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THE REPRODUCING KERNEL OF $\mathscr{H}^2$ AND RADIAL EIGENFUNCTIONS OF THE HYPERBOLIC LAPLACIAN

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Abstract

In the paper we characterize the reproducing kernel $\mathscr{K}_{n,h}$ for the Hardy space $\mathscr{H}^2$ of hyperbolic harmonic functions on the unit ball $\mathbb{B}$ in $\mathbb{R}^n$. Specifically we prove that

$$\mathscr{K}_{n,h}(x, y) = \sum_{\alpha=0}^{\infty} S_{n,\alpha}(|x|)S_{n,\alpha}(|y|)Z_\alpha(x, y),$$

where the series converges absolutely and uniformly on $K \times \mathbb{B}$ for every compact subset K of $\mathbb{B}$. In the above, $S_{n,\alpha}$ is a hypergeometric function and Z_α is the reproducing kernel of the space of spherical harmonics of degree α. In the paper we prove that

$$0 \le \mathscr{K}_{n,h}(x, y) \le \frac{C_n}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{n-1}},$$

where C_n is a constant depending only on n. It is known that the diagonal function $\mathscr{K}_{n,h}(x, x)$ is a radial eigenfunction of the hyperbolic Laplacian Δ_h on $\mathbb{B}$ with eigenvalue $\lambda_2 = 8(n-1)^2$. The result for $n = 4$ provides motivation that leads to an explicit characterization of all radial eigenfunctions of Δ_h on $\mathbb{B}$. Specifically, if g is a radial eigenfunction of Δ_h with eigenvalue $\lambda_\alpha = 4(n-1)^2\alpha(\alpha-1)$, then

$$g(r) = g(0)\frac{p_{n,\alpha}(r^2)}{(1 - r^2)^{(\alpha-1)(n-1)}},$$

where $p_{n,\alpha}$ is again a hypergeometric function. If α is an integer, then $p_{n,\alpha}(r^2)$ is a polynomial of degree $2(\alpha-1)(n-1)$.

1. Introduction

Throughout the paper we follow the notation of [9] for hyperbolic harmonic functions on the unit ball $\mathbb{B}$ in $\mathbb{R}^n$, $n \ge 2$. Let ν denote Lebesgue measure on $\mathbb{R}^n$ normalized so that $\nu(\mathbb{B}) = 1$. Also, we denote by σ the surface measure on $\mathbb{S}$, the boundary of $\mathbb{B}$, again normalized such that $\sigma(\mathbb{S}) = 1$. The hyperbolic metric on $\mathbb{B}$ is given by

$$ds = 2(1 - |x|^2)^{-1}dx,$$

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and the Laplacian Δ_h with respect to the hyperbolic metric is given by

$$\Delta_h f = (1-|x|^2)\big[(1-|x|^2)\Delta f + 2(n-2)\langle x, \nabla f\rangle\big],$$

where Δ is the usual Laplacian in $\mathbb{R}^n$, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots \frac{\partial f}{\partial x_n}\right)$ is the Euclidean gradient of f, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. It is easily shown that Δ_h satisfies $\Delta_h f(a) = \Delta(f \circ \varphi_a)(0)$, where φ_a is a Möbius transformation of $\mathbb{R}^n$ mapping $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}}$ with $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a(\varphi_a(x)) = x$.

A continuous real-valued function f is $\mathscr{H}$-harmonic on $\mathbb{B}$ if and only if

$$f(a) = \int_{\mathbb{S}} f(\varphi_a(rt))\, d\sigma(t)$$

for all $a \in \mathbb{B}$ and all r with $0 < r < 1$. If this is the case, then f is C^2 on $\mathbb{B}$ and satisfies $\Delta_h f = 0$. For $1 \le p < \infty$ let $\mathscr{H}^p$ denote the Hardy space of $\mathscr{H}$-harmonic functions f for which

$$\|f\|_p^p = \sup_{0<r<1} \int_{\mathbb{S}} |f(rt)|^p \, d\sigma(t) < \infty.$$

The hyperbolic Poisson kernel $P_h(x, t)$ is given by

$$P_h(x,t) = P_{n,h}(x,t) = \frac{(1-|x|^2)^{n-1}}{|x-t|^{2(n-1)}}, \qquad (x,t) \in \mathbb{B} \times \mathbb{S}.$$

It is well known that if $f \in \mathscr{H}^p$, $1 < p < \infty$, then there exists a function $\hat{f} \in L^p(\mathbb{S})$, the boundary function of f, such that

$$f(x) = P_h\big[\hat{f}\,\big](x) = \int_{\mathbb{S}} P_h(x,t)\hat{f}(t)\, d\sigma(t)$$

with $\|f\|_p = \big\|\hat{f}\big\|_p$. When $p = 1$, the function f is the Poisson integral of a finite signed measure ν_f on $\mathbb{S}$ with $\|f\|_1 = |\nu_f|(\mathbb{S})$ where $|\nu_f|$ denotes the total variation of ν_f ([5], [8], [9, Theorem 7.1.1]). It is easily shown that for $f \in \mathscr{H}^p$, $1 \le p < \infty$, one has

$$|f(x)|^p \le \left(\frac{1+|x|}{1-|x|}\right)^{n-1} \|f\|_p^p.$$

Similar results hold for the space H^p, $1 \le p < \infty$, of Euclidean harmonic functions on $\mathbb{B}$ [2]. In the Euclidean case, the Poisson kernel $P_e(x,t)$ is given by

$$P_e(x,t) = P_{n,e}(x,t) = \frac{1-|x|^2}{|x-t|^n}.$$

In Section 2 we compute the reproducing kernel $\mathscr{K}_{n,h}$ of $\mathscr{H}^2$. For completeness, we also include the reproducing kernel K_e of the space H^2 of Euclidean harmonic functions. As we will see, the reproducing kernel K_e of H^2 is known and is obtained by expanding the domain of the Euclidean Poisson kernel [2, 8.11]. On the other hand, the reproducing kernel of $\mathscr{H}^2$ is non-trivial and is expressed in terms of a series of hypergeometric functions. As such, explicit formulas may be obtained only for even dimensions, and for dimensions 6 and higher, even those are non-trivial. We illustrate this in dimension 4. As we will see, the diagonal function $\mathscr{K}_{n,h}(x, x)$ is a radial eigenfunction of the hyperbolic Laplacian Δ_h with eigenvalue $\lambda_2 = 8(n-1)^2$. When $n = 4$,

$$\mathscr{K}_{4,h}(x, x) = \frac{1 + 6|x|^2 + 6|x|^4 + |x|^6}{(1 - |x|^2)^3}.$$

Using this as a motivation we compute all radial eigenfunctions of Δ_h in Section 4.

