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EXTENSIONS OF DEDEKIND'S ψ-FUNCTION

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1. Introduction.

In 1877 R. Dedekind [4, p. 288] (cf. also [5, p. 123]) established the arithmetical form of $\psi(n)=\sum_{d\delta=n} d g^{-1}\varphi(g)$, where $g=(d,\delta)$ and $\varphi(n)$ is the Euler totient function. He proved that

$$\psi(n) = n \prod_{p|n}(1+p^{-1}) , \tag{1.1}$$

the product being extended over all prime divisors p of n. It is clear that $\psi(n)=\sum_{d\delta=n}\mu^2(d)\,\delta$, where $\mu(n)$ is the Möbius function. Recently, E. Cohen [3] proved an interesting result, viz.,

$$\sum_{n\in L}\psi(n)^{-1} = \pi^2/6 , \tag{1.2}$$

where L is the set of integers whose prime factors are all multiple (see Remark 2.2 below).

In this paper we define three arithmetical functions $\psi_k(n)$, $\Psi_k(n)$ and $\psi_{(k)}(n)$ as generalizations of $\psi(n)$ and establish their arithmetical forms, arithmetical identities, orders and average orders. Further, we prove two results of type (1.2).

2. Notation and preliminaries.

Let k be a fixed positive integer. Let $\varphi_k(n)$ and $\Phi_k(n)$ be Cohen's (cf. [1], also cf. [2]) and Klee's totient functions [7], respectively. They have the following arithmetical forms:

$$\varphi_k(n) = n^k \prod_{p|n}(1-p^{-k}) , \tag{2.1}$$

$$\Phi_k(n) = n \prod_{p^k|n}(1-p^{-k}) . \tag{2.2}$$

We need the following property of $\varphi_k(n)$ (cf. [1, Theorem 1]):

$$\sum_{d|n}\varphi_k(d) = n^k . \tag{2.3}$$

Let $\theta(n)$ denote the number of square-free divisors of n. It is clear that $\theta(n)=2^{w(n)}$, where $w(n)$ is the number of distinct prime factors of

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$n>1$, and $w(1)=0$. We define the functions $\psi_k(n)$, $\Psi_k(n)$ and $\psi_{(k)}(n)$ as follows:

$$(2.4) \qquad \psi_k(n) = \sum_{d\delta=n} d^k g^{-k} \varphi_k(g), \quad \text{where } g = (d,\delta),$$

$$(2.5) \qquad \Psi_k(n) = \sum_{d\delta=n} \varepsilon_k(d) \Phi_k(\delta),$$

where

$$(2.6) \qquad \varepsilon_k(n) = \theta(n) \text{ or } 0 \text{ according as } n \text{ is or is not the } k\text{th power of an integer}.$$

Clearly, $\varepsilon_1(n)=\theta(n)$. Further

$$(2.7) \qquad \psi_k(n) = \prod_{i=1}^{r} \left\{ \psi(p_i^{\alpha_i}) + \binom{k-1}{1} \psi(p_i^{\alpha_i-1}) + \ldots + \binom{k-1}{\alpha_i} \right\}$$

and $\psi_{(k)}(1)=1$, where

$$(2.8) \qquad n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} \text{ is the canonical representation of } n,$$

and

$$\binom{s}{t} = \frac{s(s-1)\ldots(s-t+1)}{1\cdot 2\cdot \ldots \cdot t}.$$

Clearly, $\binom{s}{t}=0$ if $0 \leqq s < t$, and

$$(2.9) \qquad \binom{s}{t} + \binom{s}{t+1} = \binom{s+1}{t+1}.$$

REMARK 2.1. It is clear that $\psi_1(n)=\psi_{(1)}(n)=\psi(n)$. The fact that $\Psi_1(n)=\psi(n)$ is noted in Remark 3.1 below.

