Finally, if P_a is the projection associated with a, then $\theta < P_a \circ T = \alpha T$ for some $\alpha > 0$. This implies that $\operatorname{rng} T \subset \operatorname{rng} P_a$, but $\operatorname{rng} P_a = \operatorname{LH}\{a\}$ since a is indecomposable.

Define $f \in E^b$ by $f(x) = \lambda_x$ where $\lambda_x \in R$ satisfies $\lambda_x a = T(x)$. If $g \in [\theta, f]$, then $G \in L^b(E,F)$ defined by $G(x) = g(x)a$ satisfies $G \in [\theta, T]$. Consequently, $G = \lambda T$ for some $\lambda \geqq 0$ so that $g = \lambda f$. Therefore, f is indecomposable in E^b and, by (1.1), f is a lattice homomorphism. Then it is clear that T is also a lattice homomorphism.

Conversely, let $\operatorname{rng} T = \operatorname{LH}\{a\}$ for a indecomposable in F and let T be a lattice homomorphism. If $G \in [\theta, T]$, then we have $\operatorname{rng} G \subset \operatorname{LH}\{a\}$ since a is indecomposable. Moreover, from T and G we may define $f, g \in E^b$, as in the above paragraph, with $g \in [\theta, f]$. Since T is a lattice homomorphism, f is also and, by (1.1) f is indecomposable. We conclude that $g = \alpha f$ for some $\alpha \geqq 0$ so that $G = \alpha T$.

Next, we shall give two more results about indecomposable operators which will be usefull later.

(1.3) PROPOSITION. *Suppose that E and F are order complete vector lattices and that $T \in L^b(E,F)$ is indecomposable. Then T is order continu-*

ous if and only if there exist indecomposable elements $b \in E$ and $a \in F$ such that for $x \in E$

$$T(x) = \beta_x a ,$$

where $\beta_x \in \mathsf{R}$ satisfies $\beta_x b = P_b(x)$ for P_b the projection associated with b.

PROOF. Since T is indecomposable, there exists an indecomposable $a \in F$ such that $\operatorname{rng} T = \mathrm{LH}\{a\}$ by (1.2). Let T be order continuous and let $b' \in E$ satisfy $T(b') = a$. We define

$$B = \{c \in [\theta, b'] : T(c) = a\} .$$

Clearly $B \neq \emptyset$ and we can order B with the order of E. Let C be any totally ordered subset of B. Since T is order continuous, it can be seen that $b'' = \inf(C) \in B$. By Zorn's lemma we may choose a minimal $b \in B$ and we claim that b is indecomposable. If $b = r + s$ witn $\theta \leqq r$, $\theta \leqq s$ and $r \perp s$, then since T is a lattice homomorphism with $\operatorname{rng} T = \mathrm{LH}\{a\}$, we can deduce $T(r) = \theta$ or $T(s) = \theta$. For definiteness let $T(r) = \theta$. Then $T(s) = T(b) = a$ and $s \leqq b \leqq b'$ imply that $s \in B$. Since b is minimal in B, it follows that $s = b$. Hence, $r = \theta$ and b must be indecomposable. Finally $\theta \leqq T \circ P_b \leqq T$, for P_b the projection associated with b, implies that $T \circ P_b = \lambda T$ for some $\lambda \geqq 0$. In fact $T \circ P_b(b) = T(b)$ implies that $\lambda = 1$. Thus, for any $x \in E$ we have $T(x) = \beta_x a$ where $\beta_x \in R$ satisfies $\beta_x b = P_b(x)$.

The converse is clear since it is easily seen that P_b is order continuous so that T is order continuous.

(1.4) PROPOSITION. *Let E, F be vector lattices and let F be a regular order complete vector lattice. Then $T \in L^b(E, F)$ is indecomposable if and only if T^* is indecomposable in $L^b(F^b, E^b)$.*

PROOF. Let T be indecomposable. Then, by (1.2), T is a lattice homomorphism with $\operatorname{rng} T = \mathrm{LH}\{a\}$ for some indecomposable $a \in F$. If we define $f \in E^b$ by $T(x) = f(x)a$, then we can deduce that f is indecomposable in E^b. Moreover, for $x \in E$ and $g \in F^b$ we have $T^*g(x) = g(a)f(x)$, and it follows that $\operatorname{rng} T^* = \mathrm{LH}\{f\}$. If $x \in E^+$ and $g \in F^b$, then

$$T^*(g^+)(x) = \sup\{g(y) : \theta \leqq y \leqq T(x)\} = \sup\{\lambda g(a) : 0 \leqq \lambda \leqq f(x)\} .$$

Hence, if $g(a) \leqq 0$, then

$$0 = T^*(g^+)(x) \leqq (T^*g)^+(x) ,$$

and if $g(a) > 0$, then

$$T^*(g^+)(x) = f(x)\, g(a) = T^*g(x) \leqq (T^*g)^+(x) .$$

Consequently, $T^*(g^+) \leqq (T^*g)^+$, and it follows that T^* is a lattice homomorphism since $(T^*g)^+ \leqq T^*(g^+)$. We conclude that T^* is indecomposable from (1.2).

Conversely, let T^* be indecomposable; by (1.2), T^* is a lattice homomorphism with $\operatorname{rng} T^* = \mathrm{LH}\{f\}$ for some indecomposable $f \in E^b$. Let $\theta < x_0 \in E$ satisfy $T(x_0) > \theta$. Then, since the canonical map φ of F into $(F^b)^b$ preserves the lattice operations, for any $g \in F^b$ we obtain

$$\langle \varphi(T(x_0)), |g| \rangle = \langle T^*|g|, x_0 \rangle = \langle |T^*g|, x_0 \rangle \\ = |\langle T^*g, x_0 \rangle| = |\langle \varphi(T(x_0)), g \rangle| .$$

Thus, it follows from (1.1) that $\varphi(T(x_0))$ is an indecomposable element in $(F^b)^b$, so $T(x_0)$ is indecomposable in F. If $T(y) \notin \mathrm{LH}\{T(x_0)\}$ for some $y \in E^+$, then there is an $h \in F^b$ such that $h(T(x_0)) > 0$ and $h(T(y)) = 0$. Thus, for $\lambda_h \in R$ satisfying $\lambda_h f = T^*h$ we infer that $\lambda_h \neq 0$, so $f(y) = 0$. However, as for $T(x_0)$, $T(y)$ must be indecomposable; thus, there exists $\theta < h' \in F^b$ such that $h'(T(y)) > 0$ and $h'(T(x_0)) = 0$. Hence, $f(y) > 0$, and this contradiction implies that $\operatorname{rng} T = \mathrm{LH}\{T(x_0)\}$. Finally, for $x \in E$ either $f(x^+)$ or $f(x^-) = 0$. Therefore, either $T^*g(x^+) = 0$ for all $g \in F^b$ or $T^*g(x^-) = 0$ for all $g \in F^b$. Consequently, $T(x^+) = \theta$ or $T(x^-) = \theta$ and in either case we have that $|T(x)| = T(|x|)$. We conclude from (1.2) that T is a lattice homomorphism.