2. The reproducing kernel of $\mathscr{H}^2$

The space $\mathscr{H}^2$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g\rangle = \lim_{r\to 1} \int_{\mathbb{S}} f(rt)g(rt)\, d\sigma(t) = \int_{\mathbb{S}} \hat{f}(t)\hat{g}(t)\, d\sigma(t).$$

Furthermore, since point evaluation is a bounded linear functional, $\mathscr{H}^2$ has a reproducing kernel denoted by $\mathscr{K}_{n,h}(x, y)$, i.e.,

(1) for fixed $y \in \mathbb{B}$, the function $x \mapsto \mathscr{K}_{n,h}(x, y)$ is in $\mathscr{H}^2$, and

(2) for every $f \in \mathscr{H}^2$,
$$f(y) = \langle f, \mathscr{K}_{n,h}(\cdot, y)\rangle.$$

We begin with the following theorem, the proof of which is straightforward and most likely well-known in the Euclidean case.

THEOREM 2.1. *The reproducing kernel $\mathscr{K}_{n,h}(x, y)$ of $\mathscr{H}^2$ is given by*

$$\mathscr{K}_{n,h}(x, y) = \int_{\mathbb{S}} P_h(x, t)P_h(y, t)\, d\sigma(t).$$

PROOF. For $x \in \mathbb{B}$, set $K_x(y) = \mathscr{K}_{n,h}(x, y)$. If f is continuous on $\mathbb{S}$, then $P_h[f](x) \in \mathscr{H}^2$. By the Poisson integral formula

$$P_h[f](x) = \int_{\mathbb{S}} f(t)P_h(x, t)\, d\sigma(t).$$

On the other hand, by the reproducing property,

$$P_h[f](x) = \langle P_h[f], K_x\rangle = \int_{\mathbb{S}} \widehat{P_h[f]}(t)\hat{K}_x(t)\, d\sigma(t),$$

where $\hat{K}_x$ is the boundary function of K_x. Since f is continuous

$$P_h[f](x) = \int_{\mathbb{S}} f(t)\hat{K}_x(t)\, d\sigma(t).$$

Therefore,

$$\int_{\mathbb{S}} f(t)[P_h(x,t) - \hat{K}_x(t)]\, d\sigma(t) = 0.$$

Since this holds for all continuous functions f on $\mathbb{S}$ we have

$$\hat{K}_x(t) = P_h(x,t) \qquad \text{for a.e.} \quad t \in \mathbb{S}.$$

Hence

$$\mathscr{K}_{n,h}(x,y) = \langle K_x, K_y\rangle = \int_{\mathbb{S}} P_h(x,t)P_h(y,t)\, d\sigma(t).$$

Similarly, the reproducing kernel $K_e(x,y)$ of the space H^2 of Euclidean harmonic functions is given by

$$K_e(x,y) = \int_{\mathbb{S}} P_e(x,t)P_e(y,t)\, d\sigma(t).$$

Our next step is to provide explicit formulas for $K_e(x,y)$ and $\mathscr{K}_{n,h}(x,t)$. Although not identified as the reproducing kernel of H^2 the formula for $K_e(x,y)$ is given in [2, 8.11].

THEOREM 2.2. *For $x, y \in \mathbb{B}$,*

$$K_e(x,y) = \frac{1 - |x|^2|y|^2}{(1 - 2\langle x,y\rangle + |x|^2|y|^2)^{n/2}}.$$

Even though the result is known, we include the proof since much of the terminology and results concerning spherical harmonics are required in the sequel. (See [2] for details.)

For $m = 0, 1, 2, \ldots$, we denote by $\mathscr{H}_m(\mathbb{R}^n)$ the *homogeneous harmonic polynomials of degree m* on $\mathbb{R}^n$. A *spherical harmonic of degree m* is the restriction to $\mathbb{S}$ of a harmonic polynomial in $\mathscr{H}_m(\mathbb{R}^n)$. The collection of all spherical harmonic polynomials of degree m will be denoted by $\mathscr{H}_m(\mathbb{S})$. Every element of $\mathscr{H}_m(\mathbb{S})$ has a unique extension to $\mathscr{H}_m(\mathbb{R}^n)$. Furthermore, if $m \neq k$

then $\mathcal{H}_m(\mathbb{S})$ and $\mathcal{H}_k(\mathbb{S})$ are orthogonal in $L^2(\mathbb{S})$. If $\{p_{m,1}, \dots, p_{m,d_m}\}$ is an orthonormal basis of $\mathcal{H}_m(\mathbb{S})$, where $d_m = \dim \mathcal{H}_m(\mathbb{S})$, set

$$Z_m(\eta, \zeta) = \sum_{j=1}^{d_m} p_{m,j}(\eta) p_{m,j}(\zeta). \tag{2.1}$$

The function $Z_\eta^{(m)}(\zeta) = Z_m(\eta, \zeta)$ is called the *zonal harmonic of degree m* with pole η, and Z_m is the reproducing kernel of $\mathcal{H}_m(\mathbb{S})$, i.e., if $p \in \mathcal{H}_m(\mathbb{S})$, then

$$p(\eta) = \int_{\mathbb{S}} p(\zeta) Z_m(\eta, \zeta)\, d\sigma(\zeta).$$

Since every $p \in \mathcal{H}_m(\mathbb{S})$ has a unique extension to $\mathcal{H}_m(\mathbb{R}^n)$, the function $Z_\zeta^{(m)}$ has a unique extension to $\mathcal{H}_m(\mathbb{R}^n)$ which we denote by $x \to Z_m(x, \zeta)$. Suppose $x \in \mathbb{R}^n,\ x \neq 0$. Then if $p \in \mathcal{H}_m(\mathbb{R}^n)$,

$$\begin{aligned} p(x) = |x|^m p(x/|x|) = |x|^m \int_{\mathbb{S}} p(\zeta) Z_m(x/|x|, \zeta)\, d\sigma(\zeta) \\ = \int_{\mathbb{S}} p(\zeta) Z_m(x, \zeta)\, d\sigma(\zeta). \end{aligned}$$

By orthogonality,

$$\int_{\mathbb{S}} Z_m(x, \zeta) Z_k(y, \zeta)\, d\sigma(\zeta) = 0 \qquad \text{for} \quad k \neq m. \tag{2.2}$$

Furthermore,

$$\int_{\mathbb{S}} Z_m(x, \zeta) Z_m(y, \zeta)\, d\sigma(\zeta) = Z_m(x, y). \tag{2.3}$$

The proofs of (2.2) and (2.3) are classical for $|x| = |y| = 1$ and extend immediately to $\mathbb{B}$ by homogeneity.