The following notation and terminology is needed in our present discussion: A positive integer n is said to be *k-free* if n is not divisible by the kth power of any integer >1. Clearly, 1 is k-free and if n is of the form (2.8), then n is k-free if $\alpha_i < k$ for each i. On the other hand, n is said to be *k-full* if $\alpha_i \geqq k$ for each i. 1 is also considered to be k-full. Let Q_k and L_k be the sets of k-free and k-full integers respectively.

REMARK 2.2. The set L of (1.2) is the set L_2 of 2-full integers.

We note that any positive integer n can be uniquely written as $n=n_1 n_2$, where $(n_1,n_2)=1$ and $n_1 \in Q_k$, $n_2 \in L_k$. We define an arithmetical function $\varrho_k(n)$ by setting

$$(2.10) \qquad \varrho_k(n) = \theta(n_2).$$

Since every positive integer is 1-full, $\varrho_1(n)=\theta(n)$. We introduce another function $\varrho_{(k)}(n)$ defined thus: $\varrho_{(k)}(1)=1$ and if n is of the form (2.8), then

$$\varrho_{(k)}(n) = \prod_{i=1}^{r} \binom{k}{\alpha_i}. \tag{2.11}$$

Clearly, $\varrho_{(1)}(n)=\mu^2(n)$.

Finally, we need the function $\mu_k(n)$ introduced by Klee [7] as a generalization of the Möbius μ-function, defined thus: $\mu_k(1)=1$, $\mu_k(n)=(-1)^r$ if n is of the form (2.8) with $\alpha_i=k$ for each i and $\mu_k(n)=0$, otherwise.

REMARK 2.3. We note that $\varepsilon_k(n)$, $\varrho_k(n)$, $\varrho_{(k)}(n)$ and $\mu_k(n)$ are multiplicative.

3. Arithmetical properties.

In this section we establish the arithmetical forms of $\psi_k(n)$, $\Psi_k(n)$, $\psi_{(k)}(n)$ and some arithmetical identities. We prove first

THEOREM 3.1. *$\psi_k(n)$ is multiplicative.*

PROOF. By (2.4) it is clear that $\psi_k(1)=1$. Let $(m,n)=1$. Every divisor d of mn can be expressed as d_1d_2, where $d_1|m$, $d_2|n$ and $(d_1,d_2)=1$. Further, $((d_1,m/d_1),(d_2,n/d_2))=1$ and $(d_1d_2,mn/d_1d_2)=(d_1,m/d_1)(d_2,n/d_2)$.

Hence by (2.4) and the multiplicative property of $\varphi_k(n)$,

$$\begin{aligned}
\psi_k(mn) &= \textstyle\sum_{d|mn} d^k (d,mn/d)^{-k} \varphi_k((d,mn/d)) \\
&= \textstyle\sum_{d_1|m,\, d_2|n} d_1^k d_2^k (d_1,m/d_1)^{-k} (d_2,n/d_2)^{-k} \varphi_k((d_1,m/d_1)(d_2,n/d_2)) \\
&= \textstyle\sum_{d_1|m,\, d_2|n} d_1^k (d_1,m/d_1)^{-k} \varphi_k((d_1,m/d_1))\, d_2^k (d_2,n/d_2)^{-k} \varphi_k((d_2,n/d_2)) \\
&= \textstyle\sum_{d_1|m} d_1^k (d_1,m/d_1)^{-k} \varphi_k((d_1,m/d_1)) \cdot \sum_{d_2|n} d_2^k (d_2,n/d_2)^{-k} \varphi_k((d_2,n/d_2)) \\
&= \psi_k(m)\, \psi_k(n)\,.
\end{aligned}$$

Hence $\psi_k(n)$ is multiplicative.

COROLLARY 3.1.1. $\psi_k(n)=n^k \prod_{p|n}(1+p^{-k})$.

