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DIRECT INTEGRALS OF HILBERT SPACES II

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Introduction, notation and preliminaries.

The theory of reduction of von Neumann algebras has been effectuated only for algebras $\mathfrak{A}$, which are generated by their centre $\mathfrak{Z}$ and a countable family of other elements. In this paper we hope to show, using the existence of liftings on finite measure spaces, how several of the basic results of this theory can be extended to the general case.

In the second section we shall prove that the commutant $\mathfrak{Z}'$ of the set $\mathfrak{Z}$ of diagonal operators on a constant field coincides with the set of decomposable operators (cf. [1]). The decomposition of $\mathfrak{Z}'$ will be linear and positive and it is proved that it can not be chosen so as to be multiplicative as well.

In the third section, it is shown that any commutative von Neumann algebra $\mathfrak{Z}$ is spatially isomorphic to the algebra of diagonal operators on a suitable integral of Hilbert spaces, which can be chosen so as to make a multiplicative decomposition of $\mathfrak{Z}'$ possible. The approach here is similar to the one used by I. Segal in [10].

In the last section the results of the second section are applied to obtain a partial solution of a problem posed by S. Sakai [9].

After this paper was written E. T. Kehlet informed us that Miss O. Maréchal [5] had obtained Theorem 2.1. using the same method as we do.

Added in proof.

Several more articles on reduction theory should be mentioned. Apart from [10] also in [12], [13], and [14] a decomposition of operator algebras on not necessarily separable Hilbert spaces was obtained. In these articles direct integrals on perfect measure spaces were considered. It is well known that for such spaces a natural lifting exists and the same reasoning as used in the present note can be applied. We are indebted to H. Leptin for pointing out to us the relevance of [12], [13] and [14].

It is interesting to remark that Proposition 2.5 on the impossibility of

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a multiplicative decomposition (cf. also [14, § 11]) can be derived (and generalized) from the result of Feldman and Fell [15] and Takesaki [16], which states that a separable representation of a von Neumann algebra without a direct summand of finite type I is necessarily σ-continuous.

Although this paper can be read quite independently of the foregoing one [11], we shall, in order to avoid duplication, refer to the introduction of it for most of the notation used here. Also the definition of an integrable field of Hilbert spaces is as in [11]. We need some additional information though.

Thus let (Z,Σ,μ) be a complete measure space. A linear lifting of $L^\infty(Z,\mu)$ is a linear positive map $\varrho: L^\infty(Z,\mu) \to M^\infty(Z,\mu)$ satisfying $\varrho(\tilde{1})=1$ and

$$\widetilde{\varrho(\tilde{\varphi})} = \tilde{\varphi} \quad \text{for} \quad \text{all } \tilde{\varphi} \in L^\infty(Z,\mu) .$$

A lifting of $L^\infty(Z,\mu)$ is a linear lifting ϱ of $L^\infty(Z,\mu)$, which is also multiplicative. A theorem of D. Maharam [4] asserts that every finite measure space admits a lifting. Using Maharam's result R. Ryan [6] proved that a measure space (Z,Σ,μ) admits a lifting if (Z,Σ,μ) is a direct sum of μ-summable sets. In particular if Z is locally compact and μ a positive Radon measure on Z then a lifting exists (cf. A. and C. Ionescu Tulcea [2]).

To avoid complications of a technical nature we shall state most of the theorems for finite measure spaces. It is clear from the above considerations that in fact these theorems are valid for direct sums of finite measure spaces.

As for the theory of direct integrals of Hilbert spaces we must introduce the concept of diagonal and decomposable operator.

Let $((H(z)),\Gamma)$ be an integrable family of Hilbert spaces on the finite measure space (Z,Σ,μ). If H is a Hilbert space, $L(H)$ denotes the set of bounded operators on H. The identity in $L(H)$ is denoted by I_H or I. An element $T \in \prod_z L(H(z))$ is called an operator field. For a vectorfield x, Tx is the vectorfield $Z \ni z \mapsto T(z)x(z)$. The operator field T is said to be measurable if for $x \in \Gamma$ also $Tx \in \Gamma$. It is readily seen that any measurable T defines a bounded linear operator on $\int^\oplus H(z)\,d\mu$, which is denoted by $\int^\oplus T(z)\,d\mu$ or T. Operators on $\int^\oplus H(z)\,d\mu$, which are of this form are called decomposable. The set of decomposable operators is denoted by $\mathscr{D}$. Plainly the operator field $Z \ni z \mapsto \varphi(z)I_{H(z)}$ is measurable for $\varphi \in M^\infty(Z,\mu)$ and $\int^\oplus \varphi(z)I_{H(z)}d\mu$ only depends on $\tilde{\varphi} \in L^\infty(Z,\mu)$. This operator is denoted by $T_{\tilde{\varphi}}$ or T_φ, and they are called diagonal. For the set of diagonal operators we use $\mathfrak{Z}$.

In [11] it is shown that in a global sense the study of direct integrals of Hilbert spaces reduces to consideration of so called constant families.

This is the reason why we shall mainly consider these last objects. Thus let $\mathscr{L}_\mu^2(H_0,Z)$ be a constant family on (Z,Σ,μ). For $e \in H_0$ we define $\hat{e}\colon Z \to H_0$ by $\hat{e}(z)=e$. If μ is bounded $\hat{e} \in L_\mu^2(Z,H_0)$. If $T \in L(H_0)$ then $\hat{T}$ is the operator field $\hat{T}(z)=T$ and $\hat{T}=\int^{\oplus}\hat{T}(z)\,d\mu$. -

1.2 PROPOSITION. *Suppose $\mathscr{L}_\mu^2(H_0,Z)$ is a constant field and T is a measurable operator field with* ess sup$\{\|T(z)\|\}<\infty$. *Then the following conditions are equivalent.*

1) *T is measurable.*

2) *$Z \ni z \mapsto T(z)e \in \Gamma$ for all $e \in H_0$.*

3) *$Z \ni z \mapsto (T(z)e,f)$ is measurable for all $e,f \in H_0$ and $T\hat{e}$ has essentially separable range in H_0 for all e.*

Let T be a decomposable operator and S a measurable operator field. Then $T=\int^{\oplus}S(z)\,d\mu(z)$ if and only if $(T\hat{e},\hat{f})=(S\hat{e},\hat{f})$ a.e. *for all $e,f \in H_0$.*

PROOF. 2) $\Leftrightarrow$ 3) is a standard result on measurability of vector-valued maps.