PROOF OF THEOREM 2.2. By [2, Theorem 5.2.1] for $x \in \mathbb{B}$, $\zeta \in \mathbb{S}$,

$$P_e(x, \zeta) = \sum_{m=0}^{\infty} Z_m(x, \zeta),$$

where the series converges absolutely and uniformly on $K \times \mathbb{S}$ for every compact subset K of $\mathbb{B}$. Thus

$$K_e(x, y) = \int_{\mathbb{S}} P_e(x, \zeta) P_e(y, \zeta)\, d\sigma(\zeta),$$

which by orthogonality

$$\begin{aligned} &= \sum_{m=0}^{\infty} \int_{\mathbb{S}} Z_m(x, \zeta) Z_m(y, \zeta)\, d\sigma(\zeta) \\ &= \sum_{m=0}^{\infty} Z_m(x, y) = \sum_{m=0}^{\infty} Z_m(|y|x, y/|y|) \\ &= P_e(|y|x, y/|y|) = \frac{1 - |x|^2|y|^2}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{n/2}}, \end{aligned}$$

which proves Theorem 2.2.

In the following we prove a generalization of the result obtained in Theorem 2.2. This result will be needed in several examples as well as Theorem 3.1.

COROLLARY 2.3. *For $k = 0, 1, 2, \ldots,$*

$$\sum_{\alpha=0}^{\infty} \alpha^k Z_\alpha(x, y) = \frac{P_k(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{(n/2)+k}}, \tag{2.4}$$

where $P_k(x, y)$ is a polynomial in x and y.

PROOF. By Theorem 2.2 the result is true for $k = 0$. Assume the result is true for fixed $k \geq 0$, i.e.,

$$\sum_{\alpha=0}^{\infty} \alpha^k Z_\alpha(x, y) = \frac{P_k(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{(n/2)+k}},$$

where $P_k(x, y)$ is a polynomial in x and y. Then,

$$\begin{aligned} \sum_{\alpha=0}^{\infty} \alpha^{k+1} Z_\alpha(x, y) &= \sum_{\alpha=0}^{\infty} \frac{d}{dt}\big[\alpha^k Z_\alpha(tx, y)\big]_{t=1} = \frac{d}{dt}\left[\sum_{\alpha=0}^{\infty} \alpha^k Z_\alpha(tx, y)\right]_{t=1} \\ &= \frac{d}{dt}\left[\frac{P_k(tx, y)}{(1 - 2t\langle x, y\rangle + t^2|x|^2|y|^2)^{(n/2)+k}}\right]_{t=1} \\ &= \frac{P_{k+1}(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{(n/2)+k+1}}. \end{aligned}$$

EXAMPLE 2.4. Since we will need the results in Example 2.8, we compute the sum in (2.4) for $k = 1, 2$ and $n = 4$. When $k = 0$, by Theorem 2.2

$$\sum_{\alpha=0}^{\infty} Z_\alpha(x, y) = P_e(x, y) = \frac{1 - |x|^2|y|^2}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^2}.$$

Next, for $k = 1$,

$$\sum_{\alpha=0}^{\infty} \alpha Z_\alpha(x, y) = \frac{d}{dt}\left[\sum_{\alpha=0}^{\infty} t^\alpha Z_\alpha(x, y)\right]_{t=1} = \left[\frac{d}{dt} P_e(tx, y)\right]_{t=1}$$

$$= P_1(x, y) = \frac{2[2\langle x, y\rangle - 3|x|^2|y|^2 + |x|^4|y|^4]}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^3}.$$

Similarly, for $k = 2$,

$$\sum_{\alpha=0}^{\infty} \alpha^2 Z_\alpha(x, y) = \frac{d}{dt}\left[\sum_{\alpha=0}^{\infty} t^\alpha \alpha Z_\alpha(x, y)\right]_{t=1} = \left[\frac{d}{dt} P_1(tx, y)\right]_{t=1}$$

$$= \frac{4Q(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^4},$$

where

$$Q(x, y) = \langle x, y\rangle(1 - |x|^4|y|^4) + 4\langle x, y\rangle^2 - 8\langle x, y\rangle|x|^2|y|^2 - 3|x|^2|y|^2 + 8|x|^4|y|^4 - |x|^6|y|^6. \quad (2.5)$$

We now turn our attention to $\mathscr{K}_{n,h}(x, y)$. For this we need to introduce the *hypergeometric function* $F(a, b; c; z)$ ([1], [3], [6]) which for $c \notin \mathbb{C} \setminus \{0, -1, -2, \dots\}$ is defined by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad |z| < 1. \quad (2.6)$$

The hypergeometric function is the solution of the hypergeometric equation

$$z(1 - z)\frac{d^2 w}{dz^2} + [c - (a + b + 1)z]\frac{dw}{dz} - abw = 0, \quad (2.7)$$

that is continuous at 0.

In (2.6) $(a)_0 = 1$ and for $k = 1, 2, \dots$

$$(a)_k = a(a + 1) \cdots (a + k - 1).$$

If a is not a negative integer, then

$$(a)_k = \Gamma(a+k)/\Gamma(a),$$

where Γ is the Gamma function defined on $\mathbb{C}\backslash\{0,-1,-2,\ldots\}$. If $c-a-b>0$ then the series (2.6) converges absolutely for all z with $|z|\le 1$. Also, for $c-a-b>0$, by [1, Equation 15.1.20]

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.8}$$

For $\alpha = 0,1,2,\ldots$ and dimension $n\ge 2$, set

$$\begin{aligned} S_{n,\alpha}(r) &= \frac{F\big(\alpha, 1-\frac{1}{2}n; \alpha+\frac{1}{2}n; r^2\big)}{F\big(\alpha, 1-\frac{1}{2}n; \alpha+\frac{1}{2}n; 1\big)} \\ &= \frac{\Gamma\big(\frac{1}{2}n\big)\Gamma(\alpha+n-1)}{\Gamma(n-1)\Gamma\big(\alpha+\frac{1}{2}n\big)} F\big(\alpha, 1-\tfrac{1}{2}n; \alpha+\tfrac{1}{2}n; r^2\big) \\ &= \frac{\Gamma\big(\frac{1}{2}n\big)\Gamma(\alpha+n-1)}{\Gamma(n-1)\Gamma\big(\alpha+\frac{1}{2}n\big)} \sum_{k=0}^{\infty} \frac{(\alpha)_k\big(1-\frac{1}{2}n\big)_k}{\big(\alpha+\frac{1}{2}n\big)_k k!} r^{2k}. \end{aligned}$$

Then $S_{n,\alpha}(1)=1$ and by [4], [9, Section 6.1], for $p_\alpha\in\mathscr{H}_\alpha(\mathbb{R}^n)$, the function $f(x) = S_{n,\alpha}(|x|)p_\alpha(x)$ is a solution of $\Delta_h f(x) = 0$ that is continuous on $\overline{\mathbb{B}}$ with $f(\zeta) = p_\alpha(\zeta)$. By [9, Theorem 6.2.2] the hyperbolic Poisson kernel $P_h(x,t)$ is given by

$$P_h(x,t) = \sum_{\alpha=0}^{\infty} S_{n,\alpha}(|x|) Z_\alpha(x,t),$$

where the series converges absolutely and uniformly on $K\times\mathbb{S}$ for every compact subset K of $\mathbb{B}$. Thus by the orthogonality of $\{Z_\alpha(x,t)\}$ we obtain the following.

