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SPLIT EXTENSIONS OF HOPF ALGEBRAS AND SEMI-TENSOR PRODUCTS

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Introduction.

A *homology Hopf algebra* over the field k is a (positively) graded connected Hopf algebra over k with commutative comultiplication. The category whose objects are homology Hopf algebras over k and whose morphisms are morphisms of Hopf algebras is denoted by $\mathscr{H}_*\mathscr{H}/k$.

In [4] the formal resemblance of $\mathscr{H}_*\mathscr{H}/k$ to the category of all groups was exploited to develop an analog of the derived series for homology Hopf algebras. In the present note we will exploit this formal resemblance by carrying over to $\mathscr{H}_*\mathscr{H}/k$ the relation between split short exact sequences and semi-direct products. The notion of the semi-direct product of an algebra A and a Hopf algebra H may be found in [1] and [2]. We will follow the notation $A \odot H$ of [1] and review the relevent facts in sections 1 and 2.

Our main result is the following:

THEOREM. *Suppose that*

$$\begin{tikzcd} k \arrow[r] & H' \arrow[r, "f'"] & H \arrow[r, "f''"] & H'' \arrow[r] \arrow[l, bend left, "\sigma"] & k \end{tikzcd}$$

is a split short exact sequence in $\mathscr{H}_*\mathscr{H}/k$. *Then there is a natural isomorphism of Hopf algebras* $H \cong H' \odot H''$.

Clearly there is a "dual" result for coexact sequences in the category $\mathscr{H}^*\mathscr{H}/k$ of cohomology Hopf algebras that involves the "cosemitensor" product (not a semi-cotensor product!). The formulation and details are left to the reader. They are just the formal duals of those presented here.

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1. Preliminaries.

In this section we will review some elementary facts concerning Hopf algebras. The classical reference for the basic facts is [3].

NOTATION AND CONVENTIONS:

k is a fixed field referred to as the ground field; $\otimes = \otimes_k$;

Al/k denotes the category of (positively) graded connected k-algebras;

$\mathscr{H}_*\mathscr{H}/k$ denotes the category of homology Hopf algebras over k, we refer to [4] for several of the basic properties of this category;

$\mathscr{M}/A$ is the category of graded (left) modules over the algebra $A \in \operatorname{obj} Al/k$.

If $M, N \in \operatorname{obj} \mathscr{M}/H$, $H \in \operatorname{obj} \mathscr{H}_*\mathscr{H}/k$, then as is well known [5] $M \otimes N$ may be given the structure of a left H-module via the composite

$$\begin{tikzcd} H\otimes(M\otimes N) \arrow[r, "\Delta\otimes 1\otimes 1"] & H\otimes H\otimes M\otimes N \arrow[r, "1\otimes T\otimes 1"] & H\otimes M\otimes H\otimes N \arrow[d, "\varphi_M\otimes\varphi_N"] \\ & & N\otimes M, \end{tikzcd}$$

where Δ is the comultiplication in H, T is the twisting morphism and φ_M, φ_N are the H-module structures on M and N respectively. (Recall that the twisting morphism

$$T: \quad L_1\otimes L_2 \to L_2\otimes L_1 \mid T(l_1\otimes l_2) = (-1)^{\deg l_1 \deg l_2}\, l_2\otimes l_1$$

for any $L_1, L_2 \in \operatorname{obj} \mathscr{M}/k$.)

DEFINITION. Suppose that $A \in \operatorname{obj} Al/k$ and $H \in \operatorname{obj} \mathscr{H}_*\mathscr{H}/k$. If $\varphi: H\otimes A \to A$ is an H-module structure on A then A is said to be an *algebra over the Hopf algebra H* iff the multiplication morphism

$$\mu: \quad A\otimes A \to A$$

is a morphism of H-modules.

The notion of an algebra over a Hopf algebra arose originally in algebraic topology and is due to Steenrod [5].

DEFINITION. Let A be an algebra over the homology Hopf algebra H. The *semi-tensor product* of A and H, denoted by $A \odot H$, is the object of Al/k defined as follows:

(1) as a k-module $A\odot H = A\otimes H$;

(2) the multiplication in $A\odot H$ is given by the composition

$$\begin{array}{rcl} (A\odot H)\otimes(A\odot H) & = & A\otimes H\otimes A\otimes H \\ & & \quad\downarrow 1\otimes\Delta_H\otimes 1\otimes 1 \\ & & A\otimes H\otimes H\otimes A\otimes H \\ & & \quad\downarrow 1\otimes 1\otimes T\otimes 1 \\ & & A\otimes H\otimes A\otimes H\otimes H \\ & & \quad\downarrow 1\otimes\varphi_A\otimes 1\otimes 1 \\ & & A\otimes A\otimes H\otimes H \\ & & \quad\downarrow \mu_A\otimes\mu_H \\ A\odot H & = & A\otimes H . \end{array}$$

The notion of semi-tensor product was introduced in [1] and [2] where one may find a proof that the multiplication defined above is associative.

As we shall soon see, $\odot$ plays a role in $\mathscr{H}_*\mathscr{H}/k$ analogous to the semi-direct product of two groups. (See also [1] especially the examples in the appendix.)

A more accurate notation would be $A\odot_\varphi H$, where $\varphi\colon A\otimes H\to A$ is the H-module structure on A. We have no need of this extra precision at present.

The following is an elementary consequence of the definition:

LEMMA 1.1. *Let A' be an algebra over the homology Hopf algebra H' and A'' an algebra over the homology Hopf algebra H''. Then $A'\otimes A''$ is in a natural way an algebra over the homology Hopf algebra $H'\otimes H''$ and there is a natural isomorphism of algebras*

$$(A'\otimes A'')\odot(H'\otimes H'')\cong(A'\odot H')\otimes(A''\odot H'') .$$

NOTATION. Let A be an algebra over the homology Hopf algebra H. Then there are natural morphisms of algebras

$$\alpha:\quad A\to A\odot H,\qquad \sigma:\quad H\to A\odot H$$

given by the morphisms

$$\alpha:\quad a\to a\otimes 1,\qquad \sigma:\quad h\to 1\otimes h$$

of underlying k-modules.

REMARK. (1) Note that A is an algebra over the Hopf algebra k and

$\alpha: A \cong A \odot k$. Also k is an algebra over the Hopf algebra H and $\sigma: H \cong k \odot H$.

