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DIRECT INTEGRALS OF HILBERT SPACES I

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1. Introduction, notation and preliminaries.

In the theory of direct integrals of Hilbert spaces, as it has been developed until now, it is normally assumed that the fiber spaces are separable [1]. This situation is rather unsatisfactory and has as consequence that e.g. the well-known spaces $L^2{}_\mu(H,Z)$, where H is a non-separable Hilbert space, cannot be described as integrals of Hilbert spaces. In [5] I. Segal defined so called weak direct-integrals of Hilbert spaces. No countability conditions are imposed and Segal shows how these weak direct-integrals turn up in a natural way in the theory of decomposition of von Neumann algebras of operators. Unfortunately almost no theory is developed in the general case. In the presence of separability Segal's definition is equivalent to the usual one. This lack of theory has as a consequence that e.g. the multiplicity theory in the non-separable case is entirely based on non-spatial arguments. One works with the algebra of operators rather than with the underlying Hilbert space [3], [5].

In the present paper a slight variation of the definition of Segal's for integrable families of Hilbert spaces is proposed. Some simple properties and a structure theorem are proved. An application to the multiplicity theory of spectral measures is given. In another paper [8] decomposition of von Neumann algebras will be treated.

It is a pleasure for me to thank lektor E. T. Kehlet for reading the manuscript and referring to [5].

Let (Z,Σ,μ) be a measure space and denote by $M^\infty(Z,\mu)$ the algebra of all complex-valued, bounded and measurable functions on Z, by $N^\infty(Z,\mu)$ the ideal of all $\varphi \in M^\infty(Z,\mu)$ which are locally μ-almost everywhere negligible, and by $L^\infty(Z,\mu)$ the quotient-algebra $M^\infty(Z,\mu)/N^\infty(Z,\mu)$. The canonical image of $\varphi \in M^\infty(Z,\mu)$ in $L^\infty(Z,\mu)$ we denote $\tilde{\varphi}$. By abuse of notation we shall sometimes not distinguish between φ and $\tilde{\varphi}$. For maps f and g, defined on Z, we write $f \equiv g$ if $f(z)=g(z)$ μ-locally almost everywhere. In the sequel we shall assume always that $\mu(Z)<\infty$. All

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results though are valid for direct sums of finite-measure spaces, especially for Radon measures on locally compact sets.

As for the theory of direct integrals of Hilbert spaces, we follow J. Dixmier [1]. Given (Z,Σ,μ) we call a collection $\{H(z) \mid z \in Z\}$ a field of Hilbert spaces on Z. Elements of $\prod_{z\in Z} H(z)$ are called vector fields. If $\varphi \in M^\infty(Z,\mu)$ and f is a vector field, then φf: $Z \ni z \mapsto \varphi(z)f(z)$. If f and g are vector fields, then

$$(f,g): Z \ni z \mapsto (f(z),g(z)) \quad \text{and} \quad |f|: Z \ni z \mapsto \|f(z)\| .$$

For a Hilbert space H the algebra of continuous operators on H is denoted $L(H)$.

1.1 Definition. Let $(H(z))_{z\in Z}$ be a field of Hilbert spaces on (Z,Σ,μ) and let Γ be a subspace of $\prod_{z\in Z} H(z)$. Then $\big((H(z)),\Gamma\big)$ is said to be an integrable family of Hilbert spaces if it fulfills the following conditions:

1. For every $f \in \Gamma$, the function $|f|^2$ is μ-summable.
2. If $f \in \prod_z H(z)$ is such that there is a $\varphi \in L^2(Z,\mu)$ with $|f| \leqq \varphi$ and if (f,g) is measurable for all $g \in \Gamma$, then there exists a $f' \in \Gamma$ with $(f-f',g) \equiv 0$ for every $g \in \Gamma$.
3. Let $f' \in \Gamma$ and $f \in \prod_z H(z)$ be such that $|f|^2$ is μ-summable and $f' \equiv f$. Then we have $f \in \Gamma$.

The third condition in 1.1 is of course not essential and is added for convenience. The main difference with the usual definition for measurable families of Hilbert spaces is that we do not impose the maximality condition on Γ, that if for a vector field f we have that (f,g) is measurable for all $g \in \Gamma$, then $f \in \Gamma$. This made it possible to consider subspaces Γ which are not countably generated.

That the integrable families as defined here are precisely the objects one would like them to be, follows from the following.

1.2 Theorem. *Let $(H(z))_z$ be a field of Hilbert spaces and $\Gamma \subseteqq \prod_z H(z)$. Then $\big((H(z)),\Gamma\big)$ is integrable, iff*

1. *For every $f \in \Gamma$, the function $|f|^2$ is summable.*
2. *If $\varphi \in M^\infty(Z,\mu)$ and $f \in \Gamma$, then $\varphi f \in \Gamma$.*
3. *If for $f \in \prod_z H(z)$, with $|f|^2$ summable, there exists $f' \in \Gamma$ with $f' \equiv f$, then $f \in \Gamma$.*
4. *For $f \in \Gamma$ we put $\|f\| = (\int |f|^2 d\mu)^{\frac{1}{2}}$. The semi-normed space $(\Gamma, \|\cdot\|)$ is complete.*

In the theory of integrals of Hilbert spaces, especially 1.2.2 is a crucial property.

1.3 Definition. If $((H(z)),\Gamma)$ is an integrable family of Hilbert spaces, then the Hilbert space, corresponding to $(\Gamma,\|\cdot\|)$ and obtained by forming classes of maps which are equal almost everywhere, is called the direct integral of $((H(z)),\Gamma)$ and is denoted by $\tilde{\Gamma}$ or $\int^{\oplus} H(z)\,d\mu$.

The class of elements $g \in \Gamma$ which are equal almost everywhere to a fixed $f \in \Gamma$ is denoted by $\tilde{f}$. The inner product of two elements $\tilde{f},\tilde{g} \in \tilde{\Gamma}$ is denoted by $((\tilde{f},\tilde{g}))$. (Note the difference between $(\tilde{f},\tilde{g})$ and $((\tilde{f},\tilde{g}))$.

Because integrable families have Property 2 of Theorem 1.2, the next definition makes sense.

1.4. Definition. Every element $\tilde{\varphi} \in L^{\infty}(Z,\mu)$ determines a unique operator $T_{\tilde{\varphi}}$ on $\tilde{\Gamma}$ by $T_{\tilde{\varphi}}\tilde{f} = \widetilde{\varphi f}$, $\tilde{f} \in \tilde{\Gamma}$. Such operators are called diagonal. We put $\mathfrak{Z} = \{T_{\tilde{\varphi}} \mid \tilde{\varphi} \in L^{\infty}(Z,\mu)\}$.