2. Atomic and diffuse subspaces.

Suppose that E is an Archimedean ordered vector lattice. Let $E_a{}^+$ be the set of all positive elements in E which are either finite linear combinations of indecomposable elements or are the suprema of subsets of such combinations. Then $E_a = E_a{}^+ - E_a{}^+$ is the subspace of *atomic* elements. Let $E_d{}^+$ be the subset of all positive elements which are disjoint from every indecomposable element in E. Then $E_d = E_d{}^+ - E_d{}^+$ is the subspace of *diffuse* elements. When no confusion can arise, we shall refer to E_a (respectively, E_d) as being atomic (respectively, diffuse). If E is an order complete vector lattice, then from (4.9) on page 41 in [12] we infer that E_a is a band with $P_{E_a}(x) = P_{E_a}(x^+) - P_{E_a}(x^-)$ where for each $\theta \leqq y \in E$ we have

$$P_{E_a}(y) = \sup\{h : h \text{ is a finite linear combination of} \\ \text{indecomposable elements and } \theta \leqq h \leqq y\} .$$

It is easily seen that $E_d = (E_a)^\perp$ so that E is the order direct sum of E_a and E_d.

We shall assume throughout this section that E, F are vector lattices and that F is order complete. Then $L^b(E,F)$ is an order complete vector lattice and theorem 5 in [7] can often be extended to $L^b(E,F)$.

(2.1) PROPOSITION. *Let F^b contain a family of order continuous linear functionals which is total over E. Then $T \in L^b(E,F)$ is diffuse if and only if for any $\varepsilon > 0$, $x \in E^+$, and order continuous $f \in F^b$ there exist positive pairwise disjoint operators $T_1, \ldots, T_n$ in $L^b(E,F)$ such that $|T| = T_1 + \ldots + T_n$ and $\langle f, T_i(x) \rangle < \varepsilon$ for $1 \leqq i \leqq n$.*

We shall omit a proof of this result since it can be proven with the same argument that H. Gordon gave in [7] by working with the operators as functionals on $E \otimes F^b$. (We use the order continuity of f to obtain the fact that for a directed ($\geqq$) family of operators G_α in $L^b(E,F)$ we have $(\inf_\alpha G_\alpha)(x \otimes f) = \inf_\alpha (G_\alpha(x \otimes f))$ which is needed in the adoption of Gordon's proof.) We remark that every space F which is the order dual of a vector lattice will satisfy the hypothesis for F above.

Now let us use the characterization of (2.1) to obtain the mapping properties of some of the diffuse and atomic operators.

(2.2) PROPOSITION. *If F^b contains a family of order continuous elements which is total over F, then every diffuse $T \in L^b(E,F)$ maps each indecomposable element of E into a diffuse element of F.*

PROOF. Since the diffuse operators form a sublattice, we may assume that $T > \theta$. For $b \in E$ indecomposable let $T(b) = a + d$ where a is atomic and d is diffuse in F. Assume that $a > \theta$. Then by hypothesis there exists an order continuous $\theta < f \in F^b$ such that $f(a) > 0$. Now, $f = g + h$ where g is atomic and h is diffuse, and we claim that $h(a) = 0$. Since a is atomic and $h \leqq f$ is order continuous it suffices, to show that $h(a') = 0$ for every indecomposable a' in F. Let a' be indecomposable; then since $h \in L^b(F,R) = F^b$, we infer from (2.1) that for $\varepsilon > 0$ there exist positive pairwise disjoint $h_1, \ldots, h_n$ in F^b such that $h = h_1 + \ldots + h_n$ and $\langle h_i, a' \rangle < \varepsilon$ for $1 \leqq i \leqq n$. Since the h_i's are pairwise disjoint and since a' is indecomposable, we can deduce from the definition of $h_i \hat{} h_j$ that at most one i, say $i = 1$, satisfies $h_i(a') \neq 0$. Thus, $h(a') = h_1(a') < \varepsilon$ and we conclude that $h(a') = 0$. Consequently, $g(a) = f(a) > 0$. Since g is atomic and bounded above by f, we infer that there is an indecomposable $g' \leqq g$ which is order continuous and satisfies $g'(a) = \alpha > 0$. Then, by (2.1), there exist positive pairwise disjoint $T_1, \ldots, T_n \in L^b(E,F)$ such that $T = T_1 + \ldots + T_n$ and $\langle g', T_i(b) \rangle < \alpha$ for $i = 1, \ldots, n$. Since b is indecomposable and $T_i \perp T_j$ for $i \neq j$, we can deduce that $T_i(b) \perp T_j(b)$ for $i \neq j$. Thus, by (1.1), we see that $g'(T_i(b)) \neq 0$ for at most one i, say $i=1$. Then we have

$$\alpha \leqq \langle g', T(b)\rangle = \langle g', T_1(b)\rangle < \alpha .$$

From this contradiction we conclude that $T(b) = d \in F_d$.

(2.3) PROPOSITION. *Every atomic operator in $L^b(E,F)$ maps E into F_a.*

PROOF. This is immediate from the definition of atomic elements in terms of indecomposable elements, (1.2), and the fact that F_a is a band.

Finally we have the following result for the order continuous linear operators.

(2.4) PROPOSITION. *Let F^b contain a family of positive order continuous elements which is total over F, and let $T \in L^b(E,F)$ be order continuous. Then*

i) *T is diffuse if and only if $T(E_a) \subset F_d$,*

ii) *T is atomic if and only if $T(E) \subset F_a$ and $T(E_d) = \{\theta\}$.*

PROOF. If T is atomic, then $T(E) \subset F_a$ follows from (2.3). For any indecomposable $G \leqq |T|$ we see that G is order continuous since T is order continuous. Hence, by (1.3), $G(x) = 0$ for all $x \in E_d$, and this implies that $T(E_d) = \{\theta\}$.

i) If T is diffuse, then $T(E_a) \subset F_d$ follows from (2.2), the order continuity of T, and the fact that F_d is a band. On the other hand, let $T(E_a) \subset F_d$, and let A and D be the atomic and diffuse components of T, respectively. Then for $x \in E_a{}^+$ we can see that $|T|(x) \in F_d$. Thus, $-|T|(x) \leqq |A|(x) \leqq |T|(x)$ implies that $|A|(x) \in F_d$. However, from (2.3) we have that $|A|(x) \in F_a$. Thus, $A(E_a) = \{\theta\}$ and from the first paragraph $A(E_d) = \{\theta\}$, so it follows that $A = \theta$ and $T = D$.

ii) From the first paragraph it remains only to prove the sufficiency. Let $T(E) \subset F_a$ and $T(E_d) = \{\theta\}$. We infer that similar statements hold for T^+ and T^-, so we may assume that $T > \theta$. Let $T = A + D$ where A is atomic and D is diffuse. Since $\theta \leqq D \leqq T$, we deduce that $D(E_d) = \{\theta\}$ and $D(E_a) \subset F_a$. However, from i) above $D(E_a) \subset F_d$. Thus, $D(E_a) = \{\theta\}$, and we conclude that $T = A$.

3. Extreme operators.

In this section we return to the spaces which have order units or bases. In particular, we shall extend Ellis' Theorem 1 in [5] and examine the connection between the extremal character of an operator and its adjoint. Of course, when we say extreme point we mean with respect to certain convex sets and specifically with respect to the following: If E and F are ordered vector spaces with order units e and u, respectively, then

1) $$L^b(E,F;e,[\theta,u]) = \{T \in L^b(E,F) : \theta \leqq T \text{ and } T(e) \leqq u\};$$

2) $$L^b(E,F;e,u) = \{T \in (E,F) : \theta \leqq T \text{ and } T(e) = u\}.$$

If E and F are ordered vector spaces with bases B and B', respectively, then

1′) $$L^b(E,F;B,[\theta,B']) = \{T \in L^b(E,F) : \theta \leqq T \text{ and } T(B) \subset [\theta,B']\}$$

where

$$[\theta,B'] = \{x \in F : \theta \leqq x \leqq b \text{ for some } b \in B'\};$$

2′) $$L^b(E,F;B,B') = \{T \in L^b(E,F) : \theta \leqq T \text{ and } T(B) \subset B'\}.$$

We note that the sets are convex sets of positive operators and if E,F are the A and B, respectively, mentioned in the introduction, then 1) and 2) correspond to $K_0(A,B)$ and $K_1(A,B)$, respectively.

