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ON A COMPUTATION RULE FOR POLARS

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1.

Let (E,τ) be a locally convex topological linear space over the real field R; and let E^* be the continuous dual space of E. The *polar* of a subset A of E is denoted by A^π and is defined by

$$A^\pi = \{f \in E^* : f(a) \leqq 1, \ \forall a \in A\} .$$

Let S and T be convex subsets of E containing the origin. It is well known that if S and T are closed then $(S\cap T)^\pi$ is the w^*-closed convex hull of $S^\pi \cup T^\pi$. In this note, we show that in order to draw the same conclusion the closure of S and T can be replaced by a slightly weaker condition. This generalization yields immediately a unified proof of a theorem of Grosberg–Krein [2] and a theorem of Ellis [1] in the duality theory of partially ordered Banach spaces.

2.

We first prove the following basic lemma:

LEMMA 1. *Let S and T be convex subsets of (E,τ) containing the origin. Then the following propositions hold:*

(i) *If $\bar S \cap \bar T = \overline{S\cap T}$ (for example, S and T are closed) then $(S\cap T)^\pi = \overline{\mathrm{co}}\,(S^\pi \cup T^\pi)$, where $\overline{\mathrm{co}}\,(S^\pi \cup T^\pi)$ denotes the w^*-closed convex hull of $S^\pi \cup T^\pi$.*

(ii) *If the origin is an interior point of S and of T, then $\bar S \cap \bar T = \overline{S\cap T}$.*

PROOF. Since $\bar S$ and $\bar T$ are closed convex sets containing the origin, $(\bar S \cap \bar T)^\pi$ is the w^*-closed convex hull of $\bar S^\pi \cup \bar T^\pi$ (see, for instance, [3, p. 126]). Thus, if $\bar S \cap \bar T = \overline{S\cap T}$, we then have

$$(S \cap T)^\pi = (\overline{S \cap T})^\pi = (\bar S \cap \bar T)^\pi = \overline{\mathrm{co}}\,(\bar S^\pi \cup \bar T^\pi) = \overline{\mathrm{co}}\,(S^\pi \cup T^\pi) .$$

To prove (ii), let $x \in \bar S \cap \bar T$. Since S is a convex set containing the origin as an interior point, it follows that $\lambda x \in S$ for each $0 \leqq \lambda < 1$ (λx is in fact an interior point of S, see [3, p. 38]). Similarly $\lambda x \in T$. Letting

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$\lambda \to 1$ in $\lambda x \in S \cap T$, we have $x \in \overline{S \cap T}$. This shows that $\bar{S} \cap \bar{T} \subseteqq \overline{S \cap T}$. Consequently $\bar{S} \cap \bar{T} = \overline{S \cap T}$ since it is obvious that $\bar{S} \cap \bar{T} \supseteqq \overline{S \cap T}$.

3.

Let X be a Banach space over R with a closed wedge W, let Σ and U denote the closed and open unit ball in X respectively, let $\Sigma^+ = \Sigma \cap W$ and $S = \Sigma + W$, and let X^* denote the Banach dual space of X, with the closed unit ball Σ^*. The dual wedge W^* is defined by $W^* = -W^\pi$. Let $\Sigma^{*+} = \Sigma^* \cap W^*$ and $S^* = \Sigma^* + W^*$. Since $\Sigma \subseteqq S$ and $W \subseteqq S$, $\Sigma^* \supseteqq S^\pi$ and $W^\pi \supseteqq S^\pi$. Similarly, regarding X as the continuous dual space of X^* with the $\sigma(X^*, X)$-topology, we have $S^{*\pi} \subseteqq \Sigma^{*\pi} = \Sigma$, and that $S^{*\pi} \subseteqq W^{*\pi} = -W^{\pi\pi} = -W$ (the last equality follows since W is closed). Notice that S^* and $-S^*$ are $\sigma(X^*, X)$-closed. It follows that

$$(S^* \cap -S^*)^\pi = \overline{\mathrm{co}}\,(S^{*\pi} \cup -S^{*\pi}) \subseteqq \overline{\mathrm{co}}\,(-\Sigma^+ \cup \Sigma^+)\,.$$