Normally one is especially interested in the commutant $\mathfrak{Z}'$ of $\mathfrak{Z}$, that is,

$$\mathfrak{Z}' = \{A \in L(\tilde{\Gamma}) \mid AT_{\tilde{\varphi}} = T_{\tilde{\varphi}}A, \tilde{\varphi} \in L^{\infty}(Z,\mu)\}\,.$$

The algebras $\mathfrak{Z}$ and $\mathfrak{Z}'$ are von Neumann algebras and it is shown in [5] that conversely every commutative von Neumann algebra can be identified with the algebra of diagonal operators on a suitable integral of Hilbert spaces.

We give two examples of integrable families.

1.5 Examples. 1. Suppose H_0 is a fixed Hilbert space. We put $H(z) = H_0$ for every z. For Γ we take the set of all measurable maps $f\colon Z \to H_0$ such that $|f|^2$ is summable. It is well known that this family satisfies the conditions mentioned in 1.2 and therefore is integrable. It is called a constant family, and we denote it by $(H_0, Z, \mathscr{L}^2{}_{\mu}(H_0,Z))$ or just $\mathscr{L}^2{}_{\mu}(H_0,Z)$. The corresponding integral is $L^2{}_{\mu}(H_0,Z)$.

2. Let Z be the direct sum of the measurable sets $\{Z_i\}_{i\in J}$, and $\{H_i\}_{i\in J}$ a collection of Hilbert spaces. For $z \in Z_i$ we take $H(z) = H_i$. A map $f \in \prod_z H(z)$ is in Γ, iff $|f|^2$ is summable and if $f|Z_i$ is a measurable map from Z_i into H_i. This family is said to be a direct sum of constant families, and it is denoted by $\{(Z_i,H_i)\}_{i\in J}$ and the corresponding Γ by $\Gamma = \bigoplus_i \mathscr{L}^2{}_{\mu_i}(H_i,Z_i)$, where μ_i is the restriction of μ to Z_i.

Up to a certain equivalence relation these examples exhaust the set of integrals of Hilbert spaces.

1.6 Definition. 1. Two integrable families $((H_i(z)),\Gamma_i)$, $i = 1,2$, are called equivalent if there exists a unitary operator $U\colon \tilde{\Gamma}_1 \to \tilde{\Gamma}_2$ so that $U(T_1)_{\tilde{\varphi}} = (T_2)_{\tilde{\varphi}}U$ for all $\varphi \in L^{\infty}(Z,\mu)$.

2. A direct sum of constant families $\mathcal{E}_1 = \big((H_i(z)), \Gamma_i\big)$ is called a regularisation of the integrable family $\mathcal{E}$ if $\mathcal{E}_1$ is equivalent to $\mathcal{E}$.

Then we have

1.7 THEOREM. *Any integrable family of Hilbert spaces on a finite measure space admits a unique regularisation.*

In the second section of this paper Theorem 1.2 will be proved. In the third section we consider the behaviour of integrals of Hilbert spaces under the forming of tensor products and some other properties. The proof of Theorem 1.7 will be given in the fourth section. In the fifth and last section it is shown how, by means of Theorem 1.7, the multiplicity theory of spectral measures can be treated.

Added November 1968: In a conversation with H. Leptin some more papers dealing with integrals of non-separable Hilbert spaces were brought to the attention of the author. Without going in detail too much we remark that in [2] and [7] a topological version of Definition 1.1 of the present note was used, that is, Z is assumed to be a locally compact space and the functions $|f|$, $f \in \Gamma$ are continuous. The applications are to decomposition theory. In [4] and [6] direct integrals are defined roughly stated as direct sums of constant families. The theory is applied to multiplicity theory. It is a pleasure to thank H. Leptin for mentioning these articles.

2. Integrable families of Hilbert spaces.

The proof of Theorem 1.2 will be given in two steps.

2.1 PROPOSITION. *Let* $\big((H(z)), \Gamma\big)$ *be an integrable family of Hilbert spaces. We put for* $f \in \Gamma$, $\|f\| = (\int |f|^2 d\mu)^{\frac{1}{2}}$.

1. *Suppose* $(f_n)_{n \in \mathrm{N}}$ *is a sequence in* Γ *so that* $\lim_n f_n(z) = f(z)$ *exists almost everywhere and with* $(\|f_n\|)_{n \in \mathrm{N}}$ *bounded. Then* $f\colon z \mapsto f(z)$ *is contained in* Γ.

2. Γ *is complete with respect to the semi-norm* $\|\cdot\|$.

3. *If* $\varphi \in M^\infty(Z, \mu)$ *and* $f \in \Gamma$, *then* $\varphi f \in \Gamma$.

PROOF. 1. The assumptions imply that $|f|^2$ is summable. For $g \in \Gamma$ we get $\lim_n (f_n, g)(z) = (f, g)(z)$ almost everywhere and consequently (f, g) is measurable. There exist $f' \in \Gamma$ with $(f - f', g) \equiv 0$ for all $g \in \Gamma$. In particular we find $(f - f', f_n - f') \equiv 0$ and in the limit $f \equiv f'$, whence $f \in \Gamma$.

2. Let $(f_n)_{n \in \mathrm{N}}$ be a Cauchy sequence in $(\Gamma, \|\ \|)$. There is a subsequence which converges almost everywhere to a limit $f \in \prod_z H(z)$. We obtain from 1 that $f \in \Gamma$ and it can be readily verified that $\lim_n \|f_n - f\| = 0$.

3. For any measurable set $A \subseteq Z$ and $f \in \Gamma$, there is an $f' \in \Gamma$ with

$(\chi_A f, g) = (f', g)$ for every $g \in \Gamma$, where χ_A is the characteristic function of A. For $g = f$ this becomes

$$(\chi_A f, f) = (f', f) = (f', \chi_A f) = (f', f').$$

We get $\chi_A f \equiv f'$, so $\chi_A f \in \Gamma$.

It is clear that if φ is a simple function and $f \in \Gamma$ then $\varphi f \in \Gamma$. The fact that the set of simple functions is dense in $M^\infty(Z, \mu)$, in combination with 1, readily yields the desired conclusion.

The above proposition proves the "only if" part of Theorem 1.2. Now we show the converse.