1) $\Rightarrow$ 2) is trivial.

2) $\Rightarrow$ 1). Now assume $T\hat{e} \in \Gamma$ for all $e \in H_0$. Every $x \in \Gamma$ has separable range and there exists an orthonormal set $\{e_n\}_{n\in\mathbb{N}} \subset H_0$ so that

$$x = \lim_k \textstyle\sum_{n=1}^k (x,\hat{e}_n)\hat{e}_n \in \Gamma .$$

Then

$$Tx = \lim_k T \textstyle\sum_{n=1}^k (x,\hat{e}_n)\hat{e}_n \in \Gamma$$

because Γ is complete. In interchanging T with $\lim_k$ we used the fact that $\{\|T(z)\|\}_z$ is essentially bounded.

Concerning the last assertion we remark that it is clear that if $T=\int^{\oplus}S(z)\,d\mu$ then $T\hat{e}(z)=S(z)e$ a.e. for all $e \in H$. Conversely, making use of the fact that any $x \in \Gamma$ has separable range, we easily deduce from $(T\hat{e}(z),f)=(S(z)e,f)$ a.e. for all $e,f \in H_0$ that $(Tx)(z)=S(z)x(z)$ a.e. for all $x \in \Gamma$ and we are done.

The following example shows that in the non-separable case the equality $\int^{\oplus}T(z)\,d\mu=\int^{\oplus}S(z)\,d\mu$ does not imply $T(z)=S(z)$ a.e. Take $Z=[0,1]$, μ Lebesgue measure and H_0 a Hilbertspace with an orthonormal base $\{e_z\}_{z\in Z}$. We define $S(z)$ as the projection on the span of e_z and $T(z)=0$. Then S is measurable, $S(z)\neq 0$ for all z and $\int^{\oplus}S(z)\,d\mu=0$.

2. Diagonal and decomposable operators on a constant field.

We suppose that (Z,Σ,μ) is a finite measure space and consider a fixed linear lifting ϱ of $L^\infty(Z,\mu)$ together with a constant field $\mathscr{L}_\mu(H_0,Z)$.

2.1 Proposition. *The set $\mathscr{D}$ of decomposable operators is equal to the commutant $\mathfrak{Z}'$ of $\mathfrak{Z}$. Moreover for any $T \in \mathfrak{Z}'$ and $z \in Z$ there exists a unique operator $\overline{T}(z)$ with*

$$\varrho[(T\hat{e},\hat{f})](z) = (\overline{T}(z)e,f), \quad e,f \in H_0 .$$

The operator field $Z \ni z \mapsto \overline{T}(z)$ is measurable and $T = \int^\oplus \overline{T}(z)\,d\mu$. The assignment $T \mapsto \overline{T}(z)$ is linear and positive.

Proof. It is obvious that $\mathscr{D} \subseteq \mathfrak{Z}'$. So let $T \in \mathfrak{Z}'$. For all $\varphi,\psi \in M^\infty(Z,\mu)$ and $e,f \in H_0$ we find

$$|((TT_\varphi\hat{e},T\psi\hat{f}))| \leqq \|T\|\,\|T_\varphi\hat{e}\|\,\|T_\psi\hat{f}\|,$$

that is,

$$\left|\int \varphi(z)\overline{\psi(z)}\,(T\hat{e},\hat{f})(z)\,d\mu\right|^2 \leqq \|T\|^2\,\|e\|^2\,\|f\|^2 \int |\varphi(z)|^2 d\mu \int |\psi(z)|^2\,d\mu$$

and hence

$$|(T\hat{e})(z),f)| \leqq \|T\|\,\|f\|\,\|e\| \quad \text{a.e.}$$

Consequently $|\varrho[(T\hat{e},\hat{f})]| \leq \|T\|\,\|f\|\,\|e\|$ and for every $z \in Z$ there is a unique operator $\overline{T}(z) \in L(H_0(z))$ with $\|\overline{T}(z)\| \leq \|T\|$ and $(\overline{T}(z)e,f) = \varrho[(T\hat{e},\hat{f})](z)$.

We want to show next that $\overline{T}(\cdot)$ is measurable. Observe that for $e \in H_0$, $T\hat{e}$ has separable range K. If $f \in K^\perp$ then $(Te(z),f) = 0$ a.e. and thence $(\overline{T}(z)e,f) = 0$ for all z, which shows that $\overline{T}(z)e \in K$, so that $\overline{T}(\cdot)e$ has separable range too. We see that $\overline{T}(\cdot)$ satisfies condition 3 of Proposition 1.2 and therfore is measurable. It also follows from Proposition 1.2 that $T = \int^\oplus \overline{T}(z)\,d\mu$, since $(\overline{T}(z)e,f) = (T\hat{e}(z),f)$ a.e.

Finally, the linearity of $T \to \overline{T}(z)$ is evident. Let us check the positivity. Suppose $T \geqq 0$. For $e \in H_0$ and any $\varphi \in M^\infty(Z,\mu)$ we have

$$0 \leqq ((TT_\varphi\hat{e},T_\varphi\hat{e})) = \int |\varphi(z)|^2\,(T\hat{e}(z),e)\,d\mu .$$

Therefore $(T\hat{e}(z),e) \geqq 0$ a.e. and $(\overline{T}(z)e,e) \geqq 0$ for all z. This means $\overline{T}(z) \geqq 0$. The proposition is proved.