(2) If $A \in \operatorname{obj} Al/k$ and $H \in \operatorname{obj} \mathscr{H}_* \mathscr{H}/k$, then there is a trivial action of H on A given by

$$H \otimes A \xrightarrow{\varepsilon \otimes 1} k \otimes A \cong A .$$

It is elementary to check that relative to this trivial action A is an algebra over H. Moreover, as may also be readily checked,

$$A \odot H \cong A \otimes H$$

as an algebra.

RECOLLECTION. If $H \in \operatorname{obj} \mathscr{H}_* \mathscr{H}/k$, then there is defined an anti-automorphism [3, section 8]

$$\chi_H: \quad H \to H$$

called the canonical anti-automorphism of H. A formula for χ_H is given inductively on degrees by

$$\chi_H(h) = -h - \sum h_i' \chi_H(h_i''),$$

where

$$\Delta(h) = h \otimes 1 + 1 \otimes h + \sum h_i' \otimes h_i'' .$$

An alternate definition of χ_H may be given by requiring that the diagrams

$$\begin{tikzcd} k \arrow[r, "\eta"] \arrow[d, equal] & H \arrow[d, "\chi_H"] \\ k & H \arrow[l, "\varepsilon"] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta_H"] \arrow[d, "\varepsilon"] & H \otimes H \arrow[dd, "1 \otimes \chi_H"] \\ k \arrow[d, "\eta"] & \\ H & H \otimes H \arrow[l, "\mu_H"] \end{tikzcd}$$

be commutative. The composite morphism $H \xrightarrow{\varepsilon} k \xrightarrow{\eta} H$ is often called the trivial morphism and denoted by $*$.

NOTATION. If A is an algebra over the homology Hopf algebra H and $a \in A$, $h \in H$ we will denote by $a \odot h$ the element of $A \odot H$ corresponding to the element $a \otimes h$ of $A \otimes H$ under the natural identification of these two k-modules.

Let $H \in \operatorname{obj} \mathscr{H}_* \mathscr{H}/k$ and $A \in Al/k$. Suppose that $f: H \to A$ is a morphism of algebras. Define a morphism

$$\varphi_f: \quad H \otimes A \to A$$

by the diagram

$$\begin{array}{ccccc} H\otimes A & \xrightarrow{\Delta_H\otimes 1} & H\otimes H\otimes A & \xrightarrow{1\otimes T} & H\otimes A\otimes H \\ {\scriptstyle\varphi_f}\downarrow & & & & \downarrow{\scriptstyle 1\otimes 1\otimes\chi_H} \\ A & \xleftarrow{\mu} & A\otimes A\otimes A & \xleftarrow{f\otimes 1\otimes f} & H\otimes A\otimes H \end{array}$$

One readily checks that φ_f gives A an H-module structure.

REMARK. Suppose that G and L are groups and $f\colon G \to L$ is a homomorphism. One should compare the above construction to that of letting G act on L by inner automorphisms via f.

PROPOSITION 1.2. *Let $H \in \operatorname{obj}\mathscr{H}_*\mathscr{H}/k$, $A \in \operatorname{obj} Al/k$ and $f\colon H \to A$. View A as an H-module via φ_f. Then the multiplication*

$$\mu_A : \quad A\otimes A \to A$$

is a morphism of H-modules. Hence A is an algebra over the Hopf algebra H via φ_f.

The proof is a straightforward, though messy, diagram chase and is deferred to Section 4.

2. Functorial and universal properties of $\odot$.

In order to carefully delineate the functorial properties of the semi-tensor product we find it convenient to introduce a few more categories.

DEFINITION. The category of algebras over homology Hopf algebras, denoted by $Al/\mathscr{H}_*\mathscr{H}$, has as its objects triples (A,H,φ) where

$$A \in \operatorname{obj} Al/k, \quad H \in \operatorname{obj}\mathscr{H}_*\mathscr{H}/k \quad \text{and} \quad \varphi : \; H\otimes A \to A$$

is an H-module structure on A making A into an algebra over the Hopf algebra H.

A morphism in $Al/\mathscr{H}_*\mathscr{H}$ from (A',H',φ') to (A'',H'',φ'') is a pair (f,g) where

(1) $f\colon A' \to A''$ is a morphism in Al/k;
(2) $g\colon H' \to H''$ is a morphism in $\mathscr{H}_*\mathscr{H}/k$;
(3) the diagram

$$\begin{array}{ccc} H'\otimes A' & \xrightarrow{\varphi'} & A' \\ {\scriptstyle g\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ H''\otimes A'' & \xrightarrow{\varphi''} & A'' \end{array}$$

is commutative.

If $(f,g)\colon (A',H',\varphi') \to (A'',H'',\varphi'')$ is a morphism in $Al/\mathscr{H}_*\mathscr{H}$, define a function

$$f\odot g: \quad A'\odot H' \to A''\odot H''$$

by the formula

$$(f\odot g)(a'\odot h') = f(a')\odot g(h')\,.$$

It is straightforward to check that $f\odot g\colon A'\odot H \to A''\odot H''$ is a morphism of algebras. In addition we readily obtain

PROPOSITION 1. $\odot\colon Al/\mathscr{H}_*\mathscr{H} \to Al/k$ *is a functor.*

REMARK. (1) Note that there are morphisms

$$(A,k,\cdot) \xrightarrow{(1,\eta)} (A,H,\varphi), \qquad (k,H,\cdot) \xrightarrow{(\eta,1)} (A,H,\varphi)$$

which induce the morphisms

$$\alpha: \quad A = A\odot k \to A\odot H, \qquad \sigma: \quad H = k\odot H \to A\odot H\,.$$

(2) A more precise notation for the functor $\odot$ would be

$$(A,H,\varphi) \to A\odot_\varphi H\,.$$

As previously remarked we have no need for this extra precision at present.

(3) Note that $\otimes$ provides a product on the category $Al/\mathscr{H}_*\mathscr{H}$ by setting

$$(A',H',\varphi')\times(A'',H'',\varphi'') = (A'\otimes A'',H'\otimes H'',\varphi'\otimes\varphi'')\,.$$

Lemma 1.1 then says that $\odot$ preserves products.

DEFINITION. [1] Let $(A,H,\varphi)\in \operatorname{obj} Al/\mathscr{H}_*\mathscr{H}$. An (A,H,φ)-module is a triple (M,φ_A,φ_H) where

(1) $M\in \operatorname{obj}\mathscr{M}/k$;
(2) $\varphi_A\colon A\otimes M \to M$ is an A-module structure;
(3) $\varphi_H\colon H\otimes M \to M$ is an H-module structure;
(4) $\varphi_A\colon A\otimes M \to M$ is a morphism of H-modules.