2.2. Proposition. *Let $\mathcal{E} = \big((H(z)), \Gamma\big)$ have the properties mentioned in 1.2, then $\mathcal{E}$ is an integrable family.*

Proof. The only difficulty is Condition 2 of 1.1. Suppose $f \in \prod_z H(z)$ and $\varphi \in \mathcal{L}^2(Z, \mu)$ are given as indicated in 1.1. We remark that since for $g \in \Gamma$ we have (f,g) is measurable and $|(f,g)| \leqq |g|\varphi$, (f,g) is summable. The linear functional defined on Γ by $g \mapsto \int (f,g)\, d\mu$ is continuous, because

$$\left| \int (f,g)\, d\mu \right| \leqq \|g\|\, \|\varphi\|_2 .$$

The semi-Hilbert space $(\Gamma, \|\cdot\|)$ being complete, we infer the existence of an element $f' \in \Gamma$ with

$$\int (f,g)\, d\mu = \int (f',g)\, d\mu \quad \text{for all } g \in \Gamma .$$

Thus we get, for fixed $g \in \Gamma$ and arbitrary $\psi \in M^\infty(Z, \mu)$,

$$\int \psi(f,g)\, d\mu = \int \psi(f',g)\, d\mu ,$$

whence $(f,g) \equiv (f',g)$ and we are done.

A helpful corollary to 1.2 is the following

2.3 Corollary. *If $(H(z))_z$ is a field of Hilbert spaces and $\Gamma \subseteqq \prod_z H(z)$ is a linear subspace so that for any $f \in \Gamma$, $|f|^2$ is summable, then there exists a unique smallest subspace $\bar{\Gamma} \supseteqq \Gamma$ in $\prod_z H(z)$ such that $\big((H(z), \bar{\Gamma}\big)$ is an integrable family.*

Proof. Let Γ' be the set of all finite linear combinations of products of functions $\varphi \in M^\infty(Z, \mu)$ and maps $f \in \Gamma$. Then $\Gamma' \supseteqq \Gamma$, Γ' is a$M^\infty(Z, \mu)$-module and it is easy to see that $|g|^2$ is summable for every $g \in \Gamma'$. The subspace $\bar{\Gamma}$ will consist of those elements $g \in \prod_z H(z)$ which are limits of Cauchy sequences with respect to $\|\cdot\|$ in Γ'. It is clear that $\bar{\Gamma}$ has the desired properties.

In practice, the above corollary is very useful, because often one meets the situation, where a field of Hilbert spaces is given and also a subspace $\Gamma_0 \subseteq \prod_z H(z)$. The space Γ_0' is constructed from Γ_0 as before Γ' from Γ and now we consider the subspace $\Gamma \subseteq \Gamma_0'$ which consists of those elements $f \in \Gamma_0'$ for which $|f|^2$ is summable. To Γ we find then $\overline{\Gamma}$. We see that in this way any subspace $\Gamma_0 \subseteq \prod_z H(z)$ determines a unique integrable family.

We note that the subspaces $\Gamma \subseteq \prod_z H(z)$, so that $\big((H(z)),\Gamma\big)$ is integrable, can be partially ordered by inclusion. Theorem 1.2 and Zorn's lemma enable us to conclude that there are maximal elements among the spaces Γ. These maximal elements indeed verify that if a vector field f is so that $|f|^2$ is summable and (f,g) is measurable for all $g \in \Gamma$, then $f \in \Gamma$ and they are characterized by this property. It is readily seen that in contrast to the separable case, constant families are in general not maximal. This, in fact, turns out to be the reason why it was not possible in the classical definition of integrable families to dispense with the countability hypothesis for a generating set for Γ.

3. Subfamilies and tensor products.

3.1. Definition. An integrable family $\big((H'(z)),\Gamma'\big)$ is a subfamily of the integrable family $\big((H(z)),\Gamma\big)$ if for every $z \in Z$, $H'(z) \subseteq H(z)$ and if $\Gamma' \subseteq \Gamma$.

It is obvious that $\tilde{\Gamma}'$ can be identified with a closed subspace of $\tilde{\Gamma}$ and that then $\tilde{\Gamma}'$ is invariant with respect to the set $\mathfrak{Z}$ of diagonal operators. Conversely, any closed subspace $\tilde{\Gamma}' \subseteq \tilde{\Gamma}$ which is invariant with respect to $\mathfrak{Z}$ determines a unique subfamily $\Gamma' \subseteq \Gamma$ by $f \in \Gamma'$ iff $\tilde{f} \in \tilde{\Gamma}'$.

We denote the set of all $f \in \Gamma$ for which $|f| \in M^\infty(Z,\mu)$ by Γ^∞. In the case where (Z,Σ,μ) is not a finite measure space, we have to be a little bit more careful. It is sensible to introduce then the set Γ^l of all $f \in \prod_z H(z)$ which are locally in Γ, that is, if $A \in \Sigma$ summable and χ_A is its characteristic function, then $\chi_A f \in \Gamma$. Next we define Γ^∞ as the set of bounded elements in Γ^l. It is simple to see that Γ^l, Γ^∞ and Γ determine each other uniquely.

3.2. Definition. Suppose $\mathcal{E}=\big((H(z)),\Gamma_{\mathcal{E}}\big)$ and $\mathcal{E}'=\big((H'(z)),\Gamma_{\mathcal{E}'}\big)$ are two integrable families of Hilbert spaces. There is a natural linear map $\check{}$ of $\Gamma_{\mathcal{E}}{}^\infty \otimes \Gamma_{\mathcal{E}'}{}^\infty$ onto a subspace $\check{\Gamma}$ of $\prod_z H(z) \otimes H'(z)$. It is defined by $(f \otimes f')\check{}\,(z) = f(z) \otimes f'(z)$ for all $f \in \Gamma_{\mathcal{E}}$, $f' \in \Gamma_{\mathcal{E}'}$. Let $\Gamma_{\mathcal{E}\otimes\mathcal{E}'}$ be the smallest subspace in $\prod_z H(z) \otimes H'(z)$ with $\check{\Gamma} \subseteq \Gamma_{\mathcal{E}\otimes\mathcal{E}'}$ and so that

$$\big((H(z) \otimes H'(z)),\Gamma_{\mathcal{E}\otimes\mathcal{E}'}\big)$$

is an integrable family. We denote this family by $\mathcal{E}\otimes\mathcal{E}'$ and call it the tensor product of $\mathcal{E}$ and $\mathcal{E}'$.

3.3. Proposition. *Let $\mathcal{E}=\big((H(z),\Gamma\big)$ be an integrable family and K a fixed Hilbert space. Suppose $\mathcal{E}'$ is the constant family corresponding to K. There exists a unique isomorphism $U\colon \tilde{\Gamma}_{\mathcal{E}}\bar{\otimes}K \to \tilde{\Gamma}_{\mathcal{E}\otimes\mathcal{E}'}$, which for any $\tilde{f}\in\tilde{\Gamma}$ and $x\in K$ maps $\tilde{f}\otimes x$ into $(f\otimes x)^{\sim}$.*

Proof. By $\tilde{x}$ we denote that element of $\tilde{\Gamma}_{\mathcal{E}'}$ which satisfies $\tilde{x}(z)=x$ for almost all z.