It might be useful to give an example of an operator $T \in \mathfrak{Z}'$ for which it is not immediately clear that $T \in \mathscr{D}$. Let T be an operator field such that $\sup_z \|T(z)\| < \infty$ and such that the functions $Z \ni z \mapsto (T(z)e,f)$ are measurable for all $e,f \in H_0'$. The map

$$[x,y] \to \int (T(z)x(z),y(z))\,d\mu$$

is continuous and sesquilinear on $L_\mu^2(H,Z)$. In this way an operator $\tilde{T}$ on $L_\mu^2(H,Z)$ is defined and $T \in \mathfrak{Z}'$. However the conditions imposed on T so far do not ensure that T is measurable.

2.2. Proposition. *If $T \in \mathfrak{Z}'$ and $S_0, T_0 \in L(H_0)$, then*

$$\overline{\hat{S}_0 T \hat{T}_0}(z) = S_0 \overline{T}(z) T_0$$

Proof. For $e,f \in H_0$ and $T_0 e = g$ we get

$$\begin{aligned}(\overline{\hat{S}_0 T \hat{T}_0}(z) e,f) &= \varrho[(\{\hat{S}_0 T \hat{T}_0 \hat{e}\},\hat{f})](z)\\ &= \varrho[(T\hat{g}, \hat{S}_0{}^*\hat{f})](z)\\ &= (\overline{T}(z)g, S_0{}^*f) = (S_0\overline{T}(z)T_0 e,f)\,.\end{aligned}$$

Thus the desired equality follows.

The following obvious corollary will be used later on.

2.3 Proposition. *If $\mathfrak{A} \subseteqq L(H_0)$ and $T \in \mathfrak{Z}'$ satisfies $T\hat{S} = \hat{S}T$ for all $S \in \mathfrak{A}$, then $\overline{T}(z)$ is contained in the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ for all $z \in Z$.*

Proof. Apply Proposition 2.2.

There is a natural connection between liftings and decompositions. Let $CL(H_0)$ be the set of compact operators in $L(H_0)$

2.4 Proposition. *Suppose that for every $T \in \mathfrak{Z}'$ we are given a decomposition $T \to \overline{T}(\cdot)$ into a measurable operator field such that for all $z \in Z$,*

1) $\overline{T\hat{S}}(z) = \overline{T}(z)S$ *for* $S \in CL(H_0)$,
2) $\overline{\hat{I}}_{H_0}(z) = I_{H_0}$,
3) $T \to \overline{T}(z)$ *is positive and linear.*

Then there exists a linear lifting ϱ of $L^\infty(Z,\mu)$ such that for $e,f \in H_0$ we have $\varrho[(T\hat{e},\hat{f})](z) = (\overline{T}(z)e,f)$.

Proof. Let $\tilde{\varphi} \in L^\infty(Z,\mu)$ and $S \in CL(H_0)$. Clearly $T_{\tilde{\varphi}}\hat{S} = \hat{S}T_{\tilde{\varphi}}$. Because of 1) and 3) we find $\overline{T}_{\tilde{\varphi}}(z)S = S\overline{T}_{\tilde{\varphi}}(z)$. Hence $\overline{T}_{\tilde{\varphi}}(z)$ is a scalar multiple $\varrho(\tilde{\varphi})(z)I_{H_0}$ of I_{H_0}. Since $\overline{T}_{\tilde{\varphi}}(\cdot)$ is measurable $\varrho(\tilde{\varphi})(\cdot)$ is measurable. If $e \in H_0$ is a unit vector we find

$$\tilde{\varphi} = (T_{\tilde{\varphi}}\hat{e},\hat{e}) = \widetilde{\varrho(\tilde{\varphi})}\,.$$

Since the map $\tilde{\varphi} \to \varrho(\tilde{\varphi})$ also is positive and linear and satisfies $\varrho(\tilde{1}) = 1$ because of 2) we see that ϱ is a linear lifting of $L^\infty(Z,\mu)$.

For $e,f \in H_0$ we define $e \otimes f \in CL(H)$ by $(e \otimes f)g = (g,e)f$ where $g \in H_0$. For $x,y \in \Gamma$ and $T \in \mathfrak{Z}'$ we get

$$(\widehat{f \otimes f} T \widehat{e \otimes e} x, y) = \int (T\hat{e}(z), f)(x(z), e)(f, y(z))\, d\mu = (T_{\tilde{\varphi}} \widehat{e \otimes f} x, y)$$

where $\tilde{\varphi} = (T\hat{e}(\cdot), f)$. Therefore

$$\widehat{f \otimes f} T \widehat{e \otimes e} = T_{\tilde{\varphi}} \widehat{e \otimes f} .$$

Applying 1) and 3) we get

$$f \otimes f \overline{T}(z) e \otimes e = \overline{T}_{\tilde{\varphi}}(z) e \otimes f = \varrho(\tilde{\varphi})(z) e \otimes f .$$

From this the statement in 2.4 readily follows.

The condition 2) in 2.4 is very weak, because only 1) and 3) are used to prove that

$$\bar{\tilde{I}}_{H_0}(z) = \overline{T}_{\tilde{1}}(z) = \varrho(\tilde{1})(z) I_{H_0}, \quad \text{where} \quad \widetilde{\varrho(\tilde{1})} = \tilde{1} .$$

The methods used to prove 2.1 and 2.4 readily show that "the continuous analogue of the Schur lemma" proved in [6, § 26, th. 8] holds without the condition that the space H, mentioned there, is separable.