THEOREM 2.5. *For $x, y\in\mathbb{B}$,*

$$\mathscr{K}_{n,h}(x,y) = \sum_{\alpha=0}^{\infty} S_{n,\alpha}(|x|) S_{n,\alpha}(|y|) Z_\alpha(x,y)$$

where the series converges absolutely and uniformly on $K\times\mathbb{B}$ for every compact subset K of $\mathbb{B}$.

Prior to proving Theorem 2.5 we first prove the following lemma concerning the functions $S_{n,\alpha}(r)$.

LEMMA 2.6. *For all $n = 2, 3, \ldots, \alpha = 0, 1, 2, \ldots$ and $x \in \mathbb{B}$,*

$$\left|S_{n,\alpha}(|x|)\right| \le C_n \begin{cases} \alpha^{(n/2)-1}, & \text{if } n \text{ is even,} \\ \alpha^{[n/2]}, & \text{if } n \text{ is odd,} \end{cases}$$

where C_n is a constant depending only on n.

PROOF OF LEMMA 2.6. Let $m = [n/2]$, and set

$$P_m(r) = \sum_{k=0}^{m-1} \frac{(\alpha)_k(1-\frac{1}{2}n)_k}{(\alpha+\frac{1}{2}n)_k k!} r^{2k},$$

and

$$Q_m(r) = \sum_{k=m}^{\infty} \frac{(\alpha)_k(1-\frac{1}{2}n)_k}{(\alpha+\frac{1}{2}n)_k k!} r^{2k},$$

If n is even, then $\left(1-\frac{1}{2}n\right)_k = 0$ for all $k \ge n$ and thus $Q_m(r) \equiv 0$. Also, since $(\alpha)_k/\left(\alpha+\frac{1}{2}n\right)_k \le 1$,

$$|P_m(r)| \le \sum_{k=0}^{m-1} \frac{\left|\left(1-\frac{1}{2}n\right)_k\right|}{k!} = C_n,$$

where C_n is a constant depending only on n.

We now obtain an estimate for Q_m when n is odd. For $k \ge m$ we have $(\gamma)_k = (\gamma)_m(\gamma+m)_{k-m}$. Thus

$$\begin{aligned} Q_m(r) &= \sum_{k=m}^{\infty} \frac{(\alpha)_k\left(1-\frac{1}{2}n\right)_k}{\left(\alpha+\frac{1}{2}n\right)_k k!} r^{2k} \\ &= \frac{(\alpha)_m\left(1-\frac{1}{2}n\right)_m r^{2m}}{\left(\alpha+\frac{1}{2}n\right)_m} \sum_{j=0}^{\infty} \frac{(\alpha+m)_j\left(1+m-\frac{1}{2}n\right)_j}{\left(\alpha+m+\frac{1}{2}n\right)_j (m+j)!} r^{2j}. \end{aligned}$$

Since $(m+j)! \ge j!$ and $\left(1+m-\frac{1}{2}n\right) > 0$,

$$\begin{aligned} |Q_m(r)| \le \frac{\left|\left(1-\frac{1}{2}n\right)_m\right| \Gamma(\alpha+m)\Gamma\left(\alpha+\frac{1}{2}n\right)}{\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}n+m\right)} \\ \times F\left(\alpha+m, 1+m-\tfrac{1}{2}n; \alpha+m+\tfrac{1}{2}n; r^2\right). \end{aligned}$$

But $F\left(\alpha+m, 1+m-\frac{1}{2}n; \alpha+m+\frac{1}{2}n; r^2\right)$ is an increasing function of r. Thus by (2.8),

$$F\left(\alpha+m, 1+m-\tfrac{1}{2}n; \alpha+m+\tfrac{1}{2}n; r^2\right) \le \frac{\Gamma\left(\alpha+m+\frac{1}{2}n\right)\Gamma(n-1-m)}{\Gamma\left(\frac{1}{2}n\right)\Gamma(\alpha+n-1)}.$$

Therefore

$$|Q_m(r)| \le \frac{\left|\left(1-\frac{1}{2}n\right)_m\right|\Gamma(n-1-m)}{\Gamma\left(\frac{1}{2}n\right)} \frac{\Gamma(\alpha+m)\Gamma\left(\alpha+\frac{1}{2}n\right)}{\Gamma(\alpha)\Gamma(\alpha+n-1)}.$$

But $S_{n,\alpha}(r) = c_{n,\alpha}\left[P_m(r) + Q_m(r)\right]$ where

$$c_{n,\alpha} = \frac{\Gamma\left(\frac{1}{2}n\right)\Gamma(\alpha+n-1)}{\Gamma(n-1)\Gamma\left(\alpha+\frac{1}{2}n\right)}.$$

Thus

$$|S_{n,\alpha}(r)| \le C_n \frac{\Gamma(\alpha+n-1)}{\Gamma\left(\alpha+\frac{1}{2}n\right)} + D_n \frac{\Gamma\left(\alpha+\left[\frac{n}{2}\right]\right)}{\Gamma(\alpha)},$$

where C_n and D_n are constants depending only on n with $D_n = 0$ when n is even. Since

$$\lim_{\alpha\to\infty} \alpha^{b-a} \frac{\Gamma(\alpha+a)}{\Gamma(\alpha+b)} = 1,$$

we have

$$\frac{\Gamma(\alpha+a)}{\Gamma(\alpha+b)} \approx \alpha^{a-b}. \tag{2.9}$$

Thus $|S_{n,\alpha}(r)| \le C_n \alpha^{\frac{1}{2}n-1}$ if n is even and $|S_{n,\alpha}(r)| \le C_n \alpha^{[n/2]}$ if n is odd, which proves the lemma.

NOTE. In (2.9) $A(x) \approx B(x)$ means that there exist positive constants c_1 and c_2 such that $c_1 A(x) \le B(x) \le c_2 A(x)$ for all appropriate x.