(3.1) LEMMA. *Let E,F be Archimedean ordered vector lattices with order units $e \in E$ and $u \in F$, and let $T \in L^b(E,F;e,[\theta,u])$. If T is a lattice homomorphism and if T^* maps each extreme point of the dual base $B' \subset F^b$ into either θ or into the dual base $B \subset E^b$, then T is an extreme point of $L^b(E,F;e,[\theta,u])$.*

PROOF. Let $T = \frac{1}{2}U + \frac{1}{2}V$ with $U,V \in L^b(E,F;e,[\theta,u])$ and let μ be an extreme point of B'; then μ is indecomposable in F^b. If $T^*(\mu)=0$, we have $U^*(\mu)=V^*(\mu)=0$. On the other hand, let $T^*(\mu) \in B$. Then, since $U^*, V^* \in L^b(F^b,E^b;B',[\theta,B])$ and since $\frac{1}{2}U^*(\mu)+\frac{1}{2}V^*(\mu)=T^*(\mu)$, we infer from the properties of a base that $U^*(\mu), V^*(\mu) \in B$. Since T and μ are lattice homomorphisms, we have $T^*(\mu)(|x|)=|T^*(\mu)(x)|$; consequently, by (1.1), $T^*(\mu)$ is indecomposable in E^b. It follows that $V^*(\mu)=U^*(\mu)=T^*(\mu)$, and we conclude that $T=U=V$ since the extreme points of B' are total over F.

(3.2) PROPOSITION. *Suppose that E is an Archimedean ordered vector lattice with an order unit e, and that $F=C(Y)$ where Y is a compact Hausdorff space. If $T \in L^b(E,F;e,[\theta,1])$, then T is an extreme point of $L^b(E,F;e,[\theta,1])$ if and only if T is a lattice homomorphism and T^* maps each extreme point of $B' \subset F^b$ into either θ or into an extreme point of $B \subset E^b$.*

PROOF. Let T be an extreme point of $L^b(E,F;e,[\theta,1])$. Since E can be canonically identified with a dense subspace M of $C(X)$ for some compact Hausdorff space X and since T is continuous for the order unit norms on E and F, we may regard T as a mapping on M and extend T to all of $C(X)$. We note that T is then an extreme point of $L^b(C(X),F;e,[\theta,1])$. Therefore, by theorem 3 in [2], T is multiplicative on $C(X)$ and for $a\in E$ we have

$$|T(a)|^2 = (T(a))^2 = T(a^2) = T(|a|^2) = (T(|a|))^2 .$$

Thus, $|T(a)|=T(|a|)$, so that T is a lattice homomorphism. If μ is an extreme point in B', then, as for T above, μ is multiplicative and we have

$$T^*\mu(e) = T^*\mu(e\cdot e) = T^*\mu(e)\,T^*\mu(e) .$$

Consequently, $T^*\mu(e)=0$ or 1 and it follows that $T^*\mu=\theta$ or $T^*\mu\in B$. If $T^*\mu\in B$, then since T and μ are lattice homomorphisms, we deduce that $T^*\mu$ is indecomposable in E^b by (1.1). Hence $T^*\mu$ is an extreme point of B.

The converse follows immediately from (3.1).

Now let us show that there is usually a connection between the extremal character of an operator and its adjoint for operators in the convex sets defined above.

(3.3) PROPOSITION. *Let E,F be vector lattices with bases B,B', respectively. Let F be order complete and full in the completion $\tilde{F}$ of F for the base norm topology. Then $T\in L^b(E,F;B,[\theta,B'])$ is an extreme point of this set if and only if T^* is an extreme point of $L^b(F^b,E^b;u,[\theta,e])$ where u and e are the dual order units of F^b and E^b, respectively.*

PROOF. Since $\tilde{F}$ is an (L)-space, it follows that $y_\alpha\downarrow\theta$ in F implies that $\{y_\alpha\}$ norm converges to θ in $\tilde{F}$, and since F is full in $\tilde{F}$, $\{y_\alpha\}$ norm converges to θ in F. By Theorem (39.1) in [10], we see that the canonical image $\varphi(F)$ of F is full in $F(\|\cdot\|)''$. Since the unit ball in F is full in F and since $F(\|\cdot\|)'$ is an (M)-space, it follows that

$$(F^b)^b \subset (F(\|\cdot\|)')^b = F(\|\cdot\|)'' .$$

Hence, $\varphi(F)\subset(F^b)^b$ is a full subset of $(F^b)^b$.

Let T be an extreme point of $L^b(E,F;B,[\theta,B'])$. Then it is easily seen that $T^*\in L^b(F^b,E^b;u,[\theta,e])$. Let $T^*=\frac{1}{2}U+\frac{1}{2}V$ where $U,V\in L^b(F^b,E^b;u,[\theta,e])$. Then

$$\theta \leqq \tfrac{1}{2}U^*(\varphi(x)) \leqq T^{**}(\varphi(x)) = \varphi(T(x))$$

for all $x \in E^+$. Since $\varphi(F)$ is full in $(F^b)^b$, we have that $U^*(\varphi(x)) = \varphi(y)$ for some $y \in F^+$. Therefore, we define U' on E^+ into F^+ by $U'(x) = y$. Since φ, φ^{-1} and U^* are all positive linear maps, we can extend U' to all of E by $U'(x) = U'(x^+) - U'(x^-)$ and conclude that $U' \in L^b(E,F)^+$. Similarly, we define $V' \in L^b(E,F)^+$ by $V'(x) = \varphi^{-1}(V^*(\varphi(x)))$. Since we have

$$\langle u, U'(b)\rangle = \langle U^*(\varphi(b)), u\rangle = \langle U(u), b\rangle \leqq \langle e, b\rangle = 1$$

for $b \in B$, it follows that U' and, similarly, V' is in $L^b(E,F; B,[\theta,B'])$. Moreover, for any $x \in E$ and $f \in F^b$ it can be seen that

$$\langle f, T(x) - \tfrac{1}{2}U'(x) - \tfrac{1}{2}V'(x)\rangle = 0 .$$

Consequently, $T = \frac{1}{2}U' + \frac{1}{2}V'$. Hence, $T^* = U'^* = V'^*$. But again, for $x \in E$ and $f \in F^b$ we can obtain $\langle (U'^* - U)f, x\rangle = 0$, so that $U = U'^* = T^*$ and, similarly, $V = V'^* = T^*$. We conclude that T^* is an extreme point of $L^b(F^b, E^b; u, [\theta, e])$.

Conversely, let T^* be an extreme point of $L^b(F^b,E^b; u,[\theta,e])$. Then $T \in L^b(E,F; B,[\theta,B'])$, and if $T = \frac{1}{2}U + \frac{1}{2}V$ with U, V in $L^b(E,F; B,[\theta,B'])$ then we deduce that $T^* = \frac{1}{2}U^* + \frac{1}{2}V^*$ with U^*, V^* in $L^b(F^b,E^b; u,[\theta,e])$; whence $T^* = U^* = V^*$, and it follows that $T = U = V$.

(3.4) Proposition. *Let E be an Archimedean ordered vector lattice with an order unit e, and let $F = C(X)$ where X is a compact Hausdorff space. If B and B' are the dual bases of E^b and F^b, respectively, then the following statements are equivalent*:

i) *T is an extreme point of $L^b(E,F; e,1)$*;
ii) *T^* is an extreme point of $L^b(F^b,E^b; B',B)$*;
iii) *T^{**} is an extreme point of $L^b((E^b)^b,(F^b)^b; \varphi(e), \varphi(1))$*.