NOTATION. If $f\colon A \to B$ is a morphism in Al/k, then we define a left A-module structure on B by the composition

$$\lambda_f\colon \quad A\otimes B \xrightarrow{f\otimes 1} B\otimes B \xrightarrow{\mu_B} B\,.$$

DEFINITION. Let $(A,H,\varphi) \in \operatorname{obj} Al/\mathscr{H}_*\mathscr{H}$ and $B \in \operatorname{obj} Al/k$. A φ-morphism from (A,H,φ) to B is a pair (f,g) where

$$f\colon \quad A \to B, \qquad g\colon \quad H \to B$$

are morphisms of algebras such that (B,λ_f,λ_g) is an (A,H,φ)-module.

The following result embodies the universal mapping property of $\odot$. It is a sort of restricted coproduct property.

PROPOSITION 2.2. *Let* $(A,H,\varphi) \in \operatorname{obj} Al/\mathscr{H}_*\mathscr{H}$, $B \in \operatorname{obj} Al/k$, *and* $(f,g)\colon (A,H,\varphi) \to B$ *a φ-morphism. Then there is a unique morphism of algebras*

$$f\overline{\odot} g\colon \quad A\odot H \to B$$

such that the diagram

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "f"] & \\ A\odot H \arrow[rr, "f\overline{\odot} g"] & & B \\ & H \arrow[ul, "\sigma"] \arrow[ur, "g"'] & \end{tikzcd}$$

is commutative.

PROOF. We define

$$(f\overline{\odot} g)(a\odot h) = f(a)g(h)\,.$$

We then have

$$(f\overline{\odot} g)[(a\odot h)(b\odot l)] = \sum \varepsilon_i (f\overline{\odot} g)(ah_i'b\odot h_i''l) = \sum \varepsilon_i f(ah_i'b)g(h_i''l),$$

where

$$\nabla h = \sum h_i'\otimes h_i'' \quad \text{and} \quad \varepsilon_i = (-1)^{\deg b\,\deg h'_i}\,.$$

However since (B,λ_f,λ_g) is an (A,H,φ)-module we have

$$\begin{aligned} g(h)f(a)g(l) &= \sum \varepsilon_i g(h_i')f(a)g(h_i'')g(l) \\ &= \sum \varepsilon_i g(h_i')f(a)g(h_i''l)\,. \end{aligned}$$

Thus using the formulas

$$f(h \cdot a) = g(h)f(a),$$
$$f(a \cdot h) = \sum \varepsilon_i g(h_i')f(a)g(h_i''),$$

we obtain

$$(f \overline{\odot} g)((a \odot g)(b \odot l)) = f(a)g(h)f(b)g(l)$$
$$= ((f \overline{\odot} g)(a \odot h))((f \overline{\odot} g)(b \odot l)),$$

and hence $f \odot g$ is a morphism of algebras as claimed. The remainder is routine.

DEFINITION. Let $H', H'' \in \mathrm{obj}\mathscr{H}_*\mathscr{H}/k$ and let $\varphi\colon H'' \otimes H' \to H'$ be a morphism of k-modules making H' into an H''-module. Then *H' is a homology Hopf algebra over H''* iff

(1) the multiplication morphism $\mu'\colon H' \otimes H' \to H'$ is a morphism of H''-modules and
(2) the comultiplication morphism $\Delta'\colon H' \to H' \otimes H'$ is a morphism of H''-modules.

EXAMPLES. (1) Let G be a connected topological group with

$$H^*(G; \mathsf{Z}_p) \in \mathscr{H}_*\mathscr{H}/\mathsf{Z}_p,$$

p a prime and $\mathsf{Z}_p = \mathsf{Z}/p\mathsf{Z}$. Let $\mathcal{a}^*(p)$ be the mod p-Steenrod algebra [6]. Then the Cartan formula shows that $H^*(G; \mathsf{Z}_p)$ is a homology Hopf algebra over $\mathcal{a}^*(p)$.

(2) Let $H', H'' \in \mathrm{obj}\mathscr{H}_*\mathscr{H}/k$ and $f\colon H'' \to H'$ a morphism of Hopf algebras. Let $\varphi_f\colon H'' \otimes H' \to H'$ be the H''-module structure defined in section 1. Then H' is a homology Hopf algebra over H''. (See the proof of Proposition 1.2 given in Section 4.)

DEFINITION. The category of *homology Hopf algebras over homology Hopf algebras*, denoted by $\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$, is defined as follows:

(1) the objects of $\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$ are triples (H', H'', φ) where $H', H'' \in \mathrm{obj}\mathscr{H}_*\mathscr{H}/k$ and $\varphi\colon H'' \otimes H' \to H'$ is an H''-module structure on H' turning H' into an homology Hopf algebra over H'';

(2) a morphism in $\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$ from (H', H'', φ) to (L', L'', ψ) is a pair (f, g) where

(a) $f\colon H' \to L'$ is a morphism in $\mathscr{H}_*\mathscr{H}/k$;
(b) $g\colon H'' \to L''$ is a morphism in $\mathscr{H}_*\mathscr{H}/k$;
(c) the diagram

$$\begin{tikzcd} H''\otimes H' \arrow[r, "\varphi"] \arrow[d, "g\otimes f"'] & H' \arrow[d, "f"] \\ L''\otimes L' \arrow[r, "\psi"] & L' \end{tikzcd}$$

is commutative.

PROPOSITION 2.3. *If* $(H',H'',\varphi) \in \operatorname{obj} \mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$, *then* $H' \odot H''$ *is a homology Hopf algebra in a natural way.*

PROOF. We recall that for any $H \in \operatorname{obj}\mathscr{H}_*\mathscr{H}/k$,

$$\Delta_H : \quad H \to H \otimes H$$

is a morphism of Hopf algebras. (That Δ_H is a morphism of algebras is part of the definition of a Hopf algebra. Since H has commutative comultiplication it is easily seen to be a morphism of coalgebras.) Thus we have a morphism

$$(\Delta',\Delta'') : \quad (H',H'',\varphi) \to (H'\otimes H',\, H''\otimes H'',\, \varphi\otimes\varphi)$$

in $\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$. This provides a morphism of algebras.

$$\begin{aligned} \Delta'\odot\Delta'' : \quad H'\odot H'' \to\ & (H'\otimes H')\odot(H''\otimes H'') \\ & \qquad \| \ \text{(Lemma 1.1)} \\ & (H'\odot H'')\otimes(H'\odot H'') \,. \end{aligned}$$

Thus $\Delta'\odot\Delta''$ gives $H'\odot H''$ a Hopf algebra structure. The commutativity of $\Delta'\odot\Delta''$ is immediate and thus $\Delta'\odot\Delta''$ provides a homology Hopf algebra structure on $H'\odot H''$.