PROOF OF THEOREM 2.5. By Theorem 2.6,

$$|\mathscr{K}_{n,h}(x,y)| \le C_n \sum_{\alpha=0}^{\infty} \alpha^{2[\frac{n}{2}]} |Z_\alpha(x,y)|.$$

But with $x = |x|\zeta$, $y = |y|\eta$, $\zeta, \eta \in \mathbb{S}$,

$$Z_\alpha(x,y) = |x|^\alpha |y|^\alpha Z_\alpha(\zeta,\eta).$$

But by [2, Equation 5.13] and Exercise 7 of [2, Chapter 5],

$$|Z_\alpha(\zeta,\eta)| = |\langle Z_\eta, Z_\zeta\rangle| \le \|Z_\eta\|_2 \|Z_\zeta\|_2 = d_\alpha \le C_n \alpha^{n-2}.$$

Therefore,

$$|\mathscr{K}_{n,h}(x,y)| \le C_n \sum_{\alpha=0}^{\infty} \alpha^p |x|^\alpha |y|^\alpha,$$

where $p = 2[\frac{n}{2}] + n - 2$. The above series converges uniformly for $(x, y) \in K \times \mathbb{B}$ where $K \subset \mathbb{B}$ is compact, which proves Theorem 2.5.

By (2.1),

$$Z_\alpha(x, y) = \sum_{j=1}^{d_\alpha} p_{\alpha,j}(x) p_{\alpha,j}(y),$$

where $\{p_{\alpha,j} : j = 1, \ldots, d_\alpha\}$ is an orthonormal basis of $\mathscr{H}_\alpha(\mathbb{S})$. As a consequence we obtain the following.

COROLLARY 2.7. $\{S_{n,\alpha}(|x|) p_{\alpha,j}(x) : j = 1, \ldots, d_\alpha\}_{\alpha=0}^{\infty}$ *is an orthonormal basis for the space* $\mathscr{H}^2$ *of* $\mathscr{H}$*-harmonic functions on* $\mathbb{B}$.

When n is even, say $n = 2m$, then $b = 1 - m$ and thus $(b)_k = 0$ for all $k \geq m$. Hence $S_{n,\alpha}(r)$ is a polynomial of degree $n-2$. When $n = 2$, $\mathscr{H}^2 = H^2$ and thus $\mathscr{K}_{2,h} = K_{2,e}$. In the following example we compute $\mathscr{K}_{4,h}$.

EXAMPLE 2.8. When $n = 4$,

$$S_{4,\alpha}(r) = \frac{1}{2}\big(2 + \alpha(1 - r^2)\big).$$

Therefore

$$\mathscr{K}_{4,h}(x, y) \\ = \frac{1}{4}\sum_{\alpha=0}^{\infty}\big[4 + 2\alpha[(1 - |x|^2) + (1 - |y|^2)] + \alpha^2(1 - |x|^2)(1 - |y|^2)\big] Z_\alpha(x, y).$$

By the results of Example 2.4

$$\begin{aligned}\mathscr{K}_{4,h}(x, y) = {} & \frac{1 - |x|^2|y|^2}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^2} \\ & + [(1 - |x|^2) + (1 - |y|^2)]\frac{[2\langle x, y\rangle - 3|x|^2|y|^2 + |x|^4|y|^4]}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^3} \\ & + (1 - |x|^2)(1 - |y|^2)\frac{Q(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^4},\end{aligned} \tag{2.10}$$

where $Q(x, y)$ is given by (2.5). By combining the above terms one has

$$\mathscr{K}_{4,h}(x, y) = \frac{Q_4(x, y)}{(1 - 2\langle x, y, \rangle + |x|^2|y|^2)^4},$$

where $Q_4(x, y)$ is a polynomial in x and y. In Theorem 3.1 we obtain an analogous representation of $\mathscr{K}_{n,h}(x, y)$ for all even n. Furthermore, since

$(1-|x|^2)(1-|y|^2) \leq 4(1-|x||y|)^2$ and $(1-|x|^2)+(1-|y|^2) \leq 4(1-|x||y|)$ we have

$$\mathscr{K}_{4,h}(x, y) \leq \frac{1-|x|^2|y|^2}{(1-2\langle x, y\rangle + |x|^2|y|^2)^2} + \frac{4(1-|x||y|)|Q_1(x, y)| + 4|Q(x, y)|}{(1-2\langle x, y\rangle + |x|^2|y|^2)^3},$$

where $Q(x, y)$ is given by (2.5). Therefore

$$\mathscr{K}_{4,h}(x, y) \leq \frac{C_n}{(1-2\langle x, y\rangle + |x|^2|y|^2)^3}.$$

In Theorem 3.2 we will prove that for all $n = 2, 3, \ldots,$

$$\mathscr{K}_{n,h}(x, y) \leq \frac{C_n}{(1-2\langle x, y\rangle + |x|^2|y|^2)^{n-1}}.$$

From (2.10) it immediately follows that for $\eta \in \mathbb{S}$,

$$\lim_{y \to \eta} \mathscr{K}_{4,h}(x, y) = \frac{1-|x|^2}{|x-\eta|^4} + \frac{(1-|x|^2)[2\langle x, \eta\rangle + |x|^4 - 3|x|^2]}{|x-\eta|^6} = \frac{(1-|x|^2)^3}{|x-\eta|^6} = P_h(x, \eta).$$

Also,

$$\mathscr{K}_{4,h}(x, x) = \frac{1+6|x|^2+6|x|^4+|x|^6}{(1-|x|^2)^3}.$$

Since

$$\mathscr{K}_{n,h}(x, x) = \int_{\mathbb{S}} P_h^2(x, t)\, d\sigma(t),$$

by [9, Theorem 5.5.2] $\mathscr{K}_{n,h}(x, x)$ is a radial eigenfunction of Δ_h with eigenvalue $\lambda_2 = 8(n-1)^2$. Using the result for $\mathscr{K}_{4,h}$ as a motivation, we compute all radial eigenfunctions of Δ_h in Section 4.

3. Properties of $\mathscr{K}_{n,h}(x, y)$

In this section we prove several results concerning the function $\mathscr{K}_{n,h}(x, y)$. In the following theorem we obtain a representation for $\mathscr{K}_{n,h}(x, y)$ valid for all even integers n.

THEOREM 3.1. *Let $n \geq 2$ be even. Then*

$$\mathcal{K}_{n,h}(x, y) = \frac{Q_n(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{(3n/2)-2}},$$

where $Q_n(x, y)$ is a polynomial in x and y.