If $\tilde{f}_i$, $i=1,\ldots,n$, are elements of $\tilde{\Gamma}$ and x_i, $i=1,\ldots,n$, vectors of K, then we find

$$\int\left|\sum_{i=1}^{n}\tilde{f}_i\otimes x_i\right|^2 d\mu = \sum_{i,j=1}^{n}\int(\tilde{f}_i,\tilde{f}_j)\,d\mu\,(x_i,x_j) = \left\|\sum_{i=1}^{n}(f_i\otimes x_i)^{\sim}\right\|^2.$$

The map which sends $\sum\tilde{f}_i\otimes x_i$ into $\sum(f_i\otimes x_i)^{\sim}$ is defined on the algebraic tensor product $\tilde{\Gamma}\otimes K$ and acts isometrically. We denote the continuous extension to the Hilbert space tensor product of $\tilde{\Gamma}$ and K by U.

It is readily verified that the set of elements of the form $\sum(f_i\otimes x_i)^{\sim}$ is dense in $(\Gamma_{\mathcal{E}\otimes\mathcal{E}'})^{\sim}$. Therefore U is onto and an isomorphism.

This proposition shows in particular that there is a unique isomorphism $U\colon L_\mu^2(\mathsf{C},Z)\bar{\otimes}K \to L_\mu^2(K,Z)$ so that $U(\tilde{\varphi}\otimes x)=\widetilde{\varphi x}$, where $\tilde{\varphi}\in L_\mu^2(\mathsf{C},Z)$ and $x\in K$.

4. Regularisations.

This section will be primarily devoted to proving 1.7. We start by showing that regularisations are unique in a strong sense. First a simple case is treated.

4.1. Lemma. *If the constant family $\mathcal{L}_\mu^2(H',Z)$ is a regularisation of the constant family $\mathcal{L}_\mu^2(H,Z)$, then* $\dim H=\dim H'$.

Proof. If $x\in H$, then $\tilde{x}$ will denote the class of maps $f\in\mathcal{L}_\mu^2(H,Z)$ with $f(z)=x$ almost everywhere. A similar definition holds for $x'\in H'$.

Suppose $\{e_\alpha\}_{\alpha\in J}$ is an orthonormal basis for H. Then $\{\tilde{e}_\alpha\}_{\alpha\in J}$ is a system which satisfies

1° $|\tilde{e}_\alpha|=1$,

2° $(\tilde{e}_\alpha,\tilde{e}_{\alpha'})=\tilde{0}$ for $\alpha\neq\alpha'$,

3° if $\tilde{f}\in L_\mu^2(H,Z)$ and $(\tilde{f},\tilde{e}_\alpha)=\tilde{0}$ for all $\alpha\in J$, then $\tilde{f}=\tilde{0}$.

It is clear that if $\{f_\beta \mid \beta\in J'\}$ is an orthonormal basis for H' and if $U\colon L_\mu^2(H,Z)\to L_\mu^2(H',Z)$ is the isomorphism implementing the regularisation, then $\{\tilde{f}_\beta\}_{\beta\in J'}$ and $\{Ue_\alpha\}_{\alpha\in J}$ have similar properties as $\{\tilde{e}_\alpha\}_{\alpha\in J}$.

Let H be finite dimensional. By choosing almost everywhere orthonormal representatives for $U\tilde{e}_\alpha$, $\alpha\in J$, we see $\dim H' \geqq \dim H$. The converse inequality follows from the remark that $\dim H$ is bigger than the dimension of each finite dimensional subspace of H'. This settles the problem whenever $\dim H$ or $\dim H'$ is finite. Now we assume that both H and H' are infinite dimensional and that $\mu(Z)=1$. We consider $(U\tilde{e}_\alpha, \tilde{f}_\beta)$, $\alpha\in J$, $\beta\in J'$, and note that for fixed $\alpha\in J$, there are at most countably many $\beta\in J'$ such that $(U\tilde{e}_\alpha, \tilde{f}_\beta)\neq\tilde{0}$. This follows from the fact that for each finite set $F\subseteqq J'$ we get

$$1 = \|U\tilde{e}_\alpha\| \geqq \textstyle\sum_{\beta\in F}\displaystyle\int |(U\tilde{e}_\alpha, \tilde{f}_\beta)|^2\, d\mu\,.$$

In the same way we see that for fixed $\beta\in J'$, there are not more than countably many $\alpha\in J$ with $(U\tilde{e}_\alpha, \tilde{f}_\beta)\neq 0$. The result is that

$$\dim H \leqq \aleph_0 \dim H' = \dim H'$$

and conversely, thus $\dim H = \dim H'$.

The general case follows easily from this.

4.2. Proposition. *Suppose $\big((H'(z)),\Gamma'\big)$ and $\big((H''(z)),\Gamma''\big)$ are both regularisations of the same measurable family. Then we get for almost all $z\in Z$,*

$$\dim H''(z) = \dim H'(z).$$

Proof. It suffices to show that if $\{Z_i'\}_{i\in I}$ and $\{Z_j''\}_{j\in J'}$ are the direct decompositions of Z corresponding to the two direct sums of constant families, then $\mu(Z_i\cap Z_j)\neq 0$ implies $\dim H'(z)=\dim H''(z)$ for $z\in Z_i\cap Z_j$.

The restriction of $\big((H'(z)),\Gamma\big)$ to $Z_i\cap Z_j$ is a constant family, which is a regularisation of the restriction of $\big((H''(z)),\Gamma''\big)$ to $Z_i\cap Z_j$, which is also a constant family. The conclusion thus follows from the preceding lemma.

The construction of a regularisation requires more work. The next lemma is crucial in this construction.

4.3. Lemma. *Let $\big((H(z),\Gamma\big)$ be an integrable family of Hilbert spaces. There exists an element $\tilde{g}\in\tilde{\Gamma}$ with $|\tilde{g}|^2=|\tilde{g}|$ and for $\tilde{f}\in\tilde{\Gamma}$ we have $|\tilde{g}|\tilde{f}=\tilde{f}$.*

Proof. We consider the set P of elements $\tilde{g}\in\tilde{\Gamma}$ with $|\tilde{g}|^2=|\tilde{g}|$. We find

$$\sup_{\tilde{g}\in P}\int |\tilde{g}|\, d\mu = \alpha \leqq 1\,.$$

There is a sequence $(g_n)_{n\in\mathsf{N}}$ in P such that

$$\lim_{n\to\infty} \int |\tilde{g}_n|\, d\mu = \alpha .$$

We put

$$\tilde{g}_n' = \tilde{g}_n(1-\sup_{k<n}|\tilde{g}_k|) \quad \text{and} \quad \tilde{g}_m'' = \sum_{n=1}^{m} \tilde{g}_n' .$$

Then we get $|\tilde{g}_m''| \geqq |\tilde{g}_m|$ and $\tilde{g}_m \in \Gamma$. The sequence $(\tilde{g}_m'')_{m\in\mathsf{N}}$ is a Cauchy sequence in $\tilde{\Gamma}$ and has a limit $\tilde{g}$. It can be readily verified that $|\tilde{g}|^2 = |\tilde{g}|$. Moreover we get $\alpha = \int |\tilde{g}|\, d\mu$ and from this fact it is easy to conclude that $|\tilde{g}|\tilde{f} = \tilde{f}$ for all $\tilde{f} \in \tilde{\Gamma}$, if one uses that it is possible to apply the operators $T_{\tilde{\varphi}} \in \mathfrak{Z}$ to elements of $\tilde{\Gamma}$.