COROLLARY 2.4. *There is a commutative diagram of functors*

$$\begin{tikzcd} \mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H} \arrow[r, "\odot"] \arrow[d] & \mathscr{H}_*\mathscr{H}/k \arrow[d] \\ Al/\mathscr{H}_*\mathscr{H} \arrow[r, "\odot"] & Al/k \end{tikzcd}$$

where the vertical functors are forgetful functors.

COROLLARY 2.5. *Let* $(H',H'',\varphi) \in \operatorname{obj}\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$ *and* $H \in \operatorname{obj}\mathscr{H}_*\mathscr{H}/k$. *Suppose that* $f\colon H' \to H$, $g\colon H'' \to H$ *are morphisms of Hopf algebras such that* $(f,g)\colon (H',H'',\varphi) \to H$ *is a* φ*-morphism. Then there is a unique morphism of homology Hopf algebras*

$$f \bar{\odot} g : \quad H' \odot H'' \to H$$

such that the diagram

$$\begin{tikzcd}
 & H' \arrow[dl, "\alpha"'] \arrow[dr, "f"] & \\
H' \odot H'' \arrow[rr, "f \bar{\odot} g"] & & H \\
 & H'' \arrow[ul, "\sigma"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes.

3. Split exact sequences in $\mathscr{H}_*\mathscr{H}/k$.

We will begin by recalling some elementary facts about exact sequences in $\mathscr{H}_*\mathscr{H}/k$. More complete details may be found in [4].

Suppose that

$$H' \xrightarrow{f'} H \xrightarrow{f''} H''$$

is a sequence in $\mathscr{H}_*\mathscr{H}/k$, that is, $f''f'$ is the trivial morphism $H' \to H''$. A criterion for exactness of this sequence is the following (see [4; section 2]):

View H'' as a comodule over H via the structure morphism

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f'' \otimes 1} H \otimes H'' .$$

Since $f'' \cdot f' = *$ there is a unique morphism of homology Hopf algebras (the cotensor product, $\square_H$, is defined in [3])

$$f : \quad H' \to k \square_H H'' .$$

Exactness (at H) is equivalent to f being an epimorphism.

Example. Let $(H', H'', \varphi) \in \mathrm{obj} \mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$. Then we have the sequence

$$H' \xrightarrow{\bar{f}'} H' \odot H'' \xrightarrow{\bar{f}''} H''$$

of homology Hopf algebras where

(1) $\bar{f}'$ is induced by the morphism

$$(H', k, \cdot) \to (H', H, \varphi)$$

in $\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$ and

(2) $\bar{f}''$ is induced by the universal property of $\odot$ from the φ-morphism

$$(H', H'', \varphi) \to H'' ,$$

where $H' \to H''$ is the trivial morphism, and $H'' \to H''$ is the identity morphism.

Notice that the trivial action given by

$$H'' \otimes H' \xrightarrow{\varepsilon \otimes 1} k \otimes H' \xrightarrow{\cong} H'$$

yields under this construction the trivial sequence

$$H' \xrightarrow{f'} H' \otimes H'' \xrightarrow{f''} H'' .$$

(It is easily checked that the trivial action of H'' on H' leads to an isomorphism

$$H' \odot H'' \cong H' \otimes H''$$

of homology Hopf algebras.)

Definition. A *split short exact sequence* in $\mathscr{H}_*\mathscr{H}/k$ is an exact sequence of homology Hopf algebras

$$k \to H' \xrightarrow{f'} H \xrightarrow{f''} H'' \to k$$

together with a morphism of homology Hopf algebras

$$\sigma : \quad H'' \to H, \quad \text{such that} \quad f'' \cdot \sigma = 1 : \quad H'' \to H'' .$$

We will often use the notation

$$\begin{tikzcd} k \arrow[r] & H' \arrow[r, "f'"] & H \arrow[r, "f''"] & H'' \arrow[r] \arrow[l, bend left, "\sigma"] & k \end{tikzcd}$$

to denote a split short exact sequence in $\mathscr{H}_*\mathscr{H}/k$.

Example. If $(H', H'', \varphi) \in \operatorname{obj} \mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$ then the sequence

$$\begin{tikzcd} k \arrow[r] & H' \arrow[r, "f'"] & H' \odot H'' \arrow[r, "f''"] & H'' \arrow[r] \arrow[l, bend left, "\sigma"] & k \end{tikzcd}$$

is split short exact.

Convention. Throughout the remainder of this section,

$$\begin{tikzcd} k \arrow[r] & H' \arrow[r, "f'"] & H \arrow[r, "f''"] & H'' \arrow[r] \arrow[l, bend left, "\sigma"] & K \end{tikzcd}$$

will denote a fixed split short exact sequence in $\mathscr{H}_*\mathscr{H}/k$.

Definition. Define a morphism

$$\bar{\varphi}: \quad H''\otimes H' \to H$$

by commutativity of the following diagram:

$$\begin{tikzcd} H''\otimes H' \arrow[r, "\Delta''\otimes 1"] \arrow[d, "\bar{\varphi}"'] & H''\otimes H''\otimes H' \arrow[r, "1\otimes T"] & H''\otimes H'\otimes H'' \arrow[d, "1\otimes 1\otimes\chi"] \\ H & H\otimes H\otimes H \arrow[l, "\text{multiply}"'] & H''\otimes H'\otimes H'' \arrow[l, "\sigma\otimes f'\otimes\sigma"'] \end{tikzcd}$$

Lemma 3.1.
(1) *$\bar{\varphi}$: $H''\otimes H' \to H$ is a morphism of homology coalgebras.*
(2) *$f''\bar{\varphi}$: $H''\otimes H' \to H''$ is the trivial morphism.*

Proof. By definition $\bar{\varphi}$ is a composition of morphisms of homology coalgebras and the first assertion is immediate. (Note Δ'' is a morphism of coalgebras since H'' is cocommutative.)