An interesting question is whether it is possible to decompose the operators $T \in \mathfrak{Z}'$ in such a way that apart from 1), 2) and 3) in 2.4 also $\overline{TS}(z) = \overline{T}(z)\,\overline{S}(z)$ for all $z \in Z$, S, $T \in \mathfrak{Z}'$. This is of importance among others for the decomposition of group representations. It is known [1] that if $\mathfrak{A} \subseteqq \mathfrak{Z}'$ is a *subalgebra, separable in the uniform operator topology, that then a decomposition of elements in $\mathfrak{A}$ can be found, which is multiplicative. Also it is not hard to see that if ϱ is a lifting and if we decompose with ϱ, we then find for $T \in \mathfrak{Z}'$ and $\tilde{\varphi} \in L^\infty(Z,\mu)$

$$\overline{T T_{\tilde{\varphi}}}(z) = \varrho(\tilde{\varphi})(z) \overline{T}(z) = \overline{T}_{\tilde{\varphi}}(z) \overline{T}(z) .$$

From this it is deduced, using Proposition 2.2, that if S is a measurable operator field such that $\{S(z) \mid z \in Z\}$ is μ-essentially compact in $L(H)$ and $T \in \mathfrak{Z}'$, then

$$\overline{TS}(z) = \overline{T}(z)\,\overline{S}(z) .$$

This shows that the restriction of such a decomposition to the algebra of decomposable operators with essentially compact range is multiplicative. However, we next prove that a multiplicative decomposition of all $\mathfrak{Z}'$ does not exist. Surprisingly enough we have to add an extra condition in the non-separable case.

2.5 Proposition. *There exists no decomposition $T \to \overline{T}(\cdot)$ of $\mathfrak{Z}'$ such that the maps $T \to \overline{T}(z)$ are *homomorphisms and satisfy* 1) *of* 2.4 *if $\mu \neq 0$ is non-atomic and H_0 infinite dimensional.*

In the case that H is separable, the condition that $T \to \overline{T}(z)$ satisfy 1) *of* 2.4 *can be left out.*

Proof. The proof will be given by deriving a contradiction. First let H_0 be separable and $\{e_n\}_{n\in\mathsf{N}}$ a dense set in H_0. Because H_0 is separable the theory of [1, ch. II] applies, to the effect that for the given decomposition we can find a set $Z_0 \subseteq Z$, with $\mu(Z_0{}^c)=0$, such that for $z \in Z_0$ we get

$$\overline{\widehat{e_n \otimes e_m}}(z) = e_n \otimes e_m \quad \text{for all } n,m .$$

The maps $T \to \overline{T}(z)$, being *homomorphisms, are continuous and so by continuity and linearity we get for all $S \in CL(H)$ that

$$\overline{\hat{S}}(z) = S \quad \text{for } z \in Z_0 .$$

Thus for $T \in \mathfrak{Z}'$ this means

$$\overline{T\hat{S}}(z) = \overline{T}(z)S .$$

Replacing Z by Z_0 shows that then 1) of 2.4 is satisfied.

According to the first remark following 2.4 we may as well assume that also 2) of 2.4 is satisfied.

We suppose now that we have a decomposition $T \to \overline{T}(\cdot)$ of $\mathfrak{Z}'$ as meant in 2.5.

Because of 2.4 and because the decomposition is multiplicative there exists a linear lifting ϱ of $L^\infty(Z,\mu)$ so that for $e,f \in H_0$,

$$\varrho[(T\hat{e},\hat{f})](z) = (T(z)e,f) .$$

Let $\{Z_n\}_{n\in\mathsf{N}}$ be a family of measurable and mutually disjoint subsets of Z, 1_{Z_n} the characteristic function of Z_n and $\{f_n\}_{n\in\mathsf{N}}$ an orthonormal set in H. We put

$$A(z) = \textstyle\sum_{n=1}^\infty 1_{Z_n}(z) f_1 \otimes f_n .$$

Then $A(\cdot)$ is measurable and for $k,l \in \mathsf{N}$ we get

$$\varrho[(A(\cdot)f_k,f_l)](z) = \delta_{k,1} \textstyle\sum_{n=1}^\infty \delta_{l,n}\varrho(1_{Z_n}) ,$$

which shows

$$\bar{A}(z) = \textstyle\sum_{n=1}^\infty \varrho(1_{Z_n})(z) f_1 \otimes f_n .$$

Consequently

$$(\overline{A^*A}(z)f_1,f_1) = (\overline{A(z)}^*A(z)f_1,f_1) = \textstyle\sum_{n=1}^\infty \varrho(1_{Z_n})(z) .$$

On the other hand, $A^*A(z) = (\sum_{n=1}^\infty 1_{Z_n}) f_1 \otimes f_1$ and so

$$(\overline{A^*A}(z)f_1,f_1) = \varrho\big[(\{\textstyle\sum_{n=1}^{\infty} 1_{Z_n}(\cdot)\}(f_1\otimes f_1)f_1,f_1)\big](z) = \varrho(\textstyle\sum_{n=1}^{\infty} 1_{Z_n})(z) .$$

The two expressions being equal we find $\sum_{n=1}^{\infty}\varrho(1_{Z_n})=\varrho(\sum_{n=1}^{\infty}1_{Z_n})$. It is simple to see that the above equality implies that ϱ can be extended to a linear lifting on $L^1(Z,\mu)$, which does not exist according to [3,7]. This is the desired contradiction.

3. A multiplicative decomposition.

It is our aim in this section to decompose "multiplicatively". Instead of considering the Hilbert spaces $L^2_{H_0}(Z,\mu)$ and the abelian algebra of diagonal operators, we study an abelian von Neumann algebra $\mathfrak{Z}$ on a Hilbert space H and a C^*-algebra $\mathfrak{A}$, with $\mathfrak{A}\subseteq\mathfrak{Z}'$, which admits a cyclic vector $x_0\in H$. Finally we suppose that $\mathcal{Y}\subseteq\mathfrak{Z}$ is a weakly dense C^*-subalgebra of $\mathfrak{Z}$, which contains I, and whose spectrum is Z. The assumptions that $\mathfrak{A}$ has a cyclic vector and that $I\in\mathcal{Y}$ are imposed for convenience only. For a more general theory compare [1, I, § 7].