The function (class) $|g|$ is uniquely determined and we call $|g|$ the *support function* of Γ and denote it by γ.

4.4. Proposition. *Any measurable family* $\big((H(z)),\Gamma\big)$ *(on a finite measure space) admits a regularisation.*

Proof. We consider the collection $\mathscr{F}$ of subsets $F \subseteqq \tilde{\Gamma}$ with the properties:

1° $|\tilde{h}|^2 = |\tilde{h}| \neq \tilde{0}$ for $\tilde{h} \in F$.

2° $(h,\tilde{h}') = \tilde{0}$ for $h \neq h'$, $h,\tilde{h}' \in F$.

3° Let

$$F_{\tilde{h}} = \{\tilde{g} \mid \tilde{g} \in F,\ |\tilde{g}| \geqq |\tilde{h}|\} \quad \text{for } \tilde{h} \in F$$

and

$$F_{\tilde{h}}^{\perp} = \{f \in \Gamma \mid (\tilde{f},\tilde{g}) = \tilde{0} \quad \text{for } \tilde{g} \in F_{\tilde{h}}\} .$$

The support function of the measurable family determined by $F_{\tilde{h}}^{\perp}$ is $\chi_{\tilde{h}}$ say. Then $\chi_{\tilde{h}} \leqq |\tilde{h}|$ and $\chi_{\tilde{h}} \neq |\tilde{h}|$.

4° We denote by $|F|$ the set $\{|\tilde{h}| \mid \tilde{h} \in F\}$. Any non-empty subset of $|F|$ contains a greatest element.

Let F_γ be a maximal family in $\tilde{\Gamma}$ of maps $\tilde{g}$ so that $|\tilde{g}| = \gamma$, $(\tilde{g},\tilde{g}') = \tilde{0}$ if $\tilde{g} \neq \tilde{g}'$. Such families certainly exist and they satisfy 1°, 2° and 4° of the above definition. The maximality of F_γ implies that F_γ also satisfies 3°. This proves that $\mathscr{F}$ is not empty.

We define a partial ordering on $\mathscr{F}$ by inclusion. Suppose $\{F_\nu \mid \nu \in J\}$ is a totally ordered chain in $\mathscr{F}$. We put $G = \bigcup_{\nu\in J} F_\nu$. It is easy to see that G satisfies 1°, 2° and 3° of the conditions for elements of $\mathscr{F}$.

Let $\tilde{h},\tilde{h}' \in G$ and $\tilde{h} \in F$, $\tilde{h}' \notin F$. Because $\tilde{h}' \notin F$ we get $(\tilde{h}',g) = 0$ for all $\tilde{g} \in F$ and therefore (3° applied to $\tilde{h}$) $|\tilde{h}'| < |\tilde{h}|$. For a subset $G' \subseteqq G$ and $\tilde{h} \in G'$, there is a $\nu \in J$ with $\tilde{h} \in F_\nu$. It follows from the above argument that if $\tilde{h}' \in G$ and not $|\tilde{h}'| < |\tilde{h}|$, then $\tilde{h}' \in F_\nu$. The set $|G' \cap F_\nu|$ contains, by assumption, a greatest element, $|\tilde{h}_0|$ say, with $\tilde{h}_0 \in G' \cap F_\nu$. Since $|\tilde{h}_0|$ is also greatest in $|G'|$ we proved that G has property 4° as well.

Using Zorn's lemma we infer that $\mathscr{F}$ contains a maximal element. $F = \{\tilde{e}_\alpha\}_{\alpha \in I}$.

We denote by $|F|'$ the subset of $L^\infty(Z,\mu)$ which consists of the infima of subsets of $|F|$ and of $\tilde{1}$. The set $|F|'$ is totally ordered and $\tilde{\varphi} \in |F|'$ satisfies $\tilde{\varphi}^2 = \tilde{\varphi}$. Suppose $\tilde{\varphi} \in |F|'$. Consider the set

$$\{\tilde{\psi} \mid \tilde{\psi} \in |F| \cup \{\tilde{0}\},\ \tilde{\psi} < \tilde{\varphi}\}\,.$$

This set contains a greatest element $\tilde{\varphi}^s$. Then $\tilde{\varphi}^s < \tilde{\varphi}$ and if $\tilde{\psi} \in |F|'$ is such that $\tilde{\psi} < \tilde{\varphi}$, then there exists $\tilde{\psi}' \in |F|$ with $\tilde{\varphi} > \tilde{\psi}' \geqq \tilde{\psi}$ and we get

$$\tilde{\varphi}^s \geqq \tilde{\psi}' \geqq \tilde{\psi}\,.$$

We see that if $\tilde{\varphi}, \tilde{\varphi}' \in |F|'$ and $\tilde{\varphi} < \tilde{\varphi}'$, then

$$\tilde{\varphi}^s < \tilde{\varphi} \leqq \tilde{\varphi}'^s < \tilde{\varphi}'\,.$$

Let $G \subseteqq |F|'$ and $G^s = \{\tilde{\varphi}^s \mid \tilde{\varphi} \in G\}$. Because $G^s \subseteqq |F| \cup \{\tilde{0}\}$, there is a greatest element $\tilde{\varphi}_0{}^s \in G^s$ with $\tilde{\varphi}_0 \in G$. The element $\tilde{\varphi}_0$ is greatest in G. We put

$$a_{\tilde{\varphi}} = \int (\tilde{\varphi} - \tilde{\varphi}^s)\, d\mu, \qquad \tilde{\varphi} \in |F|'\,.$$

Then $a_{\tilde{\varphi}} \neq 0$ and it follows from the foregoing discussion that $\sum_{\tilde{\varphi}} a_{\tilde{\varphi}} \leqq 1$. It is in fact not difficult to see that the sum equals 1. For let $\tilde{\varphi}_0$ be the infimum of all $\tilde{\varphi} \in |F| \cup \{\tilde{1}\}$ with $\tilde{\chi}_A = \tilde{\chi}_A \tilde{\varphi}$, where $A \subseteqq Z$ is a measurable set of positive measure. Then $\tilde{\chi}_A = \tilde{\chi}_A \tilde{\varphi}_0$ and not $\chi_A \leqq \chi_A \tilde{\varphi}_0{}^s$. This shows that $\sup_{\tilde{\varphi} \in |F|'} (\tilde{\varphi} - \tilde{\varphi}^s) = \tilde{1}$. We note that there can be at most countably many $a_{\tilde{\varphi}}$ and therefore the set $|F|'$ is denumerable.