To prove the second assertion recall that the conjugation χ satisfies

$$\sum x_i'\chi(x_i'') = 0$$

for any $x\in H''$ with $\deg x>0$ and

$$\Delta(x) = \sum x_i'\otimes x_i'' .$$

To show that $f''\bar{\varphi}$ is the trivial morphism let $a\otimes s\in H''\otimes H'$ be of positive degree. Then if $\deg s>0$ we have

$$\begin{aligned}(f''\bar{\varphi})(a\otimes s) &= \sum a_i'f''f'(s)\chi(a_i'') \\ &= \sum a_i'0\chi(a_i'') = 0 .\end{aligned}$$

If $\deg s=0$, then $\deg a>0$. Thus

$$(f''\cdot\bar{\varphi})(a\otimes s) = s\sum a_i'\chi(a_i'') = 0 .$$

Hence $f''\bar{\varphi}$: $H''\otimes H' \to H''$ is the trivial morphism.

LEMMA 3.2. *Suppose given a diagram of connected coalgebras*

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[d, "f''"] \\ & C'' \end{tikzcd}$$

with $f''g\colon B \to C''$ the trivial morphism. Then there is a unique morphism $\hat{g}\colon B \to k\square_{C''}C$ such that the diagram

$$\begin{tikzcd} & k\square_{C'}C' \arrow[d] \\ B \arrow[ur, bend left, "\hat{g}"] \arrow[r, "g"'] & C \arrow[d] \\ & C'' \end{tikzcd}$$

commutes.

PROOF. Direct from the definitions.

PROPOSITION 3.3. *There is a unique morphism of coalgebras*

$$\varphi:\quad H''\otimes H' \to H'$$

such that the diagram

$$\begin{tikzcd} H''\otimes H' \arrow[r, "\varphi"] \arrow[dr, bend right, "\bar{\varphi}"'] & H' \arrow[d, "f'"] \\ & H \end{tikzcd}$$

commutes.

PROOF. Consider the diagram of homology coalgebras

$$\begin{tikzcd} & H' = k\square_{H''}H \arrow[d, "f'"] \\ H''\otimes H' \arrow[r, "\bar{\varphi}"] & H \arrow[d, "f''"] \\ & H''. \end{tikzcd}$$

By Lemma 3.1 the morphism $f''\bar{\varphi}$ is trivial. Apply Lemma 3.2.

Proposition 3.3 provides H' with the structure of an H''-module via φ. The remainder of this section is devoted to establishing that $(H', H'', \varphi) \in \operatorname{obj} \mathscr{H}_* \mathscr{H} / \mathscr{H}_* \mathscr{H}$ and that the sequence

$$\begin{tikzcd} k \arrow[r] & H' \arrow[r, "f'"] & H \arrow[r, "f''"] & H'' \arrow[r] \arrow[l, bend left, "\sigma"] & k \end{tikzcd}$$

is isomorphic to the sequence

$$k \to H' \to H' \odot H'' \to H'' \to k;$$

the semi-tensor product being taken with respect to φ.

Convention. Henceforth the H''-module structure on H' given by the map

$$\varphi: \quad H'' \otimes H' \to H'$$

constructed in Proposition 3.3 will be referred to as simply the H''-module structure on H'.

Remark. A local formula for the H''-module structure on H' is given by

$$a \circ s = f'^{-1}\left[\sum \varepsilon_i \sigma(a_i') f'(s) \sigma \chi''(a_i'')\right],$$

where

$$\nabla(a) = \sum a_i' \otimes a_i'',$$
$$\varepsilon_i = (-1)^{\deg a_i' \deg s},$$

and χ'' is the canonical anti-automorphism on H''. The multiplication and sum on the right being taken in H.

Proposition 3.4. *The comultiplication*

$$\Delta': \quad H' \to H' \otimes H'$$

is a morphism of H''-modules.

Proof. By Proposition 3.3 the morphism

$$\varphi: \quad H'' \otimes H' \to H'$$

is a morphism of coalgebras. Thus the diagram

$$\begin{tikzcd}
H''\otimes H' \arrow[r, "\Delta''\otimes\Delta'"] \arrow[dd, "\varphi"'] & H''\otimes H''\otimes H'\otimes H' \arrow[d, "1\otimes T\otimes 1"] \\
 & H''\otimes H'\otimes H''\otimes H' \arrow[d, "\varphi\otimes\varphi"] \\
H' \arrow[r, "\Delta'"] & H'\otimes H'
\end{tikzcd}$$

commutes.

The assertion that Δ' is a morphism of H''-modules is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
H''\otimes H' \arrow[r, "\varphi"] \arrow[d, "1\otimes\Delta'"'] & H' \arrow[d, "\Delta'"] \\
H''\otimes H'\otimes H' \arrow[r, "\psi"] & H''\otimes H'
\end{tikzcd}$$

where

$$\psi: \quad H''\otimes H'\otimes H' \to H'\otimes H'$$

is the H''-module structure on $H'\otimes H'$. From the definition of ψ we are reduced to showing that the diagram

$$\begin{tikzcd}
H''\otimes H' \arrow[r, "\varphi"] \arrow[d, "1\otimes\Delta'"'] & H' \arrow[ddd, "\Delta'"] \\
H''\otimes H'\otimes H' \arrow[d, "\Delta''\otimes 1"'] & \\
H''\otimes H''\otimes H'\otimes H' \arrow[d, "1\otimes T\otimes 1"'] & \\
H''\otimes H'\otimes H''\otimes H' \arrow[r, "\varphi\otimes\varphi"] & H'\otimes H'
\end{tikzcd}$$

commutes. However this is the diagram that asserts φ is a morphism of coalgebras, and as we saw above, commutes.

PROPOSITION 3.5 *The multiplication*

$$\mu': \quad H'\otimes H' \to H'$$

is a morphism of H''-modules.

The proof of Proposition 3.5 may be simplified somewhat by the following device. Define an H''-module structure on H by the map

$$\varphi_\sigma : \quad H'' \otimes H \to H'' ,$$

where φ_σ is defined as in section 1 by the diagram

$$\begin{tikzcd}
H'' \otimes H \arrow[r, "\Delta'' \otimes 1"] \arrow[dd, "\varphi_\sigma"'] & H'' \otimes H'' \otimes H \arrow[r, "1 \otimes T"] & H'' \otimes H \otimes H'' \arrow[d, "1 \otimes 1 \otimes \chi"] \\
 & & H'' \otimes H \otimes H'' \arrow[d, "\sigma \otimes 1 \otimes \sigma"] \\
H & & H \otimes H \otimes H , \arrow[ll, "\text{multiply}"']
\end{tikzcd}$$

As a special case of Proposition 1.2 we then have:

PROPOSITION 3.6. *The multiplication*

$$\mu : \quad H \otimes H \to H$$

is a morphism of H''-modules.