PROOF. In virtue of Theorem 3.1, it is enough if we prove this corollary for $n=p^\alpha$, a prime power. By definition (2.4),

$$\psi_k(p^\alpha) = \textstyle\sum_{d\delta=p^\alpha} d^k g^{-k} \varphi_k(g), \quad \text{where } g = (d,\delta)\,.$$

By (2.1),

$$\begin{aligned}
\psi_k(p^\alpha) &= 1+(p^k-1)+(p^{2k}-p^k)+\ldots+(p^{(\alpha-1)k}-p^{(\alpha-2)k})+p^{\alpha k} \\
&= p^{\alpha k}+p^{(\alpha-1)k} = p^{\alpha k}(1+p^{-k})\,.
\end{aligned}$$

Hence the corollary follows.

THEOREM 3.2. $\Psi_k(n) = n \prod_{p^k|n}(1+p^{-k})$.

PROOF. By definition (2.5), $\Psi_k(n) = \sum_{d\delta=n} \varepsilon_k(d)\Phi_k(\delta)$. Since $\varepsilon_k(n)$ and $\Phi_k(n)$ are multiplicative, $\Psi_k(n)$ is also multiplicative, by the calculus of multiplicative arithmetical functions (cf. [8, § 7.2, Theorem 3]). Hence, it is enough if we prove the theorem for $n = p^\alpha$, a prime power.

By (2.2) and (2.6), if $\alpha < k$,

$$\Psi_k(p^\alpha) = \sum_{d\delta=p^\alpha} \varepsilon_k(d)\Phi_k(\delta) = \Phi_k(p^\alpha) = p^\alpha$$

and if $\alpha \geqq k$, then $0 \leqq \alpha - [\alpha/k]k < k$ so that

$$\begin{aligned}\Psi_k(p^\alpha) &= \Phi_k(p^\alpha) + 2\{\Phi_k(p^{\alpha-k}) + \Phi_k(p^{\alpha-2k}) + \ldots + \Phi_k(p^{\alpha-[\alpha/k]k})\} \\ &= (p^\alpha - p^{\alpha-k}) + 2\left\{\sum_{i=1}^{[\alpha/k]-1} (p^{\alpha-ik} - p^{\alpha-(i+1)k}) + p^{\alpha-[\alpha/k]k}\right\} \\ &= p^\alpha - p^{\alpha-k} + 2p^{\alpha-k} = p^\alpha + p^{\alpha-k} = p^\alpha(1+p^{-k}) .\end{aligned}$$

Hence the theorem follows.

COROLLARY 3.2.1. $\Psi_k(n^k) = \psi_k(n)$.

This follows by Corollary 3.1.1 and Theorem 3.2.

REMARK 3.1. It is clear by Theorem 3.2 and (1.1) that $\Psi_1(n) = \psi(n)$.

THEOREM 3.3. *If n is of the form* (2.8), *then*

$$\psi_{(k)}(n) = \prod_{i=1}^{r} \left\{p^{\alpha_i} + \binom{k}{1}p^{\alpha_i-1} + \ldots + \binom{k}{\alpha_i}\right\}.$$

PROOF. It is clear by definition (2.7) that $\psi_{(k)}(n)$ is multiplicative. Hence it is enough if we prove the theorem for $n = p^\alpha$, a prime power.

By (2.7), (1.1) and (2.9)

$$\begin{aligned}\psi_{(k)}(p^\alpha) &= \psi(p^\alpha) + \binom{k-1}{1}\psi(p^{\alpha-1}) + \ldots + \binom{k-1}{\alpha} \\ &= (p^\alpha + p^{\alpha-1}) + \binom{k-1}{1}(p^{\alpha-1} + p^{\alpha-2}) + \ldots + \binom{k-1}{\alpha} \\ &= p^\alpha + \binom{k}{1}p^{\alpha-1} + \binom{k}{2}p^{\alpha-2} + \ldots + \binom{k}{\alpha} .\end{aligned}$$

Hence the theorem follows.