PROOF. Since we have already proved the result for $n = 2$ and 4, we assume $n \geq 6$. Suppose $n = 2m$ where $m \geq 3$. Then

$$F(\alpha, 1 - m; \alpha + m; r^2) = \sum_{k=0}^{m-1} \frac{(\alpha)_k (1 - m)_k}{(\alpha + m)_k\, k!} r^{2k}.$$

Since $c - a - b > 0$ where $a = \alpha$, $b = 1 - m$, $c = \alpha + m$, we have

$$F(\alpha, 1 - m; \alpha + m; 1) = \frac{\Gamma(2m - 1)\,\Gamma(\alpha + m)}{\Gamma(m)\,\Gamma(\alpha + 2m - 1)}.$$

Hence

$$S_{n,\alpha}(r) = \frac{\Gamma(m)}{\Gamma(2m - 1)} \sum_{k=0}^{m-1} \frac{\Gamma(\alpha + 2m - 1)\,(\alpha)_k}{\Gamma(\alpha + m)\,(\alpha + m)_k} \frac{(1 - m)_k}{k!} r^{2k}.$$

But $\Gamma(\alpha + 2m - 1) = (\alpha + m)_{m-1}\,\Gamma(\alpha + m)$. Therefore

$$S_{n,\alpha}(r) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n - 1)} \sum_{k=0}^{m-1} \frac{(\alpha + m)_{m-1}(\alpha)_k}{(\alpha + m)_k} \frac{(1 - m)_k}{k!} r^{2k}.$$

But for $k = 0, 1, \ldots, m - 1$, $(\alpha + m)_k$ divides $(\alpha + m)_{m-1}$ and thus

$$\frac{(\alpha + m)_{m-1}(\alpha)_k}{(\alpha + m)_k}$$

is a polynomial in α of degree $m - 1$. Grouping terms in like powers of α gives

$$S_{n,\alpha}(r) = \sum_{j=0}^{m-1} a_j\, p_j(r^2)\, \alpha^j,$$

where p_j is a polynomial in r^2 of degree less than or equal to $m - 1$. Therefore

$$\begin{aligned}\mathcal{K}_{n,h}(x, y) &= \sum_{\alpha=0}^{\infty} \sum_{i,j=0}^{m-1} a_i a_j\, p_i(|x|^2) p_j(|y|^2) \alpha^{i+j} Z_\alpha(x, y)\\ &= \sum_{k=0}^{2m-2} c_k\, q_k(|x|^2, |y|^2) \sum_{\alpha=0}^{\infty} \alpha^k Z_\alpha(x, y).\end{aligned}$$

But by Corollary 2.3,

$$\sum_{\alpha=0}^{\infty} \alpha^k Z_\alpha(x, y) = \frac{P_k(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{m+k}}$$

for all $k = 0, 1, \ldots, 2m - 2$, where $P_k(x, y)$ is a polynomial in x and y. Therefore,

$$\begin{aligned}\mathscr{K}_{n,h}(x, y) &= \sum_{k=0}^{2m-2} c_k q_k(|x|^2, |y|^2) \frac{P_k(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{m+k}} \\ &= (1 - 2\langle x, y\rangle + |x|^2|y|^2)^{-3m+2} \\ &\quad \times \sum_{k=0}^{2m-2} c_k q_k(|x|^2, |y|^2)(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{2m-2-k} P_k(x, y),\end{aligned}$$

which with $m = n/2$

$$= \frac{Q_n(x, y)}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{(3n/2)-2}},$$

where Q_n is a polynomial in x and y.

Our next result is an upper bound on $\mathscr{K}_{n,h}$ valid for all integers n.

THEOREM 3.2. *For all $n = 2, 3, \ldots,$ and $x, y \in \mathbb{B}$,*

$$\mathscr{K}_{n,h}(x, y) \le \frac{2^{n+1}}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^{n-1}}.$$

where C_n is a constant depending only on n.

PROOF. We first note that

$$\begin{aligned}\left| |y|x - \frac{y}{|y|} \right|^2 &= 1 - 2\langle x, y\rangle + |x|^2|y|^2 \\ &\le \frac{1}{2}\big[2 - 4\langle x, y\rangle + |x|^2 + |y|^2\big] \\ &= \frac{1}{2}\big[|x - t|^2 + |y - t|^2 + 2\langle x, t - y\rangle + 2\langle y, t - x\rangle\big] \\ &\le \frac{1}{2}\big[|x - t|^2 + |y - t|^2 + 2|y - t| + 2|x - t|\big].\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{|x-t|^2|y-t|^2}\\&\le \frac{1}{2\big||y|x-\frac{y}{|y|}\big|^2}\left[\frac{1}{|y-t|^2}+\frac{1}{|x-t|^2}+\frac{2}{|x-t|^2|y-t|}+\frac{2}{|y-t|^2|x-t|}\right].\end{aligned}$$

Since $|y-t|\ge 1-|y|$ and $|x-t|\ge 1-|x|$ we have

$$\frac{1}{|x-t|^2|y-t|^2}\le \frac{1}{\big||y|x-\frac{y}{|y|}\big|^2}\left[\frac{2}{|y-t|^2(1-|x|^2)}+\frac{2}{|x-t|^2(1-|y|^2)}\right].$$

Therefore,

$$\begin{aligned}&\frac{1}{(|x-t|^2|y-t|^2)^{(n-1)}}\\&\qquad\le \frac{2^n}{\big||y|x-\frac{y}{|y|}\big|^{2(n-1)}}\left[\frac{(1-|x|^2)^{-(n-1)}}{|y-t|^{2(n-1)}}+\frac{(1-|y|^2)^{-(n-1)}}{|x-t|^{2(n-1)}}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{K}_{n,h}(x,y)&=\int_{\mathbb{S}}\frac{((1-|x|^2)(1-|y|^2))^{n-1}}{(|x-t|^2|y-t|^2)^{(n-1)}}\,d\sigma(t)\\&\le \frac{2^n}{\big||y|x-\frac{y}{|y|}\big|^{2(n-1)}}\\&\qquad\times\left[\int_{\mathbb{S}}\frac{(1-|y|^2)^{n-1}}{|y-t|^{2(n-1)}}\,d\sigma(t)+\int_{\mathbb{S}}\frac{(1-|x|^2)^{n-1}}{|x-t|^{2(n-1)}}\,d\sigma(t)\right]\\&\le \frac{2^{n+1}}{(1-2\langle x,y\rangle+|x|^2|y|^2)^{n-1}},\end{aligned}$$

which proves the result.