Let S_φ be the image of $\varphi\in C(Z)$ under the inverse Gelfandtransformation from $C(Z)$ onto $\mathcal{Y}$. The vector x_0 determines a measure μ on Z by $\int\varphi d\mu=(S_\varphi x_0,x_0)$ for $\varphi\in C(Z)$. The measure μ has support Z so that $C(Z)$ can be identified with a subspace of $L^\infty(Z,\mu)$. Then there is a unique extension, from $C(Z)$ to $L^\infty(Z,\mu)$ of the map $\varphi\mapsto S_\varphi$ with values in $\mathfrak{Z}$, which is continuous, $L^\infty(Z,\mu)$ being equipped with $\sigma(L^\infty(Z,\mu),L^1(Z,\mu))$ and $\mathfrak{Z}$ with the weak operator topology. This extended map, also denoted by $\varphi\mapsto S_\varphi$, is an isomorphism onto $\mathfrak{Z}$.

3.1 Proposition. *Let the notation be as above and ϱ a lifting of $L^\infty(Z,\mu)$.*

1) *There exists an integrable field of Hilbert spaces $((H(z)),\Gamma)$ on (Z,μ) and an isomorphism $U: H\to\int^\oplus H(z)d\mu$ so that for $\tilde\varphi\in L^\infty(Z,\mu)$ we get $S_{\tilde\varphi}=U^*T_{\tilde\varphi}U$.*

2) *There is a decomposition $T\to\overline{T}(\cdot)$ of $\mathfrak{Z}'$ into measurable operator fields, which is positive and linear. If $A\in\mathfrak{A}$, $T\in\mathfrak{Z}'$ and $z\in Z$, then $\overline{TA}(z)=\overline{T}(z)\overline{A}(z)$.*

Proof. 1) Let $A\in\mathfrak{A}$. By means of the Radon-Nikodym theorem we see that there is a unique $h_{A^*A}\in L^1(Z,\mu)$ with

$$\int h_{A^*A}\,d\mu = (S_\varphi A x_0,Ax_0) .$$

If $\varphi\geqq 0$ we find $\int\varphi h_{A^*A}\,d\mu\geqq 0$, so that $h_{A^*A}\geqq 0$ and we get also

$$\int \varphi h_{A^*A}\,d\mu \leqq \|A\|^2\int\varphi\,d\mu ,$$

which implies

$$h_{A^*A} \leqq \|A\|^2 = \|A^*A\| \quad \text{and} \quad h_{A^*A} \in L^\infty(Z,\mu) .$$

By linearity the map $A^*A \mapsto h_{A^*A}$ can be extended from $\mathfrak{A}^+$ to a positive linear map $A \mapsto h_A$ from $\mathfrak{A}$ into $L^\infty(Z,\mu)$. We find

$$\int \varphi h_A \, d\mu = (S_\varphi A x_0, x_0), \qquad h_I = 1, \qquad h_{S_\varphi A} = \varphi h_A .$$

For every z a state on $\mathfrak{A}$ is defined by $A \to \varrho(h_A)(z)$.

Let $H(z)$ be the Hilbert space associated with the just constructed state on $\mathfrak{A}$ and σ_z the canonical representation of $\mathfrak{A}$ into $L(H(z))$. The projection of $\mathfrak{A}$ into $H(z)$ is denoted by σ_z. Now we have a field of Hilbert space $(H(z))_{z \in Z}$. It is made integrable by taking for Γ the subspace of $\prod_z H(z)$ determined by $\Gamma_0 = \{\sigma_z(A) \mid A \in \mathfrak{A}\}$.

We put $H' = \{Ax \mid A \in \mathfrak{A}\}$. Then $\bar{H}' = H$. If $Ax_0 = 0$, then $h_{A^*A} = 0$ and so $\pi_z(A) = 0$ for all z. This shows that we can define a map U from H' into $\int^\oplus H(z) \, d\mu$ by $U(Ax_0) = \int^\oplus \sigma_z(A) \, d\mu$. The map U is linear and satisfies

$$(Ax_0, Ax_0) = \int h_{A^*A}(z) \, d\mu = \int \|\sigma_z(A)\|^2 \, d\mu .$$

Thus U is isometric and can be extended to all of H. It is obvious that this extended U is an isomorphism from H onto $\int^\oplus H(z) \, d\mu$. If $\varphi \in L^\infty(Z,\mu)$ then

$$\begin{aligned}(S_\varphi A x_0, B x_0) &= \int \varphi(z) \, h_{B^*A}(z) \, d\mu \\ &= \int \varphi(z) (\sigma_z(A), \sigma_z(B)) \, d\mu = (T_\varphi U(Ax_0), U(Bx_0)) ,\end{aligned}$$

such that indeed $U^* T_\varphi U = S_\varphi$.

2) Not to complicate matters we identify H with $\int^\oplus H(z) \, d\mu$. If $e = Ax_0$ we put $e(z) = \sigma_z(A)$. For $e, f \in H'$; $\varphi, \psi \in L^\infty(Z,\mu)$ and $T \in \mathfrak{Z}'$ we find

$$|(TT_\varphi e, T_\psi f)|^2 \leqq \|T\|^2 \, \|T_\varphi e\|^2 \, \|T_\psi f\|^2$$

that is,

$$\left| \int \varphi(z) \overline{\psi(z)} \, (Te(z), f(z)) \, d\mu \right|^2 \leqq \|T\|^2 \int |\varphi(z)|^2 \, \|e(z)\|^2 \, d\mu \cdot \int |\psi(z)|^2 \, \|f(z)\|^2 \, d\mu$$

and hence

$$|(Te(z), f(z))| \leqq \|T\| \, \|e(z)\| \, \|f(z)\| \quad \text{a.e.}$$

There exists a unique operator $\bar{T}(z) \in L(H(z))$ with $\|\bar{T}(z)\| \leqq \|T\|$ and

$$\varrho[(Te(\cdot), f(\cdot))](z) = (\bar{T}(z) e(z), f(z)) .$$

As before it is easy to see that $T \to \overline{T}(z)$ is positive and linear. There is a sequence $e_n \in H'$ with $\lim_n e_n = Te$. For every n we have

$$(\overline{T}(z)e(z), e_n(z)) = (Te(z), e_n(z)) \quad \text{a.e.}$$

The same is true for all n simultaneously, so that, since

$$Te \in \operatorname{sp}\{e_n\}_{n\in\mathbf{N}}, \quad \overline{T}(z)e(z) = Te(z) \quad \text{a.e.}$$

This shows $Z \ni z \mapsto \overline{T}(z)e(z) \in \Gamma$, $\overline{T}(\cdot)$ is measurable and $\int^{\oplus} \overline{T}(z)\,d\mu = T$.