Proof. From the definition of the dual bases B and B', we can deduce that the following are equivalent:

$$T \in L^b(E,F; e,1),$$
$$T^* \in L^b(F^b,E^b; B',B) ,$$
$$T^{**} \in L^b((E^b)^b,(F^b)^b; \varphi(e),\varphi(1)) .$$

As in (3.2), it suffices to assume that $E = C(Y)$ for some compact Hausdorff space Y.

To show that iii) follows from i), it suffices, by Theorem 1 in [5] to show that T^{**} is a lattice homomorphism. Now, we recall that $E' = E^b$ and $E'' = (E^b)^b$; moreover, we note that the topology in (11.2) and (11.3) of [8] is exactly the $o(E'',E')$ topology (see [12]). Thus, from Theorems (11.2) and (11.3) in [8] we conclude that $\varphi(E)$ is $o(E'',E')$-dense in E''

and the lattice operations are $o(E'',E')$-continuous. The same results hold for $(F^b)^b = F''$, so, by (3.12) page 177 of [12], T^{**} is continuous on $E''[o(E'',E')]$ into $F''[o(F'',F')]$. Now, if $a \in E''$, then there is a net $\{x_\alpha\} \subset E$ such that $\{\varphi(x_\alpha)\}$ $o(E'',E')$-converges to a. Since T is a lattice homomorphism by theorem 1 in [8], we obtain

$$\begin{aligned} |T^{**}(a)| = |T^{**}(\lim_\alpha \varphi(x_\alpha))| &= \lim_\alpha \varphi(|T(x_\alpha)|) \\ &= \lim_\alpha \varphi(T(|x_\alpha|)) \\ &= T^{**}|\lim_\alpha(\varphi(x_\alpha))| = T^{**}(|a|) . \end{aligned}$$

It follows that T^{**} is a lattice homomorphism.

The fact, that iii) implies ii) follows from (3.3), and an argument similar to that in the last paragraph of (3.3) shows that ii) implies i).

4. Decomposition of the extreme points.

Now, we shall decompose the space of linear operators into its atomic and diffuse subspaces and show that the components of the extreme points of the convex sets mentioned above are extreme points of the appropriate convex sets.

Let E and F be Archimedean ordered vector lattices with order units e and u, respectively. Then for u_a the atomic component of u we define

1) $\quad L^b{}_a(E,F; e,[\theta,u_a]) = \{T \in L^b{}_a(E,F) : \theta \leqq T \text{ and } T(e) \leqq u_a\};$

2) $\quad L^b{}_a(E,F; e,u_a) = \{T \in L^b{}_a(E,F) : \theta \leqq T \text{ and } T(e) = u_a\} .$

We also replace a with d in the above to define sets 3) and 4), respectively, for the diffuse component sets. We note that each of the sets is convex and that u_a and u_d are order units for F_a and F_d, respectively.

(4.1) PROPOSITION. *Let E and F be order complete vector lattices with order units $e \in E$ and $u \in F$, and let $T \in L^b(E,F; e,[\theta,u])$. Then T is an extreme point of $L^b(E,F; e,[\theta,u])$ if and only if the atomic component A of T is an extreme point of $L^b{}_a(E,F; e,[\theta,u_a])$ and the diffuse component D of T is an extreme point of $L^b{}_d(E,F; e,[\theta,u_d])$.*

PROOF. Let T be an extreme point of $L^b(E,F; e,[\theta,u])$. By (2.3), we have $A(E) \subset F_a$, and we claim that $D(E) \subset F_d$. In fact, assume that there exists an $x \in E^+$ such that $D(x) = a + d$ where $a > \theta$ is atomic and $d \geqq \theta$ is diffuse. Then there is an indecomposable $b \in F$ such that $b \leqq a$. Let P_x and P_b be the projections associated with x and b, respectively. Since F is an order complete vector lattice, we can see that if we equip it with the order unit norm, then it can be represented as a $C(X)$ space for some compact Hausdorff space X. Therefore, by (3.2), T is a lattice homomorphism. Hence, D is also a lattice homorphism by lemma 1 in [5]. Since P_x and P_b are lattice homomorphisms, we see that if G is defined from E into F by $G(z)=P_b\big(D(P_x(z))\big)$, then G is a lattice homomorphism. Moreover, since $\operatorname{rng} G \subset \operatorname{rng} P_b = \mathrm{LH}\{b\}$, it follows from (1.2) that G is indecomposable. It is clear that $G \leqq D$, but this contradicts the fact that since D is diffuse, $D \perp G'$ for every indecomposable G'. Thus, we conclude that $D(E) \subset F_d$. Since

$$u_a + u_d = u \geqq T(e) = A(e) + D(e) ,$$

it follows that $A \in L^b{}_a(E,F;e,[\theta,u_a])$ and $D \in L^b{}_d(E,F;e,[\theta,u_d])$. If $A=\frac{1}{2}U+\frac{1}{2}V$ with $U,V \in L^b(E,F;e,[\theta,u_a])$, then $T=\frac{1}{2}(U+D)+\frac{1}{2}(V+D)$. Since it is easily seen that $U+D$ and $V+D$ are in $L^b(E,F;e,[\theta,u])$, we deduce that $A=U=V$. Hence, A is an extreme point of $L^b{}_a(E,F;e,[\theta,u_a])$; similarly, D is an extreme point $L^b{}_d(E,F;e,[\theta,u_d])$.

Conversely, let $T=A+D$ with A an extreme point of $L^b{}_a(E,F;e,[\theta,u_a])$ and D an extreme point of $L^b{}_d(E,F;e,[\theta,u_d])$. If $T=\frac{1}{2}S+\frac{1}{2}W$ with S and W in $L^b(E,F;e,[\theta,u])$, then we decompose S and W into their atomic (S_a, W_a) and diffuse (S_d, W_d) components. Then because of the uniqueness of the decomposition of T into its atomic and diffuse components and because of the extremal character of A and D, we can deduce that $A=S_a=W_a$ and $D=S_d=W_d$, so that T is an extreme point of $L^b(E,F;e,[\theta,u])$.

For the situation where E and F have bases, we first prove the following lemma before defining the component sets.

(4.2) Lemma. *Let E and F be Archimedean ordered vector lattices with bases B and B', respectively. If T is an extreme point in $L^b(E,F;B,[\theta,B'])$, then T maps each extreme point of B into θ or into an extreme point of B'.*

Proof. Let c be an extreme point of B; then c is indecomposable. Then for each $\theta < x \in E$ let $P_c(x)=\sup_{\lambda \geqq 0}\{\lambda c : \lambda c \leqq x\}$. Since E is archimedean ordered, $P_c(x)$ is well defined. For any $x,y \in E^+$ we see that $P_c(x)+P_c(y) \leqq P_c(x+y)$. On the other hand, let $\lambda c \leqq P_c(x+y)$. Since E is a vector lattice, $\lambda c = a+b$ for some $a \in [\theta,x]$ and $b \in [\theta,y]$. Then, since c is indecomposable, $a=\lambda_1 c$ and $b=\lambda_2 c$ for some $\lambda_1 \geqq 0$ and $\lambda_2 \geqq 0$. We conclude that $P_c(x+y) \leqq P_c(x)+P_c(y)$. Clearly P_c is positive homogenous, so we may extend P_c linearly to all of E by $P_c(x)=P_c(x^+)-P_c(x^-)$. For each $x \in E$ we define $\lambda_x \in R$ by $\lambda_x c = P_c(x)$.