We now prove some arithmetical identities. We prove each of the following identities only for $n=p^{\alpha}$, a prime power, since all the arithmetical functions involved in the identities are multiplicative.

THEOREM 3.4. $\psi_k(n)=\sum_{d\delta=n}\mu^2(d)\,\delta^k$.

PROOF. By the definition of $\mu(n)$ and Corollary 3.1.1,

$$\begin{aligned}\sum_{d\delta=p^\alpha}\mu^2(d)\,\delta^k &= \mu^2(1)\,p^{\alpha k}+\mu^2(p)\,p^{(\alpha-1)k}\\ &= p^{\alpha k}+p^{(\alpha-1)k} = \psi_k(p^\alpha)\,.\end{aligned}$$

Hence the theorem follows.

THEOREM 3.5. $\psi_k(n)=\sum_{d\delta=n}\theta(d)\,\varphi_k(\delta)$.

PROOF. By the definition of $\theta(n)$, (2.1), (2.3) and Corollary 3.1.1,

$$\begin{aligned}\sum_{d\delta=p^\alpha}\theta(d)\,\varphi_k(\delta) &= \varphi_k(p^\alpha)+2\sum_{\delta|p^{\alpha-1}}\varphi_k(\delta)\\ &= (p^{\alpha k}-p^{(\alpha-1)k})+2p^{(\alpha-1)k}\\ &= p^{\alpha k}+p^{(\alpha-1)k} = \psi_k(p^\alpha)\,.\end{aligned}$$

Hence the theorem follows.

THEOREM 3.6. $\Psi_k(n)=\sum_{d\delta=n}\mu_k{}^2(d)\delta$.

PROOF. By the definition of $\mu_k(n)$ and Theorem 3.2,

$$\begin{aligned}\sum_{d\delta=p^\alpha}\mu_k{}^2(d)\,\delta &= p^\alpha & &\text{if } \alpha<k,\\ &= p^\alpha+p^{\alpha-k} & &\text{if } \alpha\geqq k\,,\end{aligned}$$

so that in any case

$$\sum_{d\delta=p^\alpha}\mu_k{}^2(d)\,\delta = \Psi_k(p^\alpha)\,.$$

Hence the theorem follows.

THEOREM 3.7. $\Psi_k(n)=\sum_{d^k\delta=n}\mu^2(d)\,\delta$.

PROOF. $\sum_{d^k\delta=n}\mu^2(d)\,\delta$ is multiplicative (cf. [9, Lemma 2.4]) and

$$\begin{aligned}\sum_{d^k\delta=p^\alpha}\mu^2(d)\,\delta &= p^\alpha & &\text{if } \alpha<k,\\ &= p^\alpha+p^{\alpha-k} & &\text{if } \alpha\geqq k\,.\end{aligned}$$

Hence the theorem follows.

THEOREM 3.8. $\Psi_k(n)=\sum_{d\delta=n}\varrho_k(d)\,\varphi(\delta)$.

PROOF. By the definition of $\varrho_k(n)$ in (2.10) and $\sum_{d|n}\varphi(d)=n$, we have if $\alpha<k$,

$$\sum_{d\delta=p^\alpha} \varrho_k(d)\varphi(\delta) = \sum_{d\delta=p^\alpha} \varphi(\delta) = p^\alpha ,$$

and if $\alpha \geqq k$,

$$\begin{aligned}\sum_{d\delta=p^\alpha} \varrho_k(d)\varphi(\delta) &= \varphi(p^\alpha) + \ldots + \varphi(p^{\alpha-k+1}) + \\ &\quad + 2\{\varphi(p^{\alpha-k}) + \ldots + \varphi(1)\} \\ &= \sum_{d\delta=p^\alpha} \varphi(\delta) + \sum_{d\delta=p^{\alpha-k}} \varphi(\delta) \\ &= p^\alpha + p^{\alpha-k} .\end{aligned}$$

Hence in any case,

$$\sum_{d\delta=p^\alpha} \varrho_k(d)\varphi(\delta) = \Psi_k(p^\alpha) ,$$

by Theorem 3.2. Hence the theorem follows.