Finally, if $A, B, C \in \mathfrak{A}$, then we have

$$\begin{aligned}
\varrho[(ABx_0(\cdot), Cx_0(\cdot))](z) &= \varrho[h_{C^*AB}](z) \\
&= (\sigma_z(AB), \sigma_z(C)) \\
&= (\pi_z(A)\sigma_z(B), \sigma_z(C)) \\
&= (\pi_z(A)(Bx_0)(z), (Cx_0(z))) \\
&= (\bar{A}(z)(Bx_0)(z), (Cx_0)(z)) .
\end{aligned}$$

Therefore $\pi_z(A) = \bar{A}(z)$ and $(ABx_0)(z) = \bar{A}(z)(Bx_0)(z)$.

For $A \in \mathfrak{A}$, $T \in \mathfrak{Z}'$, and $e, f \in H'$ we get

$$\begin{aligned}
(\overline{TA}(z)e(z), f(z)) &= \varrho[((TA)e(\cdot), f(\cdot))](z) \\
&= \varrho[(T(Ae(\cdot), f(\cdot)))](z) \\
&= (\overline{T}(z)(Ae)(z), f(z)) = (\overline{T}(z)\bar{A}(z)e(z), f(z)) .
\end{aligned}$$

An easy corollary is the following

3.2. Corollary. *If* $\mathfrak{A}'(z) = \{\overline{T}(z) \mid T \in \mathfrak{A}'\}$, *then* $\mathfrak{A}'(z) \subseteq \mathfrak{A}(z)'$.

If we take $\mathfrak{A} = \mathfrak{Z}'$, then the described decomposition has some nice properties.

3.3. Proposition. *Let* $\mathfrak{A} = \mathfrak{Z}'$.

1) *The maps* $T \mapsto \overline{T}(z)$ *of* $\mathfrak{A}$ *into* $L(H(z))$ *are *homomorphisms.*
2) *The spaces* $\{\pi_z(A) \mid A \in \mathfrak{Z}'\}$ *are complete.*
3) *The algebra* $\{\overline{T}(z) \mid T \in \mathfrak{Z}'\}$ *contains all compact operators on* $H(z)$.

Proof. 1) This follows directly from 2) of Proposition 3.1.

2) Let $N(z) = \{A \in \mathfrak{Z}' \mid \varrho(h_{A^*A})(z) = 0\}$. The quotient norm on $\mathfrak{Z}'/N(z) = \{\sigma_z(A) \mid A \in \mathfrak{Z}'\}$ is defined by

$$\|\sigma_z(A)\|_q = \inf\{\|B\| \mid B \in A + N(z)\} .$$

We show that

$$\|\sigma_z(A)\|_q^2 = \|\sigma_z(A)\|^2 = \varrho(h_{A^*A})(z) .$$

This proves that $\mathfrak{Z}'/N(z)$ is complete, because a quotient space of a Banach space is complete in the quotient topology.

It is clear that $\|\pi_z(A)\|_q \geqq \|\pi_z(A)\|$. Suppose P is the projection on $\{\mathfrak{Z}x_0\}^-$ and $Q=I-P$. For $\tilde{\varphi} \in L^\infty$, $A \in \mathfrak{Z}'$ we find $(T_{\tilde{\varphi}}AQx_0, AQx_0)=0$ and therefore $AQ \in N(z)$. We also get

$$\begin{aligned} \|AP\|^2 &= \sup\{\|APT_{\tilde{\varphi}}x_0\|^2 \mid \|T_{\tilde{\varphi}}x_0\| \leqq 1\} \\ &= \sup\left\{\int |\varphi|^2(z) h_{A^*A}(z)\, d\mu \;\Big|\; \int |\varphi|^2(z)\, d\mu \leqq 1\right\} \\ &= \|h_{PA^*AP}\|_\infty . \end{aligned}$$

We notice that $h_{PA^*AP}=h_{A^*A}$ because $AQ \in N(z)$ for all z.

Suppose $\|\sigma_z(A)\|=\alpha$. There is a $T_{\tilde{\varphi}} \in \mathfrak{Z}$ with $\varrho(\tilde{\varphi})(z)=1$ and such that

$$h_{T_{\tilde{\varphi}}^2 A^*A} = \tilde{\varphi}^2 h_{A^*A} \leqq \alpha^2 .$$

Also $A-T_{\tilde{\varphi}}A \in N(z)$ because $I-T_{\tilde{\varphi}} \in N(z)$. We consider next $T_{\tilde{\varphi}}AP$ then $A-T_{\tilde{\varphi}}AP \in N(z)$ and $\|T_{\tilde{\varphi}}AP\| \leqq \alpha$ because of what we proved just before.

3) For $A,B,C,D \in \mathfrak{Z}'$ we get

$$\begin{aligned} \varrho\big[\big((Ax_0(\cdot)\otimes Bx_0(\cdot)), Cx_0(\cdot)\otimes Dx_0(\cdot)\big)\big)\big](z) &= \big(Ax_0(z), Cx_0(z)\big)\big(Bx_0(z), Dx_0(z)\big) \\ &= \big(\big(Ax_0(z)\otimes Bx_0(z)\big)Cx_0(z), Dx_0(z)\big) \end{aligned}$$

so that

$$\big(Ax_0(\cdot)\otimes Bx_0(\cdot)\big)(z) = Ax_0(z)\otimes Bx_0(z) .$$

In 2) it is proved that the space $\{Ax_0(z) \mid A \in \mathfrak{Z}'\}$ is all of $H(z)$ so that $\mathfrak{Z}'(z)$ contains all finite dimensional operators on $H(z)$. Because $\mathfrak{Z}'(z)$ is closed, it contains all compact operators as well.