Now $c \in B$ implies that $T(c) \in [\theta,B']$, so $T(c)=\lambda_0 b$ for some $b \in B'$

and $0 \leqq \lambda_0 \leqq 1$. If $\lambda_0 = 0$ we are finished; therefore, let $\lambda_0 > 0$. Assume that $b = \frac{1}{2}r + \frac{1}{2}s$ with $r, s \in B'$, and define U, V from E into F by

$$U(x) = \lambda_x \lambda_0 r + T \circ (I - P_c)(x) ,$$
$$V(x) = \lambda_x \lambda_0 s + T \circ (I - P_c)(x) .$$

It is not difficult to see that $U, V \in L^b(E,F)^+$. If $t \in B$, then $t = \lambda_t c + \gamma v$ for $\lambda_t, \gamma \in [0,1]$ and some $v \in B$ with $v \perp c$. From the properties of bases, $\gamma = (1 - \lambda_t)$. Thus,

$$U(t) = \lambda_t(\lambda_0 r) + (1 - \lambda_t)T(v) ,$$

and since $T(v), \lambda_0 r \in [\theta, B']$, it follows that U and, similarly, V is in $L^b(E,F; B, [\theta, B'])$. Finally, $T(x) = \frac{1}{2}U(x) + \frac{1}{2}V(x)$ for all $x \in E$ which implies that $U = V$ since T is an extreme point of $L^b(E,F; B, [\theta, B'])$. Hence, $r = s = b$, and it follows that b is an extreme point of B. To complete the proof we must show that $\lambda_0 = 1$. However, if $\lambda_0 < 1$, then there exists an $\alpha \in (0,1)$ such that $\lambda_0 + \alpha\lambda_0 < 1$. Hence, by defining U, V from E into F by

$$U(x) = (1 + \alpha) T \circ P_c(x) + T \circ (I - P_c)(x) ,$$
$$V(x) = (1 - \alpha) T \circ P_c(x) + T \circ (I - P_c)(x) ,$$

and by arguing as above we may again deduce that $U = V = T$ which contradicts the fact that $\alpha \in (0,1)$.

Let E and F be vector lattices with bases $B \subset E$ and $B' \subset F$. Then, we define

1) $L^b{}_a(E,F; B_a, [\theta, B']) = \{T \in L^b{}_a(E,F) : \theta \leqq T \text{ and } T(E_a \cap B) \subset [\theta, B']\}$;
2) $L^b{}_a(E,F; B_a, B') = \{T \in L^b{}_a(E,F) : \theta \leqq T \text{ and } T(E_a \cap B) \subset B'\}$.

We also define sets 3) and 4) of diffuse operators by replacing a with d in 1) and 2) respectively.

(4.3) Proposition. *Suppose that E is an order complete vector lattice with a base B and that F is an (L)-space with a base B'. If T is an order continuous element in $L^b(E,F; B, [\theta, B'])$, then T is an extreme point of this convex set if and only if $T = A + D$ where A is an extreme point of $L^b{}_a(E,F; B_a, [\theta, B'])$, D is an extreme point of $L^b{}_d(E,F; B_d, [\theta, B'])$, and $D(E_a) = \{\theta\}$.*

Proof. Let T be an extreme point of $L^b(E,F; B, [\theta, B'])$, and let A and D be the atomic and diffuse components of T, respectively. Since

$A,D \in [\theta,T]$ and T is order continuous, it follows that A and D are order continuous. Furthermore, since F is an (L)-space, there is a family of positive order continuous linear functionals in F^b which is total over F. Thus, by (2.4), we have $D(E_a) \subset F_d$. On the other hand, for each indecomposable $b \in E$ we have $\theta \leqq D(b) \leqq T(b)$, and since each indecomposable b is a scalar multiple of an extreme point of B, we have $T(b) \in F_a$ by (4.2). Thus, we can deduce that $D(E_a) \subset F_a$, and it follows that $D(E_a) = \{\theta\}$. Hence, if $b \in B \cap E_a$, then $A(b) = T(b) \in [\theta,B']$ so that $A \in L^b{}_a(E,F;B_a,[\theta,B'])$. Let $A = \frac{1}{2}U' + \frac{1}{2}V'$ with

$$U', V' \in L^b{}_a(E,F;B_a,[\theta,B']) .$$

We note that each $b' \in B$ can be written in the form $\lambda a + (1-\lambda)d$ where $a \in B \cap E_a$, $d \in B \cap E_d$ and $\lambda \in [0,1]$. Also, since $\theta \leqq \frac{1}{2}U' \leqq A \leqq T$ implies that U' is order continuous, we infer that $U'(E_d) = \{\theta\}$ from (2.4). Thus, we have that

$$(U'+D)(b') \;=\; \lambda U'(a) + (1-\lambda)D(d) .$$

Since $U'(a)$ and $D(d)(\leqq T(d))$ are in $[\theta,B']$, it follows that $U'+D$ and, similarly, $V'+D$ is in $L^b(E,F;B,[\theta,B'])$. Since T is an extreme point of this convex set and since $T = \frac{1}{2}(U'+D) + \frac{1}{2}(V'+D)$, we conclude that $U' = V' = A$. By a similar use of (2.4) we can also conclude that D is an extreme point of $L^b{}_d(E,F;B_d,[\theta,B'])$.

Conversely, let $T = A + D$ where A is an extreme point of $L^b{}_a(E,F;B_a,[\theta,B'])$, D is an extreme point of $L^b{}_d(E,F;B_d,[\theta,B'])$, and $D(E_a) = \{\theta\}$. Then, as for $U'+D'$ above, we have $T \in L^b(E,F;B,[\theta,B'])$. Let $T = \frac{1}{2}U + \frac{1}{2}V$ with U and V in $L^b(E,F;B,[\theta,B'])$, and decompose U and V into their atomic (U_a, V_a) and diffuse (U_d, V_d) components. Now,

$$A+D \;=\; T \;=\; \tfrac{1}{2}(U_a+V_a) + \tfrac{1}{2}(U_d+V_d) ,$$

and since the decomposition of T into its components is unique, we have that $A = \frac{1}{2}(U_a+V_a)$ and $D = \frac{1}{2}(U_d+V_d)$. Let $\theta \leqq x \in E_a$; then $\theta \leqq \frac{1}{2}U_d(x) \leqq D(x) = \theta$ implies that $U_d(E_a) = \{\theta\}$. Similarly, $V_d(E_a) = \{\theta\}$. Thus, for $b \in E_a \cap B$ we have that $U_a(b) = U(b) \in [\theta,B']$ and $V_a(b) = V(b) \in [\theta,B']$. Hence, $U_a, V_a \in L^b{}_a(E,F;B_a,[\theta,B'])$, and $A = U_a = V_a$ since A is an extreme point of this convex set. On the other hand, by (2.4), $U_a(E_d) = V_a(E_d) = \{\theta\}$. Therefore, by similar arguments, we infer that $U_d = V_d = D$, and we conclude that $U = V = T$.

(4.4) Corollary. *Let E and F be (L)-spaces with bases B and B', respectively. If $T \in L^b{}_a(E,F;B_a,[\theta,B'])$, then T is an extreme point of this* *convex set if and only if T maps each extreme point of B into either θ or an extreme point of B'.*

PROOF. Since E is an (L)-space, $T \geqq \theta$ is order continuous. Thus, since θ is an extreme point of $L^b{}_d(E,F; B_d[\theta,B'])$, the necessary conditions are immediate from (4.3) and (4.2). To show the sufficiency we note that if $T=\frac{1}{2}U+\frac{1}{2}V$ with $U,V \in L^b{}_a(E,F; B_a,[\theta,B'])$, then T, U, and V will agree on the indecomposable elements of E. We conclude that $T=U=V$ from the fact that T, U, and V agree on E_a since they are order continuous, and they agree on E_d since they map E_d into θ by (2.4).