By a simple modification of the proof of Proposition 3.4 we also obtain:

PROPOSITION 3.7. *The comultiplication*

$$\Delta : \quad H \to H \otimes H$$

is a morphism of H''-modules.

COROLLARY 3.8. $(H, H'', \varphi_\sigma) \in \operatorname{obj} \mathscr{H}_* \mathscr{H} / \mathscr{H}_* \mathscr{H}$.

PROOF OF PROPOSITION 3.5. We have a diagram

$$\begin{tikzcd}
H'' \otimes H' \arrow[r, "1 \otimes f'"] \arrow[dddd, bend right=60, "\psi"'] & H'' \otimes H \arrow[dddd, bend left=60, "\varphi_\sigma"] \\
H'' \otimes H' \otimes H' \arrow[u, "1 \otimes \mu"] \arrow[r, "1 \otimes f' \otimes f''"] \arrow[d] & H'' \otimes H \otimes H \arrow[u, "1 \otimes \mu"'] \arrow[d] \\
H' \otimes H' \arrow[r, "f' \otimes f'"] \arrow[d, "\mu'"] & H \otimes H \arrow[d, "\mu"] \\
H' \arrow[r, "f'"] & H \\
 &
\end{tikzcd} ,$$

where the right hand rectangle commutes (Proposition 3.6) and the horizontal morphisms are injections ([4, Proposition 2.1(2)]). It therefore follows that the left hand rectangle commutes.

Corollary 3.9.
$(H',H'',\varphi) \in \operatorname{obj} \mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$ and $(H',H'',\varphi) \xrightarrow{(f',1)} (H,H'',\varphi_\sigma)$ is a morphism in $\mathscr{H}_*\mathscr{H}/\mathscr{H}_*\mathscr{H}$.

Analogous to the proof of Proposition 3.5 we also obtain:

Proposition 3.10. *The morphism*

$$H' \otimes H \xrightarrow{f' \otimes 1} H \otimes H \xrightarrow{\mu} H$$

is a morphism of H''-modules.

Proposition 3.11. *The pair of morphisms*

$$f' : \quad H' \to H \leftarrow H'' : \sigma$$

is a φ-morphism of (H',H,φ) to H.

The proof of Proposition 3.11 is a tedious diagram chase and is deferred to section 5.

It follows from Proposition 3.11 and the universal property of $\odot$ (Corollary 2.5) that there is a commutative diagram

$$\begin{tikzcd}
k \arrow[r] & H' \arrow[r, "f'"] \arrow[d, equal] & H \arrow[r, "f''"] & H'' \arrow[r] \arrow[d, equal] & k \\
k \arrow[r] & H' \arrow[r, "\bar{f}'"] & H' \odot H'' \arrow[u, "f' \odot \sigma"'] \arrow[r, "\bar{f}''"] & H'' \arrow[r] & k
\end{tikzcd}$$

of homology Hopf algebras. It follows from [3; Proposition 4.4] that $f' \odot \sigma$ is a bijection of graded k-modules and hence is an isomorphism of homology Hopf algebras. Thus we have proved:

Theorem 3.12. *A split exact sequence of homology Hopf algebras*

$$\begin{tikzcd}
k \arrow[r] & H' \arrow[r, "f'"] & H \arrow[r, "f''"] & H'' \arrow[l, bend left, "\sigma"] \arrow[r] & k
\end{tikzcd}$$

is isomorphic to the split exact sequence of homology Hopf algebras

$$k \to H' \xrightarrow{j'} H' \odot H'' \xrightarrow{j''} H'' \to k,$$

where $(H', H'', \varphi) \in \mathscr{H}_* \mathscr{H} / \mathscr{H}_* \mathscr{H}$ *is as constructed above.*

This establishes the result announced in the introduction.

4. Some diagram lemmas and the proof of Proposition 2.1.

LEMMA 4.1. *If C is a commutative coalgebra then the diagram*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[r, "\Delta\otimes\Delta"] & C\otimes C\otimes C\otimes C \arrow[d, "1\otimes T\otimes 1"] \\
C\otimes C \arrow[d, "\Delta\otimes\Delta"'] & & C\otimes C\otimes C\otimes C \arrow[d, "T\otimes 1\otimes 1"] \\
C\otimes C\otimes C\otimes C & & C\otimes C\otimes C\otimes C \arrow[ll, "1"']
\end{tikzcd}$$

is commutative.

PROOF. Immediate from the definitions.

LEMMA 4.2. *If $H \in \operatorname{obj} \mathscr{H}_* \mathscr{H} / k$ and $*: H \to H$ is the trivial morphism then the diagram*

$$\begin{tikzcd}
H\otimes H \arrow[r, "1\otimes\chi"] & H\otimes H \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[r, "*"] & H
\end{tikzcd}$$

is commutative.

PROOF. This is essentially the definition of χ (see [3; Section 8]).

LEMMA 4.3. *If $H \in \operatorname{obj} \mathscr{H}_* \mathscr{H} / k$ then the diagram*

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H\otimes H \arrow[r, "\Delta\otimes\Delta"] & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1\otimes\chi"] \\
H\otimes H \arrow[d, "1\otimes\chi"'] & & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes\mu\otimes 1"] \\
H\otimes H \arrow[rr, "1\otimes k\otimes 1"] & & H\otimes H\otimes H
\end{tikzcd}$$

commutes.