On the other hand, let $x \in \Sigma^+$. Let $f \in S^* \cap -S^*$. Then there exists h in Σ^* such that $f \leqq h$. Hence $f(x) \leqq h(x) \leqq 1$. This shows that $x \in (S^* \cap -S^*)^\pi$ and hence that $\Sigma^+ \subseteqq (S^* \cap -S^*)^\pi$. Since $(S^* \cap -S^*)^\pi$ is symmetric, convex and closed, it follows that $\overline{\mathrm{co}}\,(\Sigma^+ \cup -\Sigma^+) \subseteqq (S^* \cap -S^*)^\pi$. Therefore

$$(S^* \cap -S^*)^\pi = \overline{\mathrm{co}}\,(\Sigma^+ \cup -\Sigma^+)\,. \tag{1}$$

Similarly, by lemma 1, we have

$$(S \cap -S)^\pi = \overline{\mathrm{co}}\,(S^\pi \cup -S^\pi) \subseteqq \overline{\mathrm{co}}\,(-\Sigma^{*+} \cup \Sigma^{*+}) = \mathrm{co}\,(\Sigma^{*+} \cup -\Sigma^{*+})\,.$$

(The last equality holds since $\mathrm{co}\,(\Sigma^{*+} \cup -\Sigma^{*+})$ is $\sigma(X^*, X)$-compact.) Further, by an argument similar to the one above, we can verify that $\mathrm{co}\,(\Sigma^{*+} \cup -\Sigma^{*+}) \subseteqq (S \cap -S)^\pi$. Consequently,

$$(S \cap -S)^\pi = \mathrm{co}\,(\Sigma^{*+} \cup -\Sigma^{*+})\,. \tag{2}$$

Definition. Let c be a positive constant. W is said to be *c-normal* if

$$S \cap -S \subseteqq c\Sigma\,,$$

equivalently,

$$x, y, z \in X,\ x \leqq y \leqq z \ \Rightarrow\ \|y\| \leqq c \max\{\|x\|, \|z\|\}\,.$$

W is said to be *c-generating* if

$$c^{-1}\Sigma \subseteqq \mathrm{co}\,(\Sigma^+ \cup -\Sigma^+)\,.$$

We now give an alternative proof of the following theorem:

THEOREM (Grosberg–Krein [2] and Ellis [1]). *Let X be a Banach space with a closed wedge W. Then:*

(i) *W is c-normal if and only if W^* is c-generating.*

(ii) *W^* is c-normal if and only if W is $(c+\varepsilon)$-generating for each $\varepsilon > 0$.*

PROOF. (i) By simple computation rules for polars and formula (2), we have

$$S \cap -S \subseteqq c\Sigma \Leftrightarrow (S \cap -S)^\pi \supseteqq c^{-1}\Sigma^* \Leftrightarrow \mathrm{co}(\Sigma^{*+} \cup -\Sigma^{*+}) \supseteqq c^{-1}\Sigma^* .$$

(ii) Since X is a Banach space, it follows from a theorem of Klee (cf. [1, lemma 7]) that $\mathrm{co}(\Sigma^+ \cup -\Sigma^+)$ contains every open ball in which it is dense. Thus it follows from (1) that

$$\begin{aligned} S^* \cap -S^* \subseteqq c\Sigma^* &\Leftrightarrow (S^* \cap -S^*)^\pi \supseteqq c^{-1}\Sigma \\ &\Leftrightarrow \overline{\mathrm{co}}(\Sigma^+ \cup -\Sigma^+) \supseteqq c^{-1}\Sigma \\ &\Leftrightarrow \mathrm{co}(\Sigma^+ \cup -\Sigma^+) \supseteqq c^{-1}U \\ &\Leftrightarrow \mathrm{co}(\Sigma^+ \cup -\Sigma^+) \supseteqq (c+\varepsilon)^{-1}\Sigma \quad \text{for each } \varepsilon > 0 . \end{aligned}$$

REFERENCES

1. A. J. Ellis, *The duality of partially ordered normed linear spaces*, J. London Math. Soc. 39 (1964), 730–744.
2. J. Grosberg and M. Krein, *Sur la décomposition des fonctionnelles en composantes positives*, Doklady Akad. Nauk SSSR (N. S.) 25 (1939), 723–726.
3. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966.

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