We shall conclude this section by investigating the relationship between the extremal character of the atomic and diffuse operators and their adjoints. In the case where the underlying spaces E and F each have an order unit we have difficulties, for it is not necessarily true that a diffuse operator will have an adjoint which is diffuse. For example it is not difficult to see that $L^\infty(\mathsf{R})$ is diffuse and, therefore, that the identity operator I on $L^\infty(\mathsf{R})$ is diffuse by (2.4). However, the adjoint I^* is not diffuse, by (2.4), since $L^\infty(\mathsf{R})'$ contains indecomposable elements and I^* does not map these elements into diffuse elements. In the case where E and F are (L)-spaces with bases the situation is as we would expect.

(4.5) PROPOSITION. *Let E and F be (L)-spaces with bases B and B', respectively. If T is an extreme point of $L^b(E,F; B,[\theta,B'])$, then $(T_a)^*=(T^*)_a$ and $(T_d)^*=(T^*)_d$ where the subscripts a and d denote the atomic and diffuse components of the corresponding operators.*

PROOF. We recall that T, T_a, and T_d are order continuous as are $f \in E^b$ and $g \in F^b$. By (4.3), T_a is an extreme point of $L^b{}_a(E,F; B_a[\theta,B'])$, T_d is an extreme point of $L^b{}_d(E,F; B_d[\theta,B'])$, and $T_d(E_a)=\{\theta\}$. Therefore, $T(E_a)=T_a(E_a)\subset F_a$ by (2.4). Moreover, by (2.4), $T_a(E_d)=\{\theta\}$ so that $T(E_d)=T_d(E_d)$.

Let $T^*=(T^*)_a+(T^*)_d$ be the unique decomposition of T^* into its atomic and diffuse components. We need only show that $(T_a)^*$ is atomic and $(T_d)^*$ is diffuse. Since T_a is atomic, it is the supremum of all finite sums H of indecomposable $G \in L^b(E,F)$ which are majorized by T. By (1.4), for each G we have that G^* is indecomposable in $L^b(F^b,E^b)$ and is majorized by T^*. Thus, since $(T^*)_a$ is also the supremum of finite sums of indecomposable elements in $L^b(F^b,E^b)$ which are majorized by T^* and since the elements in F^b are order continuous over F, we may deduce that $(T_a)^* \leqq (T^*)_a$. It follows that $(T_a)^*$ is atomic since $L^b{}_a(F^b,E^b)$ is an order ideal. Next, let us assume that $(T_d)^*$ is not diffuse. Then there is an indecomposable $A \in L^b(F^b,E^b)$ such that $A = P_A((T_d)^*)$. By (1.2), $\operatorname{rng} A = \mathrm{LH}\{a\}$ for some indecomposable $a \in E^b$. For $x \in F^b$ let $\lambda_x \in R$ satisfy $\lambda_x a = A(x)$. Since $T_d(E_a) = \{\theta\}$, we have that

$$\theta \leqq \langle A(x),y\rangle \leqq \langle (T_d)^*(x),y\rangle = \langle x,T_d(y)\rangle = \theta$$

for all $\theta < x \in F^b$ and $\theta \leqq y \in E_a$. On the other hand, if $\theta < x \in F^b$ and $y \in E_d$, then by replacing F with R in (2.4) we infer that $\langle a,y\rangle = 0$. Hence, $0 = \lambda_x\langle a,y\rangle = \langle A(x),y\rangle$. Consequently, $A = \theta$ which contradicts the fact that A is indecomposable, and we conclude that $(T_d)^*$ is diffuse.

(4.6) Corollary. *Suppose that E,F are (L)-spaces with bases $B \subset E$ and $B' \subset F$. If $T \in L^b{}_a(E,F;B_a,[\theta,B'])$, then T is an extreme point of this convex set if and only if T^* is an extreme point of $L^b{}_a(F^b,E^b;u,[\theta,e_a])$ where u and e are the dual order units of F^b and E^b, respectively. Moreover, the same sort of equivalence is valid for the diffuse operators.*

Proof. Since θ is an extreme point of $L^b{}_d(E,F;B_d,[\theta,B'])$, we infer from (4.3) that T is an extreme point of $L^b{}_a(E,F;B_a,[\theta,B'])$ if and only if T is an extreme point of $L^b(E,F;B,[\theta,B'])$. Thus, the result follows from (3.3), (4.1), and (4.5). The argument for the diffuse operators is similar.

We conclude these last two sections by remarking that the results which were given in terms of the convex sets occurring in 1), 1′), or 3) in the definitions can be established for the convex sets occurring in 2), 2′), or 4) in the definitions, respectively.

5. Mapping an (L)-space into an (M)-space.

Let us now examine the atomic operators mapping an (L)-space L with a base B and equipped with the base norm topology into an order complete (M)-space M with an order unit e and equipped with the order unit norm.

(5.1) Proposition. *$L^b(L,M)(\mathscr{T}_0)$ is an (M)-space with an order unit E, and $L^b(L,M)(\mathscr{T}_0) = \mathscr{L}(L,M)(\mathscr{T})$ where $\mathscr{L}(L,M)$ is the space of all continuous linear operators from L into M and $\mathscr{T}$ is the uniform operator topology.*

Proof. Since M is equipped with an order unit norm, it is clear that $\mathscr{L}(L,M) \subset L^b(L,M)$. On the other hand, it is easily deduced from (2.16) page 86 of [12] that $L^b(L,M) \subset \mathscr{L}(L,M)$. Thus, from [6], E defined from

L into M by $E(x) = \|x^+\|e - \|x^-\|e$ is an order unit for $L^b(L, M)$. Moreover, $L^b(L, M)$ is Archimedean ordered since M is. Therefore, $\mathscr{T}_0$ is a normable topology with $[-E, E]$ as the unit ball. If $T \in [-E, E]$ and $x \in L$ with $\|x\| \leqq 1$, we have

$$\|T(x)\| \leqq \||T|(|x|)\| \leqq \|E(|x|)\| = \| |x| \| \leqq 1 .$$

Hence, $[-E, E] \subset U$ where U is the unit ball for $\mathscr{L}(L, M)$ equipped with the uniform operator topology. On the other hand, for $T \in U$ and $x \geqq \theta$ we have that $-E(x) \leqq T(x) \leqq E(x)$, so $U \subset 2[-E, E]$. We conclude that $L^b(L, M)(\mathscr{T}_0) = \mathscr{L}(L, M)(\mathscr{T})$, and since $\mathscr{T}$ is complete, it follows that $L^b(L, M)(\mathscr{T}_0)$ is a Banach space. Moreover, for the order unit norm $L^b(L, M)(\mathscr{T}_0)$ is an (M)-space.

We shall continue to denote the order unit of $L^b(L, M)$ by E. Thus, E_a (respectively, E_d) will denote the atomic (respectively, diffuse) component of E.