A disadvantage though of taking $\mathfrak{A}=\mathfrak{Z}'$ is that the spaces $H(z)$ become very big. This can readily be verified by looking at $L_\mu^2(H_0,Z)$, $\mathfrak{Z}$ being the algebra of diagonal operators. If H_0 is separable, and the measure $\mu \neq 0$ is completely non-atomic then the Hilbert dimension of the spaces $H(z)$ equals the cardinality of $\mathbf{R}$. If we decompose with respect to the algebra $\mathfrak{A}$, where $\mathfrak{A}$ is equal to the set of decomposable operators T which have essentially compact range in $L(H_0)$, then the spaces $H(z)$ can be identified in a natural way with H_0 and $\mathfrak{A}(z)=L(H(z))$.

A very interesting problem is whether the algebra $\mathfrak{A}(z)$ acts irreducibly on $H(z)$ if we assume that $\mathfrak{A}$ is weakly dense in $\mathfrak{Z}'$. This problem is related to the question whether every representation of a locally compact group can be decomposed into irreducible representations and it is also closely connected with the question whether every W^*-algebra can be written as a direct integral of factors.

4. On a problem of Sakai's.

In this section we shall discuss a problem posed by S. Sakai [9]. Let (Z,μ) be a finite measure space and E_* a Banach space with dual E. $L_\mu^1(Z,E_*)$ is the set of equivalence classes of strongly measurable and integrable E_*-valued functions on Z. $\mathscr{L}_{E_s}^\infty(Z,\mu)$ is the set of functions $S\colon Z \to E$ such that $\langle p,S(\cdot)\rangle \in M^\infty(Z,\mu)$ for all $p \in E_*$. $N_{E_s}^\infty(Z,\mu)$ is the set of functions in $\mathscr{L}_{E_s}^\infty(Z,\mu)$ for which $\langle p,S(\cdot)\rangle = 0$ a.e. (μ) for every $p \in E_*$. Finally the quotient space $\mathscr{L}_{E_s}^\infty(Z,\mu)/N_{E_s}^\infty(Z,\mu)$ denoted by $L_{E_s}^\infty(Z,\mu)$, is a Banach space under the norm

$$\|\tilde{S}\| = \inf\{\|T\| \mid T \in \mathscr{L}_{E_s}^\infty(Z,\mu),\ \tilde{T} = \tilde{S}\},$$

where $T \to \tilde{T}$ denotes the canonical map from $\mathscr{L}_{E_s}^\infty(Z,\mu)$ in $L_{E_s}^\infty(Z,\mu)$ and $\|T\| = \sup\{\|T(z)\| \mid z \in Z\}$.

The bilinear form $\langle\cdot,\cdot\rangle$ on $L_\mu^1(Z,E_*) \times L_{E_s}^\infty(Z,\mu)$ defined by

$$\langle f,S\rangle = \int \langle f(z),S(z)\rangle\, d\mu(z)$$

establishes a duality between the two spaces, under which $L_{E_s}^\infty(Z,\mu)$ is the dual of $L_\mu^1(Z,E_*)$ [3]. Sakai proved that if E is an algebra with separately $\sigma(E,E_*)$-continuous multiplication and if E_* is separable, $\mathscr{L}_{E_s}^\infty(Z,\mu)$ is an algebra under pointwise operations [9]. This implies that $L_{E_s}^\infty(Z,\mu)$ is a Banach algebra, since in the separable case $N_{E_s}^\infty(Z,\mu)$ is an ideal in $\mathscr{L}_{E_s}^\infty(Z,\mu)$. In case that E is a W^*-algebra with separable predual, $L_{E_s}^\infty(Z,\mu)$ becomes in a natural way a C^*-algebra, and because it is a dual space, it is a W^*-algebra. We shall deal with the case, where E is a W^*-algebra with not necessarily separable predual.

Before we prove that $L_{E_s}^\infty(Z,\mu)$ is a W^*-algebra, we state a preliminary theorem of independent interest. If $\mathfrak{A}$ and $\mathscr{B}$ are von Neumann algebras $\mathfrak{A}\overline{\otimes}\mathscr{B}$ denotes the von Neumann algebra tensor product.

4.1. THEOREM. *Let H_0 be a Hilbert space and $\mathfrak{A} \subseteq L(H_0)$ a von Neumann algebra. Let $\int^\oplus \mathfrak{A}\, d\mu$ be the set of decomposable operators $T = \int^\oplus T(z)\, d\mu$ on the constant field $\mathscr{L}_\mu^2(H_0,Z)$ for which $T(z) \in \mathfrak{A}$ for almost all z. Let M be the algebra of multiplication operators on $L_\mu^2(Z)$. We identify $L_\mu^2(Z,H_0)$ and $H_0\overline{\otimes}L_\mu^2(Z)$. Then*

1) *$\int^\oplus \mathfrak{A}\, d\mu$ is a von Neumann algebra and $(\int^\oplus \mathfrak{A}\, d\mu)' = \int^\oplus \mathfrak{A}'\, d\mu$,*
2) *$\mathfrak{A}\overline{\otimes} M = \int^\oplus \mathfrak{A}\, d\mu$,*
3) *$(\mathfrak{A}\overline{\otimes} M)' = \mathfrak{A}'\overline{\otimes} M$.*

PROOF. 1) $\int^{\oplus}\mathfrak{A}d\mu$ is a *-algebra of decomposable operators. Let $T\in L(H)$. If T commutes with $\int^{\oplus}\mathfrak{A}d\mu$ then T is decomposable say $T=\int^{\oplus}\bar{T}(z)d\mu(z)$ where $\bar{T}(\cdot)$ denotes the decomposition of 2.1. From 2.2. it follows that $\bar{T}(z)\in\mathfrak{A}'$, hence $T\in\int^{\oplus}\mathfrak{A}'d\mu$. If conversely $T\in\int^{\oplus}\mathfrak{A}'d\mu$, clearly we have that T commutes with $\int^{\oplus}\mathfrak{A}d\mu$. Hence $(\int^{\oplus}\mathfrak{A}d\mu)'=\int^{\oplus}\mathfrak{A}'d\mu$. Interchanging the roles of $\mathfrak{A}$ and $\mathfrak{A}'$ we see that $\int^{\oplus}\mathfrak{A}d\mu$ is a von Neumann algebra.