Proof. By Lemma 4.1 the diagram

$$
\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H\otimes H \arrow[r, "\Delta\otimes\Delta"] & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes T\otimes 1"] \\
H\otimes H \arrow[dr, "\Delta\otimes\Delta"'] & \mathrm{I} & H\otimes H\otimes H\otimes H \arrow[dl, "T\otimes 1\otimes 1"] \\
 & H\otimes H\otimes H\otimes H &
\end{tikzcd}
$$

commutes. Thus it suffices to show that the diagram

$$
\begin{tikzcd}
H \arrow[r] \arrow[d, "\Delta"'] & H\otimes H \arrow[r, "\Delta\otimes\Delta"] & H\otimes H\otimes H\otimes H \arrow[r, "1\otimes T\otimes 1"] & H\otimes H\otimes H\otimes H \arrow[d, "T\otimes 1\otimes 1"] \\
H\otimes H \arrow[dd, "1\otimes\chi"'] & & \mathrm{II} & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1\otimes\chi"] \\
 & & & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes\mu\otimes 1"] \\
H\otimes H \arrow[rrr, "1\otimes k\otimes 1"] & & & H\otimes H\otimes H
\end{tikzcd}
$$

commutes. Applying Lemma 4.2 we obtain the commutative diagram

$$
\begin{tikzcd}
H \arrow[r] \arrow[d, "\Delta"'] & H\otimes H \arrow[r, "\Delta\otimes\Delta"] & H\otimes H\otimes H\otimes H \arrow[r, "1\otimes T\otimes 1"] & H\otimes H\otimes H\otimes H \arrow[d, "T\otimes 1\otimes 1"] \\
H\otimes H \arrow[d, "\Delta\otimes\varepsilon"'] & & \mathrm{III} & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1\otimes\chi"] \\
H\otimes H\otimes k \arrow[d, "1\otimes T"'] & & & H\otimes H\otimes H\otimes H \arrow[d, "1\otimes\mu\otimes 1"] \\
H\otimes k\otimes H \arrow[r, "1\otimes 1\otimes\chi"] & H\otimes k\otimes H \arrow[rr, "1\otimes\eta\otimes 1"] & & H\otimes H\otimes H
\end{tikzcd}
$$

which is easily seen to be equivalent to the diagram II as required.

Proof of Proposition 1.2. The assertion is equivalent to the commutativity of the following horrendous diagram

$$\begin{tikzcd}
H\otimes A\otimes A \arrow[r, "\Delta\otimes 1\otimes 1"] \arrow[d, "1\otimes\mu"'] & H\otimes H\otimes A\otimes A \arrow[r, "1\otimes T\otimes 1"] & H\otimes A\otimes H\otimes A \arrow[d, "\Delta\otimes 1\otimes\Delta\otimes 1"] \\
H\otimes A \arrow[d, "\Delta\otimes 1"'] & & H\otimes H\otimes A\otimes H\otimes H\otimes A \arrow[d, "1\otimes T\otimes 1\otimes T"] \\
H\otimes H\otimes A \arrow[d, "1\otimes T"'] & & H\otimes A\otimes H\otimes H\otimes A\otimes H \arrow[d, "1\otimes 1\otimes\chi\otimes 1\otimes 1\otimes\chi"] \\
H\otimes A\otimes H \arrow[d, "1\otimes 1\otimes\chi"'] & \mathscr{H} & H\otimes A\otimes H\otimes H\otimes A\otimes H \arrow[d, "f\otimes 1\otimes f\otimes f\otimes 1\otimes f"] \\
H\otimes A\otimes H \arrow[dd, "f\otimes 1\otimes f"'] & & (A\otimes A\otimes A)\otimes(A\otimes A\otimes A) \arrow[d, "\mu\otimes\mu"] \\
 & & A\otimes A \arrow[d, "\mu"] \\
A\otimes A\otimes A \arrow[rr, "\mu"] & & A
\end{tikzcd}$$

Since f is a map of algebras, the above diagram will commute if the following diagram commutes.

$$\begin{tikzcd}
H\otimes A\otimes A \arrow[r, "\Delta\otimes 1\otimes 1"] \arrow[d, "\Delta\otimes 1"'] & H\otimes H\otimes A\otimes A \arrow[r, "\Delta\otimes\Delta\otimes 1\otimes 1"] & H\otimes H\otimes H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\chi\otimes 1\otimes\chi\otimes 1\otimes 1"] \\
H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\chi\otimes 1\otimes 1"'] & & H\otimes H\otimes H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\mu\otimes 1\otimes 1"] \\
H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\mu"'] & & H\otimes H\otimes H\otimes A\otimes A \arrow[d, "1\otimes 1\otimes T\otimes 1"] \\
H\otimes H\otimes A \arrow[d, "1\otimes T"'] & \mathscr{H}^{+} & H\otimes H\otimes A\otimes H\otimes A \arrow[d, "1\otimes T\otimes T"] \\
H\otimes A\otimes H \arrow[dd, "f\otimes 1\otimes f"'] & & H\otimes A\otimes H\otimes A\otimes H \arrow[d, "f\otimes 1\otimes f\otimes 1\otimes f"] \\
 & & A\otimes A\otimes A\otimes A\otimes A \arrow[d, "\mu"] \\
A\otimes A\otimes A \arrow[rr, "\mu"] & & A
\end{tikzcd}$$

This diagram is seen to commute by applying Lemma 4.3 to deduce that

$$\begin{tikzcd}
H\otimes A\otimes A \arrow[r, "\Delta\otimes 1\otimes 1"] \arrow[d, "\Delta\otimes 1"'] & H\otimes H\otimes A\otimes A \arrow[r, "\Delta\otimes\Delta\otimes 1\otimes 1"] & H\otimes H\otimes H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\chi\otimes 1\otimes\chi\otimes 1\otimes 1"] \\
H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\chi\otimes 1\otimes 1"'] & & H\otimes H\otimes H\otimes H\otimes A\otimes A \arrow[d, "1\otimes\mu\otimes 1\otimes 1\otimes 1"] \\
H\otimes H\otimes A\otimes A \arrow[rr, "1\otimes k\otimes 1\otimes 1\otimes 1"] & & H\otimes H\otimes H\otimes A\otimes A
\end{tikzcd}$$

commutes and applying elementary considerations to the lower portion of $\mathscr{H}^+$.

5. More diagram lemmas and the proof of proposition 3.11.

LEMMA 5.1. *If C is a commutative coalgebra then the diagram*

$$\begin{tikzcd}
C \arrow[r] & C\otimes C \arrow[r, "\Delta\otimes 1"] & C\otimes C\otimes C \arrow[d, "1\otimes T"] \\
 & & C\otimes C\otimes C \arrow[d, "1\otimes T"] \\
C\otimes C \arrow[uu, "\Delta"] \arrow[rr, "\Delta\otimes 1"] & & C\otimes C\otimes C
\end{tikzcd}$$

is commutative.

PROOF. Direct from the definitions.

LEMMA 5.2. *If $H \in \mathrm{obj}\,\mathscr{H}_*\mathscr{H}/k$ then the diagram*

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[dd, "k\otimes 1"'] & H\otimes H \arrow[r, "\Delta\otimes 1"] & H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1"] \\
 & & H\otimes H\otimes H \arrow[d, "\mu\otimes 1"] \\
k\otimes H \arrow[rr, "\eta\otimes 1"] & & H\otimes H
\end{tikzcd}$$

commutes.