(5.2) Remark. Let X be a compact Hausdorff space.

i) Since X is completely regular, it is easily deduced that g is indecomposable in $C(X)$ if and only if there is an isolated point $x \in X$ such that $g(x) > 0$ and g maps $X \setminus \{x\}$ into zero. Moreover, if 1 is the order unit of $C(X)$, then $f \in [\theta, 1]$ is an extreme point of $[\theta, 1]$ if and only if $\operatorname{rng} f \subset \{0, 1\}$.

ii) If $C(X)$ is order complete, then by arguments similar to those in the latter parts of (4.1) we may deduce that $f \in [\theta, 1]$ is an extreme point if and only if the atomic component of f is an extreme point of $[\theta, 1_a]$ and the diffuse component of f is an extreme point of $[\theta, 1_d]$ where 1_a and 1_d are the atomic and diffuse components of 1, respectively.

iii) Let $C(X)$ be order complete. Since an element $d \in C(X)$ is diffuse if and only if it is disjoint from all the indecomposable elements of $C(X)$, it is clear that d is diffuse if and only if $d(x) = 0$ for every isolated $x \in X$.

On the other hand, an element $a \in C(X)$ is atomic if and only if the only nonisolated y for which $a(y) \neq 0$ are those which are in the closure of Y where

$$Y = \{x \in X : x \text{ is an isolated point of } X \text{ and } a(x) \neq 0\} .$$

In order to see this, let a be atomic and let $a(y_0) \neq 0$ for some nonisolated $y_0 \in X$. If $y_0 \notin \operatorname{Cl} Y$, then there exists an open set $U \subset X$ such that $y_0 \in U$ and $U \cap Y = \emptyset$. Hence, if $x \in U$ is an isolated point of X, then $a(x) = 0$. Let

$$G = \{g : g \leqq |a| \text{ and } g \text{ is indecomposable}\} .$$

Then $|a| = \sup\{g \in G\}$ and it is clear that every $g \in G$ is supported on an isolated $x_g \in Y \subset X \setminus U$. Choose $h \in [\theta, 1]$ such that $h(X \setminus U) = 1$ and $h(y_0) = 0$. Then $0 < h|a| \leqq |a|$, but since $h|a| \geqq g$ for all $g \in G$, we conclude that $h|a| = |a|$. It follows that $|a|(y_0) = 0$, so that $a(y_0) = 0$.

Conversely, assume that $a(y) \neq 0$ only if $y \in \mathrm{Cl}\, Y$, and let $a = a' + d'$ where d' is diffuse. We infer from above that $d'(x) = 0$ for all isolated $x \in Y$. By continuity $d'(x) = 0$ for all $x \in \mathrm{Cl}\, Y$. Since $|a| = |a'| + |d'|$ and $|a|(y) = 0$ for all nonisolated $y \in X \setminus \mathrm{Cl}\, Y$, we see that d' also maps such y's into zero. Finally, since from above d' is already zero on the isolated points, it follows that $d' = \theta$, so $a = a'$.

The following corollary is an immediate consequence of (5.1) and (5.2) (ii).

(5.3) COROLLARY. *Let $T \in L^b(L, M)(\mathscr{T}_0)$. Then T is an extreme point of $[\theta, E]$ if and only if the atomic and diffuse components of T are extreme points of $[\theta, E_a]$ and $[\theta, E_d]$, respectively.*

We remark that there is an algebraic isomorphism between $L \otimes M'$ and $\mathscr{L}_s(L, M)'$ where $\mathscr{L}_s(L, M)$ is $\mathscr{L}(L, M)$ equipped with the topology of simple convergence (see corollary 4 page 139 in [13]). In particular, for $x \in E$, $f \in M$, and $T \in \mathscr{L}(L, M)$ we have $(x \otimes f)(T) = f(T(x))$. Thus, in preparation for a description of the extreme points in $[\theta, E_a]$ we establish two results dealing with the indecomposable elements in the dual space of $L^b(L, M) = \mathscr{L}(L, M)$.

(5.4) PROPOSITION. *Let x be indecomposable in L and let f be indecomposable in M'. Then $x \otimes f$ is an indecomposable element in $L^b(L, M)(\mathscr{T}_0)'$.*

PROOF. We note that $\mathscr{L}_s(L, M)' \subset \mathscr{L}(L, M)(\mathscr{T}_0)' = L^b(L, M)(\mathscr{T}_0)'$. Since $L^b(L, M)(\mathscr{T}_0)'$ is an (L)-space (as the dual of an (M)-space), we have $\mathscr{L}_s(L, M)' \subset L^b(L, M)^b$. For $T \in L^b(L, M)$ we have $(x \otimes f)(|T|) = \langle f, |T|(x) \rangle$. However, $|T|(x) = \sup\{T(y) : y \in [-x, x]\}$ implies that $|T|(x) = |T(x)|$ since x is indecomposable. Thus, since f is indecomposable, we have from (1.1) that

$$(x \otimes f)(|T|) = \langle f, |T(x)| \rangle = |\langle f, T(x) \rangle| = |(x \otimes f)(T)| .$$

Consequently, by (1.1), $x \otimes f$ is indecomposable in

$$L^b(L, M)^b = L^b(L, M)(\mathscr{T}_0)' .$$

(5.5) PROPOSITION. *Let $\mathscr{L}_s(L, M)'$ be ordered with the order induced* *from $L^b(L,M)^b$. If $x \otimes f$ is indecomposable in $\mathscr{L}_s(L,M)'$, then either x and f or their negatives are indecomposable elements in L and M', respectively.*

PROOF. If $x \leqq \theta$, we claim that $f \leqq \theta$; for if $y \in M^+$ and $f(y) > 0$, then we choose an $h > \theta$ in L^b such that $h(x) < 0$. Then for $z \in L$ define T on L into M by $T(z) = h(z) \cdot y$. Clearly $T \in L^b(L,M)^+$, but $(x \otimes f)(T) = f(y) \cdot h(x) < 0$ which contradicts the fact that $x \otimes f$ is indecomposable. Similar arguments yield the following: if $f \leqq \theta$, then $x \leqq \theta$; if $x \nleqq \theta$, then $x \geqq \theta$; and if $f \nleqq \theta$, then $f \geqq \theta$. Thus, either $x \geqq \theta$ and $f \geqq \theta$, or $x \leqq \theta$ and $f \leqq \theta$. Since $x \otimes f = (-x) \otimes (-f)$, we may assume that $\theta \leqq x$ and $\theta \leqq f$. Then for $\theta \leqq g \leqq f$ we have that $\theta \leqq x \otimes g \leqq x \otimes f$. Since $x \otimes f$ is indecomposable, there is a $\gamma \geqq 0$ such that $x \otimes g = \gamma (x \otimes f)$. Hence, $x \otimes (g - \gamma f) = 0$, and since $x \neq \theta$, it follows that $g = \gamma f$. Consequently, f is indecomposable, and by a similar argument we conclude that x is also indecomposable.

(5.6) LEMMA. *Suppose that B is the base for L, X is the set of extreme points in the dual base in M', and Y is the set of extreme points in the dual base in $L^b(L,M)(\mathscr{T}_0)'$. Then there is an isolated point $\mu \in Y$ if and only if there exists an extreme point b of B and an isolated $f \in X$ such that $b \otimes f = \mu$.*