Choose sets $A_{\tilde{\varphi}}$ such that $A_{\tilde{\varphi}} \cap A_{\tilde{\varphi}'} = \emptyset$ for $\tilde{\varphi} \neq \tilde{\varphi}'$, $\bigcup_{\tilde{\varphi} \in |F|'} A_{\tilde{\varphi}} = Z$ and with $\chi_{A_{\tilde{\varphi}}} = \tilde{\varphi} - \tilde{\varphi}^s$. We define $d_{\tilde{\varphi}}$ as the cardinal number of the set

$$\{\tilde{e}_\alpha \in F \mid |\tilde{e}_\alpha| \geqq \tilde{\varphi}\}$$

and take Hilbert spaces $H_{\tilde{\varphi}}$ with dimension $d_{\tilde{\varphi}}$. The sets $A_{\tilde{\varphi}}$ and the Hilbert spaces $H_{\tilde{\varphi}}$ determine a direct sum of constant families $\big((H'(z)), \Gamma'\big)$ with $H'(z) = H_{\tilde{\varphi}}$ if $z \in A_{\tilde{\varphi}}$ and $f \in \Gamma'$ iff $|f|^2$ is summable and $f|A_{\tilde{\varphi}}$ is a measurable map from $A_{\tilde{\varphi}}$ into $H_{\tilde{\varphi}}$.

The idea is to prove that this direct sum of constant families is a regularisation of the original family.

The maximality of F implies that for $\tilde{f} \in \tilde{\Gamma}$ with $(\tilde{f}, \tilde{e}_\alpha) = \tilde{0}$ for $\alpha \in I$ we have $\tilde{f} = \tilde{0}$. For a finite subset $B \subseteqq I$ and $f \in \tilde{\Gamma}$ we get

$$\|\breve{f}\|^2 \geqq \sum_{\alpha\in B}\|(\breve{f},\breve{e}_\alpha)\|^2$$

which shows that there are at most countably many $\alpha \in I$ with $(\breve{f},\breve{e}_\alpha) \neq \breve{0}$ and that $\sum_\alpha(\breve{f},\breve{e}_\alpha)\breve{e}_\alpha$ exists. We infer from the remark made just before that

$$\breve{f} = \sum_\alpha(\breve{f},\breve{e}_\alpha)\,\breve{e}_\alpha$$

and also

$$\breve{f} = \sum_{\breve{\varphi}\in|F|'}\sum_{\alpha\in I}(\breve{f},\breve{e}_\alpha)\breve{\chi}_{A_{\breve{\varphi}}}\breve{e}_\alpha\,.$$

If $I_{\breve{\varphi}}$ is the subset of I, which consists of the indices $\alpha \in I$ with $|\breve{e}_\alpha| \geqq \breve{\varphi}$, then $|\breve{\chi}_{A_{\breve{\varphi}}}\breve{e}_\alpha| = |\breve{\chi}_{A_{\breve{\varphi}}}|$ for $\alpha \in I_{\breve{\varphi}}$ and $\breve{\chi}_{A_{\breve{\varphi}}}\breve{e}_\alpha = \breve{0}$ for $\alpha \notin I_{\breve{\varphi}}$.

In each of the Hilbert spaces $\underbrace{H_{\breve{\varphi}}}$ we take an $\underbrace{\text{orthonormal}}$ base $\{\breve{v}_\alpha^{\varphi} \mid \alpha \in I_{\breve{\varphi}}\}$. We denote by $\underbrace{\breve{v}_\alpha^{\varphi}}$ the element of $\breve{\Gamma}'$ with $\underbrace{\breve{v}_\alpha^{\varphi}}(z) = \breve{v}_\alpha^{\varphi}$ for almost all $z \in A_{\breve{\varphi}}$ and 0 almost everywhere else. For $\breve{f} \in \breve{\Gamma}$ we put

$$U\!f = \sum_{\breve{\varphi}\in|F|'}\sum_{\alpha\in I_{\breve{\varphi}}}(\breve{f},\breve{e}_\alpha)\,\overbrace{\breve{v}_\alpha^{\varphi}}\,.$$

Then U is a unitary map from $\breve{\Gamma}$ onto $\breve{\Gamma}'$ which commutes with all diagonal operators. This ends the proof.

4.5. Corollary. *Let (Z,Σ,μ) be a direct sum of summable sets. Any measurable family on (Z,Σ,μ) admits a regularisation.*

Proof. We apply Proposition 4.4 to each of the summable sets which belong to the direct decomposition of Z and fit the resulting families together in the obvious way.

The corollary applies in particular to the case where Z is a locally compact space and μ a Radon measure.

Theorem 1.7 can also be derived from the well-known structure theorem for discrete von Neumann algebras, which states that every discrete von Neumann algebra is the product of homogeneous ones. The homogeneous von Neumann algebras are known [1, III. § 3].

For the discrete von Neumann algebra one must take the commutant $\mathfrak{Z}'$ of the commutative algebra $\mathfrak{Z}$. The proof of the above mentioned theorem is not spatial in nature and involves typical von Neumann algebra techniques.

5. Multiplicity theory.

Two operators A and B on a Hilbert space H are said to be (unitarily) equivalent if there exists a unitary operator U on H such that $UBU^{-1} = A$.

Two equivalent operators are geometrically indistinguishable. Our problem is to find a complete set of invariants for equivalence classes of normal operators on H.

There is a one-to-one correspondence between normal operators and spectral measures on the Borel subsets of the complex plane. Equivalent normal operators are associated with equivalent spectral measures.

In general a *spectral measure*, abbreviated s.m., on a measurable space (Z,Σ) is a map E, defined on Σ, with values in the set of orthogonal projections on a fixed Hilbert space H, so that $E(Z)=I$, the identity operator, and $E(\bigcup_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty} E(A_n)$, whenever $(A_n)_{n\in\mathrm{N}}$ is a sequence of mutually disjoint elements of Σ.

Two s.m. E and E' are said to be *equivalent* if there exists a unitary operator $U\colon H \to H'$, where H and H' are the underlying Hilbert spaces, such that $UE(A)=E'(A)U$ for all $A \in \Sigma$.