As a consequence of the previous theorem we have the following:

COROLLARY 3.3. *For all $n=2,3,\ldots,x,y\in\mathbb{B}$,*

$$\mathscr{K}_{n,h}(x,y)\le \frac{2^{n+1}}{(1-|x||y|)^{2n-2}}.$$

As we will see, when n is even we can improve on the above. When $n=2$,

$$\mathscr{K}_{2,h}(x,y)=\frac{1-|x|^2|y|^2}{(1-2\langle x,y\rangle+|x|^2|y|^2)}\le\frac{2(1-|x||y|)}{(1-|x||y|)^2}\le\frac{2}{(1-|x||y|)}.$$

When $n = 4$, if we write $Q(x, y)$ in (2.5) as

$$Q(x, y) = \langle x, y\rangle(1 - |x|^4|y|^4) + 4\big[\langle x, y\rangle^2 - 2\langle x, y\rangle|x|^2|y|^2 + |x|^4|y|^4\big] \\ - 4\left[|x|^2|y|^2 - |x|^4|y|^4\right] + |x|^2|y|^2 - |x|^6|y|^6,$$

we have

$$|Q(x, y)| \le 8(1 - |x|^2|y|^2) + (1 - 2\langle x.y\rangle + |x|^2|y|^2).$$

Therefore,

$$\begin{aligned}
&\mathcal{K}_{4,h}(x, y) \\
&\quad \le \frac{2(1 - |x||y|)}{(1 - |x||y|)^4} + \frac{4(1 - |x||y|)|Q_1(x, y)|}{(1 - |x||y|)^6} + \frac{4|Q(x, y)|}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^3} \\
&\quad \le \frac{2}{(1 - |x||y|)^3} + \frac{4|Q_1(x, y)|}{(1 - |x||y|)^5} \\
&\qquad + \frac{64}{(1 - |x||y|)^5} + \frac{4}{(1 - 2\langle x, y\rangle + |x|^2|y|^2)^2} \\
&\quad \le \frac{C_4}{(1 - |x||y|)^5}.
\end{aligned}$$

Thus for the special cases $n = 2, 4$,

$$\mathcal{K}_{n,h}(x, y) \le \frac{C_n}{(1 - |x||y|)^{2n-3}}.$$

We now prove that this is the case for all even integers n.

THEOREM 3.4. *If n is even, then*

$$\mathcal{K}_{n,h}(x, y) \le \frac{C_n}{(1 - |x||y|)^{2n-3}}.$$

PROOF. By Lemma 2.6, since n is even,

$$\mathcal{K}_{n,h}(x, y) \le C_n \sum_{\alpha=0}^{\infty} \alpha^{n-2}|Z_\alpha(x, y)|.$$

But as in the proof of Theorem 2.5, $|Z_\alpha(x, y)| \le C_n |x|^\alpha |y|^\alpha \alpha^{n-2}$. Therefore,

$$\mathscr{K}_{n,h}(x, y) \le C_n \sum_{\alpha=0}^{\infty} \alpha^{2n-4} |x|^\alpha |y|^\alpha,$$

which by (2.9)

$$\le C_n \sum_{\alpha=0}^{\infty} \frac{\Gamma(\alpha + 2n - 3)}{\Gamma(\alpha + 1)} |x|^\alpha |y|^\alpha$$

$$= \frac{C_n}{(1 - |x||y|)^{2n-3}}.$$

REMARKS 3.5. (a) A similar proof using Lemma 2.6 for odd n yields no improvement on Corollary 3.3.

(b) As a consequence of Theorem 3.4, for even n, one has

$$\mathscr{K}_{n,h}(x, x) \le \frac{C_n}{(1 - |x|^2)^{2n-3}},$$

with an analogous result for odd n. However,

$$\|\mathscr{K}_{n,h}(x, \cdot)\|_2^2 = \mathscr{K}_{n,h}(x, x) = (1 - |x|^2)^{2(n-1)} \int_{\mathbb{S}} \frac{d\sigma(t)}{|x - t|^{4(n-1)}},$$

which by [9, Theorem 5.5.7]

$$\le \frac{C_n}{(1 - |x|^2)^{n-1}}$$

for all n. An explicit formula for $\mathscr{K}_{n,h}(x, x)$ will be derived in the next section.

4. Radial Eigenfunctions of Δ_h

Eigenfunctions of the invariant Laplacian on real hyperbolic spaces were initially investigated by K. Minemura in [7]. In this section we provide a characterization of the radial eigenfunctions of the hyperbolic Laplacian. For $\alpha \in \mathbb{R}$ set

$$g_{n,\alpha}(x) = \int_{\mathbb{S}} P_h^\alpha(x, t)\, d\sigma(t). \tag{4.1}$$

Then by [9, Theorem 5.5.2] $g_{n,\alpha}$ is a radial eigenfunction of Δ_h with eigenvalue λ_α given by

$$\lambda_\alpha = 4(n-1)^2 \alpha(\alpha - 1).$$

Furthermore, if f is a radial eigenfunction of Δ_h with eigenvalue λ_α, then by [9, Theorem 5.5.5] $f(x) = f(0) g_{n,\alpha}(x)$. As a consequence one has $g_{n,\alpha} = g_{n,1-\alpha}$.

Also, by [9, Corollary 5.5.8]

$$g_{n,\alpha}(x) \approx \begin{cases} (1-|x|^2)^{\alpha(n-1)}, & \text{if } \alpha < \frac{1}{2}, \\ (1-|x|^2)^{\frac{1}{2}(n-1)} \log \frac{1}{(1-|x|^2)}, & \text{if } \alpha = \frac{1}{2}, \\ (1-|x|^2)^{(1-\alpha)(n-1)} & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

In the previous section we obtained that

$$g_{4,2}(x) = \mathscr{K}_{4,h}(x,x) = \frac{1+6|x|^2+6|x|^4+|x|^6}{(1-|x|^2)^3}.$$

In this section we prove the following.

THEOREM 4.1. *For* $\alpha \geq \frac{1}{2}$,

$$g_{n,\alpha}(r) = \frac{p_{n,\alpha}(r^2)}{(1-r^2)^{(\alpha-1)(n-1)}}$$

where

$$\begin{aligned} p_{n,\alpha}(r^2) &= F\big((1-\alpha)(n-1), \tfrac{n}{2}-\alpha(n-1); \tfrac{n}{2}; r^2\big) \\ &= \sum_{k=0}^{\infty} \frac{\big((1-\alpha)(n-1)\big)_k \big(\frac{n}{2}-\alpha(n-1)\big)_k}{\big(\frac{n}{2}\big)_k k!} r^{2k}. \end{aligned} \tag{4.2}$$

REMARKS 4.2. (a) For $\alpha < \frac{1}{2}$ we use the fact that $g_{n,\alpha} = g_{n,1-\alpha}$.

(b) If $\alpha > \frac{1}{2}$, then $c-a-b = (2\alpha-1)(n-1) > 0$ and the series (4.2) converges absolutely for $|r| \leq 1$.