PROOF. We note that $M \cong C(X)$ and $L^b(L,M)(\mathscr{T}_0) \cong C(Y)$. Let μ be an isolated point of Y. By (5.2)(i), we define an indecomposable $G \in C(Y)$ such that $G(\mu) = 1$ and $G(Y \setminus \{\mu\}) = 0$. Then, as an element of $L^b(L,M)$, G is positive on L and is, therefore, order continuous. Consequently, by (1.3), there exist indecomposable elements $b \in L$ and $a \in M$ such that $G(x) = \lambda_x a$ where $\lambda_x \in \mathsf{R}$ is defined by $\lambda_x b = P_b(x)$ for P_b the projection associated with b. By choosing a suitable multiple of b if necessary, we may assume that $b \in B$ and is, therefore, an extreme point of B. Now $G \leqq E$, since E corresponds to 1 and $G \leqq 1$. Therefore, $G(b) = a \leqq E(b) = e$. If $\|a\| < 1$, then we have $\|G(x)\| = \lambda_x \|a\| < \|x\|$ since $\lambda_x \leqq \|x\|$. This implies that $\|G\| < 1$ which contradicts the fact that $G(\mu) = 1$; hence, $\|a\| = 1$. Since a is indecomposable, we infer from (5.2)(i) that there is an isolated $f \in X$ such that $\langle f, a \rangle = 1$ and $\langle g, a \rangle = 0$ for all $g \in X \setminus \{f\}$. We claim that $b \otimes f = \mu$. Since $b \otimes f(G) = 1 = \mu(G)$, we need only prove that the null space of μ is contained in the null space of $b \otimes f$. Let $\mu(T) = 0$ for $T \in L^b(L,M)$. Then for P_G the projection associated with G, we have $\mu(P_G(T)) = 0$; hence, $T \perp G$. Since b is indecomposable, we can deduce that $T(b) \perp G(b)$. Hence, since f is indecomposable and $f(G(b)) = 1$, we infer from (1.1) that $f(T(b)) = 0$, and we conclude that T is in the null space of $b \otimes f$.

Conversely, let b be an extreme point of B and let f be an isolated point of X. Then, by (5.4), $b \otimes f$ is an indecomposable element in B, and we have $b \otimes f \in Y$ because $(b \otimes f)(E) = \|b\| f(e) = 1$. Since f is an isolated point of X, there is an indecomposable $a \in M$ such that $\langle f, a \rangle = 1$. Let G be defined on L into M by $G(z) = \lambda_z a$ where $\lambda_z \in R$ is defined by $\lambda_z b = P_b(z)$. Then from (1.2) it is easily deduced that G is an indecomposable element of $L^b(L, M)$. Since $b \otimes f(G) = 1$, it follows from (5.2)(i) that $b \otimes f$ is an isolated point of Y.

We conclude our discussion of the space $L^b(L, M)$ with the following description of the atomic operators which are extreme points of $[\theta, E_a]$.

(5.7) Proposition. *Let B, X, and Y be as in* (5.6) *and let Y_1 be the set of all weak* isolated points of Y. Then $A \in L^b{}_a(L, M)$ is an extreme point of $[\theta, E_a]$ if and only if A maps every extreme point of B into an extreme point of $[\theta, e_a]$.*

Proof. As noted in (5.6) we have $M \cong C(X)$ and $L^b(L, M) \cong C(Y)$. Let A be an extreme point of $[\theta, E_a]$ and let $b \in B$ be indecomposable in L. Since for any $f \in X$ we have $b \otimes f \in Y$, it follows that $f(A(b)) \in \{0, 1\}$ for all $f \in X$. Thus, by (5.2)(i), $A(b)$ is an extreme point of $[\theta, e]$. Finally, since, by (2.3), $A(b) \in [\theta, e_a]$, we conclude that $A(b)$ is an extreme point of $[\theta, e_a]$.

Conversely, let $A(b)$ be an extreme point of $[\theta, e_a]$ for each $b \in B$ which is indecomposable in L. Then $f(A(b)) \in \{0, 1\}$ for all $f \in X$. Since for each $\mu \in Y_1$ we have that $\mu = b \otimes f$ for some extreme point b of B and some $f \in X$ by (5.6), it follows that $A(Y_1) \subset \{0, 1\}$. By the continuity of A we have $A(\mathrm{Cl}\, Y_1) \subset \{0, 1\}$, and since A is atomic, we infer from (5.2)(iii) that $A(Y \setminus \mathrm{Cl}\, Y_1) = \{0\}$. Thus, $A(Y) \subset \{0, 1\}$ and it follows from (5.2)(i) that A is an extreme point of $[\theta, E_a]$.

6. The space $L^b(M, L)$.

We now turn to the remaining case where the domain space has an order unit and the range space has a base. In particular, we shall briefly examine $L^b(M, L)$ where M and L are as in section 5 above except that M need not be order complete.

(6.1) Proposition. *There exists a base B' in $L^b(M, L)$ and for the base norm topology $L^b(M, L)$ is an (L)-space.*

Proof. It is easily seen that $B' = \{T \in L^b(M, L) : \theta < T \text{ and } T(e) \in B\}$ is a base in L^b since B is a base in L. We equip $L^b(M, L)$ with the base norm topology $\|\cdot\|$ and note that this norm is additive on the positive cone

$L^b(M,L)^+$. Thus, it remains only to show that $L^b(M,L)(\|\cdot\|)$ is complete. If U is the unit ball, then it is clear that U is solid and that $L^b(M,L)^+$ is a strict b-cone. Hence, by (1.12) page 146 in [12], to show the completeness, it suffices to show that $\|\cdot\|$ is coarser than the order topology $\mathscr{T}_0$ and that every monotonically increasing sequence in $L^b(M,L)$ which is norm bounded has a supremum. Since $L^b(M,L)(\|\cdot\|)$ is a locally convex lattice, it follows from (1.17) page 124 of [12] that $\|\cdot\| \geqq \mathscr{T}_0$. Now let $\{T_n\}$ be monotonically increasing in $L^b(M,L)$ and let $\{T_n\} \subset \alpha U$ for some $\alpha > 0$. Since U is solid, we have $\{T_n{}^+\}, \{T_n{}^-\} \subset \alpha U$, so for each n there exist $\lambda_n, \gamma_n \in [0,1]$ and $B_n, C_n \in B'$ such that $T_n{}^+ = \alpha\lambda_n B_n$ and $T_n{}^- = \alpha\gamma_n C_n$. Hence,

$$T_n[-e,e] \subset \alpha B_n[-e,e] + \alpha C_n[-e,e] \subset 2\alpha U'$$

where U' is the unit ball of L. If $\theta \leqq x \in M$, then $\{T_n(x)\}$ is monotonically increasing in L and $\{T_n(x)\} \subset 2\alpha\|x\|U'$. Since, by (1.8) on page 145 of [12], the canonical images $\{\varphi(T(x_n))\} \subset L''$ have a supremum in L'' and since L is a band in L'', it follows that $\sup_n\{T_n(x)\}$ exists in L. For each $x \in M^+$ let $y_x = \sup_n\{T_n(x)\}$ and define T_0 on M^+ into L by $T_0(x) = y_x$. Since L is an (L)-space, we infer that $(T_n(a) - T_0(a)) \downarrow \theta$ for $a \in M^+$ implies that $\{T_n(a)\}$ norm converges to $T_0(a)$. Consequently, for $a,b \in M^+$ an any $\varepsilon > 0$ there exists an $N > 0$ such that $n > N$ implies that

$$\begin{aligned}&\|T_0(a+b) - T_0(a) - T_0(b)\| \\ &\qquad \leqq \|T_0(a+b) - T_n(a+b)\| + \|T_0(a) - T_n(a)\| + \|T_0(b) - T_n(b)\| < \varepsilon .\end{aligned}$$

It follows that T_0 is additive on M^+, and since it is clearly positive homogenous, we extend T_0 linearly to all of M by defining $T_0(x) = T_0(x^+) - T_0(x^-)$. Finally, since we have $(T_0 - T_1)(x) \geqq \theta$ for all $x \in M^+$, it follows that $T_0 - T_1$ is order bounded, and, therefore, T_0 is order bounded since it is the sum of two order bounded linear operators. We conclude that $L^b(M,L)(\|\cdot\|)$ is an (L)-space.

As a last remark, we note that the indecomposable elements of $L^b(M,L)$ are scalar multiples of the extreme points of the base B' and that each such extreme point is indecomposable.

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