We remark that with every integrable family of Hilbert spaces there is associated a *natural spectral measure* (n.s.m.), that is, $\Sigma \ni A \mapsto T_{\hat{\chi}_A}$. It follows from 1.6 that two integrable families are equivalent iff their n.s.m. are equivalent.

Every discrete von Neumann algebra $\mathfrak{A}$ can be represented in such a way that its commutant $\mathfrak{Z}$ is commutative. Let Z be the spectrum of $\mathfrak{Z}$ in the hull-kernel topology and Σ the set of Borel subsets of Z. To every open and closed set $A \subseteq Z$ corresponds a unique projection $P_A \in \mathfrak{Z}$ so that $P_A(z)=1$ iff $z \in A$. The map $A \mapsto P_A$ defines a spectral measure on (Z,Σ). Two discrete von Neumann algebras are isomorphic iff their associated s.m. are equivalent.

We shall show how a relatively short proof can be given of the existence of a complete set of invariants for equivalent spectral measures. The proof is based on Theorem 1.7. For a different approach cf. P .Halmos [3, ch. III] and I. Segal [5].

We need some notation. As usual we write $\nu \ll \mu$ for two measures ν and μ on a measurable space (Z,Σ) if $\mu(A)=0$ implies $\nu(A)=0$ for $A \in \Sigma$. We indicate the situation $\nu \ll \mu$ and $\mu \ll \nu$ by writing $\mu \equiv \nu$. We shall make use of the fact that the partially ordered set F^+ of all finite positive measures on a measurable space (Z,Σ) is a boundedly complete lattice. The infimum of two measures ν and μ is denoted by $\nu \wedge \mu$. Two measures ν and μ are called *orthogonal*, $\nu \perp \mu$, if $\nu \wedge \mu$ is the zero measure 0. For any subset $A \in F^+$ we put

$$A' = \{\nu \in F^+ \mid \nu \perp \mu,\ \mu \in A\}.$$

A subset $A \subseteq F^+$ is called a *band* iff $A=A''$. The band $b(A)$ generated by $A \subseteq F^+$ is by definition $b(A)=A''$. Finally a *direct decomposition* of F^+

is a family $\{A_i\}_{i\in J}$ of mutually orthogonal bands such that any element $\nu\in F^+$ can be uniquely represented in the form $\nu=\sum_{i\in J}\nu_i$ where $\nu_i\in A_i$ and with order convergence.

To every spectral measure E we shall associate a direct decomposition $\{A_i\}_{i\in J}$ of F^+ where the index set J is a set of cardinal numbers. The invariant, as constructed here, is closely related to the multiplicity function of a s.m. introduced by Plessner and Rohlin [3, ch. III].

5.1. PROPOSITION. *To every spectral measure E on (Z,Σ) corresponds a unique direct decomposition $\{A_i\}_{i\in J}$ of F^+, where J is the set of cardinal numbers not exceeding the Hilbert dimension of H.*

PROOF. We start by constructing to each $\mu\in F^+$ an integrable family $\mathcal{E}_\mu$ of Hilbert spaces on (Z,Σ). Let $\mu\in F^+$. For $x\in H$ we define $\nu_x\in F^+$ by

$$\nu_x\colon\ \Sigma\ni A\mapsto (E(A)x,x)$$

and put $H_\mu=\{x \mid \nu_x\ll\mu\}$. Obviously H_μ is closed and invariant under E. The relation

$$\nu_{\alpha x+\beta y}(A)\ \leqq\ 2|\alpha|^2\nu_x(A)+2|\beta|^2\nu_y(A)$$

shows that H_μ is a linear subspace of H. There is a maximal set of unit vectors $\{e_\alpha\}_{\alpha\in T}$ in H so that $(E(A)e_\alpha,e_{\alpha'})=0$ for $A\in\Sigma$ and $\alpha\neq\alpha'$. We denote the Radon-Nikodym derivative of ν_{e_α} with respect to μ by $\tilde{f}_\alpha$. Then

$$\tilde{g}_\alpha=\tilde{f}_\alpha^{\frac12}\in L^2(Z,\mu)\,.$$

For

$$x\in H_\mu,\quad x=\sum_{j=1}^{n}a_jE(A_j)e_\alpha$$

we put

$$V_\mu x=\sum_{j=1}^{n}a_jT_{\tilde{\chi}_{A_j}}\tilde{g}_{\alpha_j}\tilde{e}_{\alpha_j}\,,$$

where the notation is as before. It is readily verified that V_μ is a well-defined, linear and isometric map from a dense subspace of H in $L^2{}_\mu(H,Z)$, which satisfies $V_\mu E(A)x=T_{\tilde{\chi}_A}V_\mu x$ for all $A\in\Sigma$, $x\in H$. The continuous extension of V_μ to H_μ is also denoted by V_μ.

We remark that $V_\mu H_\mu\subseteqq L_\mu{}^2(H,Z)$ is invariant with respect to the n.s.m. F_μ of $\mathcal{L}_\mu{}^2(H,Z)$. The subspace $V_\mu H_\mu$ corresponds to an integrable subfamily $\mathcal{E}_\mu$ of $\mathcal{L}_\mu{}^2(H,Z)$, which by 1.7 is equivalent to a direct sum of constant families $\{(Z_i,H_i)\}_{i\in J}$. The elements $\chi_{Z_i}\in L^1(Z,\mu)$ are uniquely determined by the condition $\dim H_i\neq\dim H_i{}'$ if $i\neq i'$.

Next we construct a direct decomposition of F^+. If for a cardinal number $d\leqq\dim H$, there is a $i\in J_\mu$ with $\dim H_i=d$, then we put $\mu_d=\tilde{\chi}_{Z_i}$

and $\mu_d = 0$ otherwise. The measures μ_d are mutually orthogonal, at most countably many differ from zero and $\mu = \sum_d \mu_d$.

Now suppose $\nu \neq 0$ and $\nu \ll \mu_d$ for some d and $\mu \in F^+$. We have $H_\nu \subseteq H_\mu$. The integrable family $\mathcal{E}_\nu$, associated with H_ν, is equivalent to the family $\mathcal{E}_\nu'$ associated with $V_\mu H_\nu$. There is a unique $\tilde{\chi} \in L'(Z,\mu)$ so that $\nu \equiv \tilde{\chi}\mu$. There follows that $\mathcal{E}_\nu'$ can be gotten out of $\mathcal{E}_\mu$ by multiplying the elements of $\mathcal{E}_\mu$ by χ. The family $\mathcal{E}_\nu'$ is thus a constant family of dimension d since $\chi \leqq \chi_{Z_i}$, where $\dim H_i = d$, the notation being as before. We obtain that $\mathcal{E}_{\nu'}$ is a regularisation of $\mathcal{E}_\nu$ and so $\nu = \nu_d$.

