MATH. SCAND. 26 (1970), 103–106

ON SOME MEASURES ANALOGOUS TO HAAR MEASURE

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1.

Let M be a locally compact Hausdorff space and consider a fixed base $\mathcal{U}$ for a uniform structure on M compatible with the topology and suppose that $\mathcal{U}$ consists of open sets. By a measure on M we mean a Radon measure defined on the Borel field of M. Usually we study positive measures, not identically zero.

A positive measure u on M is called *right uniform* (with respect to $\mathcal{U}$) if

$$\forall\, x,y \in M \ \forall\, U \in \mathcal{U}:\ \ u(U[x]) \;=\; u(U[y])\,.$$

It is *left uniform* if

$$\forall\, x,y \in M \ \forall\, U \in \mathcal{U}:\ \ u(U^{-1}[x]) \;=\; u(U^{-1}[y])\,,$$

and it is *uniform* if it is both right and left uniform.

Loomis is probably the first to have studied uniform measures (see [1], [2]). He obtained simultaneously existence and uniqueness of a uniform measure in a space satisfying a combinatorial axiom and some further conditions. By a different approach we obtain uniqueness without further assumptions.

2.

THEOREM 1. *Let u and v be positive measures on M with u right uniform and v left uniform. Then there exists $\lambda \in \mathsf{R}_+$ such that $v = \lambda u$. In particular a uniform measure is unique.*

It is known (see [3, theorem 7,2, p. 187]) that there is a unique G invariant measure if G is a transitive group of homeomorphisms of M satisfying

$$\forall\, \varphi \in G \ \forall\, x,y \in M \ \forall\, U \in \mathcal{U}:\ \ (x,y) \in U \iff \big(\varphi(x),\varphi(y)\big) \in U\,.$$

Since a G invariant measure is uniform this is also the only uniform measure on M.

Received September 23, 1968.

Proof. In the sequel Fubini's theorem is used several times. This can always be justified by observing that from a suitable stage all integrations are carried out over a compact set and the functions involved are well defined. For $U \in \mathscr{U}$ we define the kernel $K_U(x,y)$ by

$$\begin{aligned} K_U(x,y) &= (c_U(u))^{-1} \quad \text{for} \quad (x,y) \in U\,, \\ &= 0 \qquad\qquad\quad \text{for} \quad (x,y) \notin U\,, \end{aligned}$$

where $c_U(u) = u(U[x])$, $x \in M$ arbitrary.

Let φ be a continuous function of compact support on M. Define $K_U\varphi$ by

$$(K_U\varphi)(x) = \int_M K_U(x,y)\, \varphi(y)\, du(y)\,.$$

We then have

$$|K_U\varphi(x) - \varphi(x)| \leqq \int_M K_U(x,y)\, |\varphi(y) - \varphi(x)|\, du(y) \leqq W_\varphi(U)\,,$$

where

$$W_\varphi(U) = \sup \{|\varphi(x) - \varphi(y)| \mid (x,y) \in U\}\,.$$

$K_U\varphi$ is uniformly of compact support and bounded for U sufficiently small. Let B be the filter base on $\mathscr{U}$

$$\{\{U \in \mathscr{U} \mid U \subseteqq V\} \mid V \in \mathscr{U}\}\,.$$

Since φ is uniformly continuous, we see from the above estimate that

$$\int_M K_U\varphi(x)\, dv(x) \to \int_M \varphi(x)\, dv(x) \quad \text{along } B\,.$$

We define $c^U(v) = v(U^{-1}[x])$ and see by Fubini's theorem that

$$\int_M K_U\varphi(x)\, dv(x) = (c^U(v)/c_U(u)) \int_M \varphi(x)\, du(x)\,.$$

If $\int_M \varphi(x) du(x) \neq 0$ this shows that $c^U(v)/c_U(u)$ has a limit λ along B and we have

$$\int_M \varphi(x)\, dv(x) = \lambda \int_M \varphi(x)\, du(x)\,.$$

Since λ is independent of φ, theorem 1 is proved.

3.

For simplicity we shall now confine ourselves to the case of a locally compact metric space (M,d), although similar results could be obtained in a more general setup. Let $S(x,r)$ denote the open ball with center x and radius r.

An interesting and seemingly open question is whether a measure is uniquely determined by its values on the balls. By applying the Hahn–Banach theorem it is easily seen that the locally compact metric space (M,d) has this property if and only if the space of functions of the form $\sum_i \lambda_i(d(a_i,x))$ (where the sum is finite and the λ_i's are continuous functions such that $\lambda_i(d(a_i,x))$ has support in a compact ball) is dense in the space of continuous functions of compact support with the usual inductive limit topology. In the special case where a uniform measure exists, we have the following result.

THEOREM 2. *Let (M,d) be a locally compact metric space and u a uniform measure on M. If m is a signed measure with $m(S(x,r))=0$ for all $x \in M$ and $r>0$ such that $m(S(x,r))$ is defined, then $m=0$.*

PROOF. The kernel function is now

$$K_\varepsilon(x,y) = (c_\varepsilon(u))^{-1} \quad \text{for} \quad d(x,y)<\varepsilon\,,$$
$$= 0 \quad \text{for} \quad d(x,y)\geqq \varepsilon\,,$$

where $c_\varepsilon(u)=u(S(x,\varepsilon))$. Using the same argument as in the proof of theorem 1 we find that

$$\lim_{\varepsilon\to 0} \int_M K_\varepsilon\varphi(x)\,dm(x) = \int_M \varphi(x)\,dm(x)\,,$$

where φ is continuous and of compact support. Fubini's theorem shows that $\int_M K_\varepsilon\varphi(x)\,dm(x)=0$ for all ε, hence $\int_M \varphi(x)\,dm(x)=0$, and theorem 2 is proved.

We call a positive measure u on (M,d) an *almost uniform measure* if, for every compact set $K\subseteq M$,

$$\lim_{\varepsilon\to 0} u(S(x,\varepsilon))/u(S(y,\varepsilon)) = 1 \quad \text{uniformly in } (x,y)\in K^2\,.$$

With only minor modifications the proof of the uniqueness theorem 1 carries over and we have the result that there is at most one almost uniform measure.

In the case of a Riemannian space the well-known Riemannian measure is almost uniform with respect to the Riemann metric.

A positive measure u is called *relatively uniform with modulus $\Delta : M^2 \to \mathbf{R}$* if

$$\forall x,k\in M\ \forall r>0:\ \Delta(x,y)\,u(S(x,r)) = u(S(y,r))\,.$$

One easily shows that Δ must be strictly positive, continuous and satisfy

$$\Delta(x,y)\Delta(y,z) = \Delta(x,z) \quad \text{for all } x,y,z \in M .$$

Hence Δ has the form $\Delta(x,y)=\varphi(y)/\varphi(x)$, where φ is unique up to a positive factor and continuous. The measure $\hat{u}$ with density $(\varphi(x))^{-1}$ with respect to u can easily be shown to be almost uniform. Applying the uniqueness theorem for almost uniform measures we obtain the following result:

If u and v are relatively uniform measures with modulus $\Delta(x,y)=\varphi(y)/\varphi(x)$, then there exists $\lambda \in \mathsf{R}_+$ such that $v=\lambda u$. If m is a relatively uniform measure with modulus $\Delta'(x,y)=\psi(y)/\psi(x)$, then m has density $\psi(x)/\varphi(x)$ with respect to u (up to a positive factor of proportionality).

Acknowledgements. In the development of the present results I had helpful discussions with F. Topsøe, B. Fuglede and C. Berg.

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