2) Any operator of the form $\sum T_i\otimes f_i$ is via the identification of the form $\int^{\oplus}\sum f_i(z)T_i d\mu(z)$ with $f_i\in L_\mu{}^\infty$. Hence it is in $\int^{\oplus}\mathfrak{A}d\mu$ and since the latter is a von Neumann algebra we have $\mathfrak{A}\bar{\otimes}M\subseteqq\int^{\oplus}\mathfrak{A}d\mu$. If $T\in(\mathfrak{A}\bar{\otimes}M)'$ then by the same procedure as above we get $T\in\int^{\oplus}\mathfrak{A}'d\mu=(\int^{\oplus}\mathfrak{A}d\mu)'$. Hence $(\mathfrak{A}\bar{\otimes}M)'\subseteqq(\int^{\oplus}\mathfrak{A}d\mu)'$ or $\int^{\oplus}\mathfrak{A}d\mu\subseteqq\mathfrak{A}\bar{\otimes}M$ so finally $\int^{\oplus}\mathfrak{A}d\mu=\mathfrak{A}\bar{\otimes}M$.

3) Follows immediately from 1) and 2).

4.2. COROLLARY. *Let $\mathscr{C}$ and $\mathscr{D}$ be two von Neumann algebras of which $\mathscr{C}$ is of type* I. *Then $(\mathscr{C}\bar{\otimes}\mathscr{D})'=\mathscr{C}'\bar{\otimes}\mathscr{D}'$.*

PROOF. From the structure theorem for type I algebras it follows that it suffices to consider $\mathscr{C}$ maximal abelian and σ-finite. Then apply 3) of Theorem 4.1.

4.3. THEOREM. *Let E be a W^*-algebra and (Z,μ) a finite measure space. Then $L^\infty_{E_s}(Z,\mu)$ is isometrically *isomorphic to $E\bar{\otimes}L_\mu{}^\infty(Z)$ and is therefore a von Neumann algebra.*

PROOF. We represent E faithfully as a von Neumann algebra on some Hilbert space H_0 and identify $L^\infty_{E_s}(Z,\mu)$ with $L^\infty_{\mathfrak{A}_s}(Z,\mu)$. By the isomophism E_* is transformed into the set of ultraweakly continuous linear functionals on $\mathfrak{A}$, that is, the set of functionals of the form $\omega: T\in\mathfrak{A}\mapsto\sum(Te_i,f_i)$, where $e_i,f_i\in H_0$, $\sum\|e_i\|^2$ and $\sum\|f_i\|^2<\infty$. Since any ω is a uniform limit of weakly continuous functionals on $\mathfrak{A}$ we infer that a field of operators T is weakly measurable if and only if it is ultraweakly measurable.

By the remark after 2.2 we get that an element of $L^\infty_{\mathfrak{A}_s}(Z,\mu)$ defines a decomposable operator on the constant field (H_0,Z,μ), and it is easily seen that it belongs to $(\int^{\oplus}\mathfrak{A}'d\mu)'$, therefore to $\int^{\oplus}\mathfrak{A}d\mu$. Hence we have a linear *-preserving map from $L^\infty_{\mathfrak{A}_s}(Z,\mu)$ onto $\int^{\oplus}\mathfrak{A}d\mu$ and its kernel is $N^\infty_{\mathfrak{A}_s}(Z,\mu)$. Its image is all of $\int^{\oplus}\mathfrak{A}d\mu$. Consequently, $L^\infty_{\mathfrak{A}_s}(Z,\mu)$ is isometrically isomorphic to $\int^{\oplus}\mathfrak{A}d\mu$, which is isomorphic to $\mathfrak{A}\bar{\otimes}L_\mu{}^\infty(Z)$. Hence $L^\infty_{E_s}(Z,\mu)$ is isomorphic $E\bar{\otimes}L_\mu{}^\infty(Z)$, and the theorem is proved.

REMARKS. 1) The difficult thing in making $L^{\infty}_{E_s}(Z,\mu)$ an algebra is that $\mathscr{L}_{E_s}(Z,\mu)$ is in general not an algebra under pointwise operations. This was overcome by considering the measurable operator fields. This notion is easily seen to be invariant under algebraic isomorphism between von Neumann algebras. They form an algebra under pointwise multiplication, and each class in $L^{\infty}_{\mathfrak{A}_s}(Z,\mu)$ admits a measurable operator field as representative. Hence, denoting the measurable fields by $\mathscr{L}^{\infty}_{\mathfrak{A}_m}(Z,\mu)$, $L^{\infty}_{\mathfrak{A}_s}(Z,\mu)$ is isomorphic to

$$\mathscr{L}^{\infty}_{\mathfrak{A}_m}(Z,\mu)/\mathscr{L}^{\infty}_{\mathfrak{A}_m}(Z,\mu) \cap N^{\infty}_{\mathfrak{A}_s}(Z,\mu) ,$$

and $\mathscr{L}^{\infty}_{\mathfrak{A}_m}(Z,\mu) \cap N^{\infty}_{\mathfrak{A}_s}(Z,\mu)$ is an ideal in $\mathscr{L}^{\infty}_{\mathfrak{A}_m}(Z,\mu)$. This constitutes another way of equipping $L^{\infty}_{\mathfrak{A}_m}(Z,\mu)$ with an algebraic structure.

Theorem 4.3 generalizes Theorem 2.5, page 3.21 in [8].

2) Essentially the same methods as used here work to solve Sakai's problem in the affirmative in the case where $E = L(X)$, the algebra of continuous linear maps on a reflexive Banach space.

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