Finally we put $A_d = \{\mu_d \mid \mu \in F^+\}$ for every cardinal number $d \leqq \dim H$ and we show that $\{A_d \mid d \leqq \dim H\}$ is a direct decomposition of F^+.

Because every $\mu = \sum_d \mu_d$ with $\mu_d \in A_d$, it suffices to prove that the sets A_d are mutually orthogonal. Suppose $d \neq d'$ and $\nu = \mu_d \wedge \mu'_{d'}$ for some $\mu, \mu' \in F^+$. Since $\nu \ll \mu_d$ we get $\nu = \nu_d$ and similarly $\nu = \nu_{d'}$. But we also have that ν_d is orthogonal to $\nu_{d'}$ and thus

$$\nu = \nu_d \wedge \nu_{d'} = 0 = \mu_d \wedge \mu_{d'}' .$$

Therefore $\{A_d \mid d \leqq \dim H\}$ does the trick.

5.2. Proposition. *Equivalent spectral measures E and E' are associated with the same direct decomposition of F^+.*

Proof. If E and E' are equivalent, then the restrictions E_μ and E_μ' to the spaces H_μ and H_μ' are equivalent. Consequently the restrictions F_μ and F_μ' of the n.s.m. of $\mathscr{L}_\mu^2(H,Z)$ and $\mathscr{L}_\mu^2(H',Z)$ to $V_\mu H_\mu$ and $V_\mu H_\mu'$ are equivalent and thus $\mathcal{E}_\mu$ is equivalent to $\mathcal{E}_\mu'$. The decomposition of μ with respect to $\mathcal{E}_\mu$ is therefore equal to the decomposition with respect to $\mathcal{E}_\mu'$. The statement readily follows from this.

The next proposition shows that we in fact have a return ticket.

5.3. Proposition. *To every direct decomposition $\{A_d \mid d \leqq d_0\}$ of F^+ corresponds a spectral measure E such that the direct decomposition of F^+, which is associated with E, equals $\{A_d \mid d \leqq d_0\}$.*

Proof. For each $d \leqq d_0$ there is a maximal family $\{\mu_{\alpha,d} \mid \alpha \in J_d\}$ of mutually orthogonal elements in A_d. Suppose H_d is a Hilbert space of dimension d. We put

$$H = \sum_{d \leqq d_0} \sum_{\alpha \in J_d} L^2_{\mu_{\alpha,d}}(H_d, Z) .$$

To each $L^2_{\mu_{\alpha,d}}(H_d, Z)$ there is the n.s.m. $F_{\alpha,d}$. We consider

$$F = \sum_{d \leqq d_0} \sum_{\alpha \in J_d} F_{\alpha, d} \,.$$

Then F is a s.m. on (Z, Σ).

If $\{A_d' \mid d \leqq \dim H\}$ is the direct decomposition of F^+ associated with F, then clearly $\mu_{\alpha, d} \in A_d'$ for $\alpha \in J_d$ and also if ν is the supremum of measures equivalent to the measures $\mu_{\alpha, d}$, d fixed, then $\nu \in A_d'$. This implies $A_d \subseteqq A_d'$ for all $d \leqq d_0$. Because $\{A_d \mid d \leqq d_0\}$ and $\{A_d' \mid d \leqq \dim H\}$ are direct decompositions of F^+, this implies $A_d = Ad'$ for all $d \leqq d_0$.

Finally we have

5.4. Proposition. *If the* s.m. *E and E' are associated with the same direct decomposition, then E and E' are equivalent.*

Proof. Let E be a s.m. and $\{A_d \mid d \leqq \dim H\}$ the associated direct decomposition. We take $\{\mu_{\alpha, d} \mid \alpha \in J_d\}$ as in 3.3 and note that if $\mu \perp \nu$, then $H_\mu \perp H_\nu$, where H_μ and H_ν are as in 3.1. We get

$$H = \sum_d \sum_{\alpha \in J_d} H_{\mu_{\alpha, d}}, \qquad E = \sum_d \sum_{\alpha \in J_d} E_{\mu_{\alpha, d}} \,.$$

The s.m. $E_{\mu_{\alpha, d}}$ is the restriction of E to $H_{\mu_{\alpha, d}}$ and, as remarked in 3.3, it is equivalent to $F_{\mu_{\alpha, d}}$, the n.s.m. of $\mathscr{L}_\mu^2(H_d, Z)$ where $\dim H_d = d$. Thus E is equivalent to

$$F = \sum_d \sum_{\alpha \in J_d} F_{\mu_{\alpha, d}} \,,$$

which is the s.m. canonically associated with $\{A_d \mid d \leqq \dim H\}$.

If E' is another s.m. with the same decomposition of F^+ corresponding to it, then also E' is equivalent to F. This implies that E' is equivalent to E.

In total we have proved the following.

5.5. Theorem. *Let (Z, Σ) be a measurable space. The set of direct decompositions of F^+, which are indexed by the cardinal numbers, is a complete set of invariants for the spectral measures on (Z, Σ).*

The relation between the direct decompositions of F^+ associated with a s.m. E and the multiplicity function U_E of E is very simple [3, ch. III]. Let $\mu \in F^+$ and $\mu = \sum_d \mu_d$, then we put $U_E(\mu) = \inf\{d \mid \mu_d \neq 0\}$. It is readily verified that U_E thus defined is a multiplicity function and that U_E is the multiplicity function of E.

The reason why we have chosen direct decompositions rather than multiplicity functions is that it seems to us that direct decompositions are a little bit easier to handle and are maybe intuitively more clear.

It is obvious that the above multiplicity theory is to a large extent inspired by [3, ch. III].

In the case of a separable Hilbert space the theory simplifies considerably, since it is then easy to show the existence of a measure $\mu \in F^+$ so that $H = H_\mu$.

In general it is easy to see that for a spectral measure E one has

$$A_0 = \{\mu \mid H_\mu = 0\} = \{\nu_x \mid x \in H\}',$$

which shows that if E is simple, that is, $A_d = \emptyset$ for $d \geqq 2$, then $\{\nu_x \mid x \in H\}'' = A_1$ and $\{\nu_x \mid x \in H\}$ is a complete invariant for E. We note that E is simple iff the smallest weakly closed algebra in $L(H)$ which contains the operators $\{E(A) \mid A \in \sum\}$ is maximal abelian.

The fact that two simple spectral measures E and E' are equivalent iff the sets $\{\nu_x \mid x \in H\}$ and $\{\nu_x' \mid x \in H\}$ coincide is a theorem due to Wecken and to Plessner and Rohlin.

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