(c) If α is an integer, then $p_{n,\alpha}(r^2)$ is a polynomial of degree $2(\alpha-1)(n-1)$.

PROOF. Since $g_{n,\alpha}(x) \approx (1-|x|^2)^{(1-\alpha)(n-1)}$ for $\alpha > \frac{1}{2}$, we assume

$$g_{n,\alpha}(x) = \frac{p(|x|^2)}{(1-|x|^2)^{(\alpha-1)(n-1)}} = \frac{p(|x|^2)}{(1-|x|^2)^{\beta}},$$

where $\beta = (\alpha-1)(n-1)$. In terms of β, the eigenvalue $\lambda_\alpha = 4[\beta(\beta+1) + (n-2)\beta]$. Set

$$u(t) = \frac{p(t)}{(1-t)^{\beta}}.$$

Then

$$\begin{aligned}(1-t)^{\beta+1}u'(t) &= (1-t)p'(t)+\beta p(t),\\ (1-t)^{\beta+2}u''(t) &= (\beta+1)(1-t)^{\beta+1}u'(t)+(1-t)^2p''(t)+(\beta-1)p'(t)\\ &= (1-t)^2p''(t)+2\beta(1-t)p'(t)+\beta(\beta+1)p(t).\end{aligned} \tag{4.3}$$

Since $g_{n,\alpha}$ is a radial function ([8, 2.1.7], [9, 3.1.4]),

$$\begin{aligned}&(1-r^2)^\beta\Delta_h g_{n,\alpha}(x)\\ &\quad=(1-r^2)^{\beta+2}g''_{n,\alpha}(r)\\ &\qquad+(1-r^2)^{\beta+1}\frac{g'_{n,\alpha}(r)}{r}\left\{(n-1)(1-r^2)+2(n-2)r^2\right\},\end{aligned}$$

which since $g_{n,\alpha}(r)=u(r^2)$

$$=4r^2(1-r^2)^{\beta+2}u''(r^2)+(1-r^2)^{\beta+1}u'(r^2)[2n(1-r^2)+4(n-2)r^2].$$

Replacing r^2 by t and using equations (4.3) above gives

$$\begin{aligned}&(1-t)^\beta\Delta_h g_{n,\alpha}\\ &\quad=4t(1-t)^{\beta+2}u''(t)+(1-t)^{\beta+1}u'(t)[2n(1-t)+4(n-2)t]\\ &\quad=4t(1-t)^2p''(t)+[8\beta t+2n(1-t)+4(n-2)t](1-t)p'(t)\\ &\qquad\quad+[4\beta(\beta+1)t+2n\beta(1-t)+4\beta(n-2)t]p(t).\end{aligned}$$

Now, using the fact that $(1-t)^\beta\Delta_h g_{n,\alpha}=4[\beta(\beta+1)+(n-2)\beta]p$, we obtain

$$\begin{aligned}4t(1-t)^2p''(t)+\big[2n-4\big(-\tfrac12 n-2\beta+2\big)t\big](1-t)p'(t)&\\ +4\beta\big[-\tfrac12 n-\beta+1\big](1-t)p(t)&=0\end{aligned}$$

Dividing by $4(1-t)$ yields

$$t(1-t)p''(t)+\big[\tfrac12 n-\big(2-2\beta-\tfrac12 n\big)t\big]p'(t)+\beta\big[-\tfrac12 n-\beta+1\big]p(t)=0.$$

This however is the hypergeometric equation (2.7) with

$$a=-\beta=(1-\alpha)(n-1),\quad b=-\tfrac12 n-\beta+1=\tfrac12 n-\alpha(n-1),\quad\text{and}\quad c=\tfrac12 n,$$

for which the solution that is continuous at 0 is given by

$$F(a,b;c;z) = F\big((1-\alpha)(n-1), \tfrac{n}{2}-\alpha(n-1); \tfrac{n}{2}; z\big)$$
$$= \sum_{k=0}^{\infty} \frac{\big((1-\alpha)(n-1)\big)_k \big(\tfrac{n}{2}-\alpha(n-1)\big)_k}{\big(\tfrac{n}{2}\big)_k\, k!}\, z^k,$$

from which the result follows.

EXAMPLES 4.3. (a) When $\alpha = 2$, as above,

$$g_{4,2}(r) = \frac{1+6r^2+6r^4+r^6}{(1-r^2)^3},$$

whereas

$$g_{3,2}(r) = \frac{1+\frac{10}{3}r^2+r^4}{(1-r^2)^2}$$

and

$$g_{5,2}(r) = \frac{1+\frac{44}{5}r^2+\frac{594}{35}r^4+\frac{44}{5}r^6+r^8}{(1-r^2)^4}.$$

(b) When $n = 4$ and $\alpha = 3$,

$$g_{4,3}(r) = \frac{1+21r^2+105r^4+175r^6+105r^8+21r^{10}+r^{12}}{(1-r^2)^6}.$$

(c) When $\alpha = \frac{1}{2}$

$$g_{n,\frac{1}{2}}(r) = (1-r^2)^{\frac{1}{2}(n-1)} \frac{\Gamma\big(\frac{n}{2}\big)}{\Gamma\big(\frac{1}{2}(n-1)\big)\Gamma\big(\frac{1}{2}\big)} \sum_{k=0}^{\infty} \frac{\Gamma\big(\frac{1}{2}(n-1)+k\big)\Gamma\big(\frac{1}{2}+k\big)}{\Gamma\big(\frac{n}{2}+k\big)\Gamma(k+1)}\, r^{2k},$$

which since $\Gamma(k+a)/\Gamma(k+b) \approx k^{a-b}$

$$\approx C_n(1-r^2)^{\frac{1}{2}(n-1)}\left[1+\sum_{k=1}^{\infty}\frac{r^{2k}}{k}\right]$$

$$= C_n(1-r^2)^{\frac{1}{2}(n-1)}\left[1+\log\frac{1}{(1-r^2)}\right]$$

$$\approx C_n(1-r^2)^{\frac{1}{2}(n-1)}\log\frac{1}{(1-r^2)}.$$

APPLICATION 4.4. The above results can be used in the evaluation of certain integrals in $\mathbb{R}^n$. As an example, by (4.1) and Theorem 4.1,

$$\int_{\mathbb{S}} \frac{d\sigma(t)}{|x-t|^{2\alpha(n-1)}} = \frac{g_{n,\alpha}(x)}{(1-|x|^2)^{\alpha(n-1)}} = \frac{p_{n,\alpha}(|x|^2)}{(1-|x|^2)^{(2\alpha-1)(n-1)}}.$$

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