PROOF. Tensor the diagram

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "*"'] & H\otimes H \arrow[d, "1\otimes\chi"] \\ H & H\otimes H \arrow[l, "\mu"'] \end{tikzcd}$$

with H. This gives the commutative diagram

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dd, "k\otimes 1"'] & H\otimes H \arrow[r, "\Delta\otimes 1"] \arrow[dd, "*\otimes 1"] & H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1"] \\ & & H\otimes H\otimes H \arrow[d, "\mu\otimes 1"] \\ k\otimes H \arrow[r, "\eta\otimes 1"] & H\otimes H \arrow[r, leftrightarrow] & H\otimes H \end{tikzcd}$$

and the result follows.

LEMMA 5.3. *If $H \in \mathrm{obj}\mathscr{H}_*\mathscr{H}/k$ then the diagram*

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dd, "1\otimes k"'] & H\otimes H \arrow[r, "\Delta\otimes 1"] & H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1"] \\ & & H\otimes H\otimes H \arrow[d, "1\otimes\mu"] \\ H\otimes H \arrow[rr, "1\otimes\eta"] & & H\otimes H \end{tikzcd}$$

commutes.

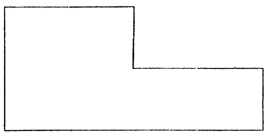
PROOF. By Lemma 5.1 we have the commutative diagram

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dd, "\Delta\otimes 1"'] & H\otimes H \arrow[r, "\Delta\otimes 1"] & H\otimes H\otimes H \arrow[d, "1\otimes T"] \\ & & H\otimes H\otimes H \arrow[d, "1\otimes T"] \\ H\otimes H \arrow[rr, "\Delta\otimes 1"] & & H\otimes H\otimes H \end{tikzcd}$$

This gives a commutative diagram

$$
\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[ddd, "1"'] & H\otimes H \arrow[r, "\Delta\otimes 1"] & H\otimes H\otimes H \arrow[r, "1\otimes T"] \arrow[d, "1\otimes\chi\otimes 1"] & H\otimes H\otimes H \arrow[r, "1\otimes T"] & H\otimes H\otimes H \arrow[d, "1\otimes\chi\otimes 1"] \\
 & & H\otimes H\otimes H \arrow[d, "1\otimes\mu"] & & H\otimes H\otimes H \arrow[d, "1\otimes\mu"] \\
 & & H\otimes H \arrow[rr, leftrightarrow, "1"] & & H\otimes H \\
H \arrow[rrrr, "1\otimes k"] & & & & H\otimes k \arrow[u, "1\otimes\eta"']
\end{tikzcd}
$$

by applying Lemma 5.2 to obtain the outside rectangle. The commutativity of the



shaped region now follows and is the desired result.

PROOF OF PROPOSITION 3.11. The conclusion is equivalent to the commutativity of the diagram

$$
\begin{tikzcd}
H''\otimes H'\otimes H \arrow[r, "1\otimes f'\otimes 1"] \arrow[d, "\Delta''\otimes 1\otimes 1"'] & H''\otimes H\otimes H \arrow[rr, "\sigma\otimes 1\otimes 1"] & & H\otimes H\otimes H \arrow[dddddd, "\mu"] \\
H''\otimes H''\otimes H'\otimes H \arrow[d, "1\otimes T\otimes 1"'] & & & \\
H''\otimes H'\otimes H''\otimes H \arrow[d, "\Delta''\otimes 1\otimes 1\otimes 1"'] & & & \\
H''\otimes H''\otimes H'\otimes H''\otimes H \arrow[d, "1\otimes T\otimes 1\otimes 1"'] \arrow[rr, phantom, "\mathrm{I}"] & & & \\
H''\otimes H'\otimes H''\otimes H''\otimes H \arrow[d, "1\otimes 1\otimes\chi\otimes 1\otimes 1"'] & & & \\
H''\otimes H'\otimes H''\otimes H''\otimes H \arrow[d, "1\otimes 1\otimes\mu\otimes 1"'] & & & \\
H''\otimes H'\otimes H''\otimes H \arrow[rr, "\sigma\otimes f'\otimes\sigma\otimes 1"] & & H\otimes H\otimes H\otimes H \arrow[r, "\mu"] & H\,.
\end{tikzcd}
$$

From Lemma 5.3 we obtain the commutative diagram

$$\begin{tikzcd}
H'' \arrow[d, "\Delta''"'] \arrow[rrdd, "1\otimes k"] & & \\
H''\otimes H'' \arrow[d, "\Delta''\otimes 1"'] & & \\
H''\otimes H''\otimes H'' \arrow[d, "1\otimes\chi\otimes 1"'] & \mathrm{II} & H''\otimes k \\
H''\otimes H''\otimes H'' \arrow[d, "1\otimes\mu"'] & & \\
H''\otimes H'' \arrow[rruu, "1\otimes\eta"'] & &
\end{tikzcd}$$

This then gives the commutative diagram

$$\begin{tikzcd}
H''\otimes H'\otimes H \arrow[rr, "1\otimes f'\otimes 1"] \arrow[d, "\Delta''\otimes 1\otimes 1"'] & & H''\otimes H\otimes H \\
H''\otimes H''\otimes H'\otimes H \arrow[d, "1\otimes T\otimes 1"'] & & \\
H''\otimes H'\otimes H''\otimes H \arrow[d, "\Delta''\otimes 1\otimes 1\otimes 1"'] & & \\
H''\otimes H''\otimes H'\otimes H''\otimes H \arrow[d, "1\otimes T\otimes 1\otimes 1"'] & \mathrm{III} & \\
H''\otimes H'\otimes H''\otimes H''\otimes H \arrow[d, "1\otimes 1\otimes\chi\otimes 1\otimes 1"'] & & \\
H''\otimes H'\otimes H''\otimes H''\otimes H \arrow[d, "1\otimes 1\otimes\mu\otimes 1"'] & & \\
H''\otimes H'\otimes H''\otimes H \arrow[r, "1\otimes 1\otimes\varepsilon\otimes 1"] & H''\otimes H'\otimes k\otimes H \arrow[r, equal] & H''\otimes H'\otimes H \arrow[uuuuuu, "1\otimes f'\otimes 1"']
\end{tikzcd}$$

and the result follows from III by elementary considerations.

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