THEOREM 3.9. $\psi_{(k)}(n) = \sum_{d\delta=n} \varrho_{(k)}(d)\,\delta$.

PROOF. By the definition of $\varrho_{(k)}(n)$ in (2.11) and Theorem 3.3,

$$\sum_{d\delta=p^\alpha} \varrho_k(d)\,\delta = p^\alpha + \binom{k}{1} p^{\alpha-1} + \ldots + \binom{k}{\alpha} = \psi_{(k)}(p^\alpha) .$$

Hence the theorem follows.

THEOREM 3.10. *If* $k \geqq 2$, *then* $\psi_{(k)}(n) = \sum_{d\delta=n} \mu^2(d)\,\psi_{(k-1)}(\delta)$.

PROOF. By (2.9) and Theorem 3.3,

$$\begin{aligned}\sum_{d\delta=p^\alpha} \mu^2(d)\,\psi_{(k-1)}(\delta) &= \psi_{(k-1)}(p^\alpha) + \psi_{(k-1)}(p^{\alpha-1}) \\ &= \left\{p^\alpha + \binom{k-1}{1} p^{\alpha-1} + \ldots + \binom{k-1}{\alpha}\right\} + \\ &\quad + \left\{p^{\alpha-1} + \binom{k-1}{1} p^{\alpha-2} + \ldots + \binom{k-1}{\alpha-1}\right\} \\ &= p^\alpha + \binom{k}{1} p^{\alpha-1} + \ldots + \binom{k}{\alpha} \\ &= \psi_{(k)}(p^\alpha) .\end{aligned}$$

Hence the theorem follows.

THEOREM 3.11. *If* $k \geqq 2$, *then* $\psi_{(k)}(n) = \sum_{d\delta=n} \varrho_{(k-1)}(d)\psi(\delta)$.

PROOF. By (2.11) and (2.7),

$$\begin{aligned}\sum_{d\delta=p^\alpha} \varrho_{(k-1)}(d)\,\psi(\delta) &= \psi(p^\alpha) + \binom{k-1}{1} \psi(p^{\alpha-1}) + \binom{k-1}{\alpha} \\ &= \psi_{(k)}(p^\alpha) .\end{aligned}$$

Hence the theorem follows.

THEOREM 3.12. *If* $k \geqq 2$, *then* $\varrho_{(k)}(n) = \sum_{d\delta=n} \mu^2(d)\,\varrho_{(k-1)}(\delta)$.

PROOF. By (2.11) and (2.9),

$$\sum_{d\delta=p^\alpha} \mu^2(d)\,\varrho_{(k-1)}(\delta) = \binom{k-1}{\alpha} + \binom{k-1}{\alpha-1} = \binom{k}{\alpha} = \varrho_{(k)}(p^\alpha)\,.$$

Hence the theorem follows.

REMARK 3.2. Theorems 3.4 to 3.12 can also be obtained by using generating series of arithmetical functions involved and multiplication of Dirichlet series.

4. Orders and Average orders.

In this section we establish the orders and average orders of $\psi_k(n)$, $\Psi_k(n)$ and $\psi_{(k)}(n)$.

THEOREM 4.1. *For every* $\varepsilon > 0$,

$$\psi_k(n) = O(n^{k+\varepsilon})\,, \tag{4.1}$$

$$\Psi_k(n) = O(n^{1+\varepsilon})\,, \tag{4.2}$$

$$\psi_{(k)}(n) = O(n^{1+\varepsilon})\,. \tag{4.3}$$

PROOF. We first prove (4.2). Writing $f(n) = \Psi_k(n)/n^{1+\varepsilon}$, we see that $f(n)$ is multiplicative.

If $m < k$,

$$f(p^m) = \Psi_k(p^m)/p^{m(1+\varepsilon)} = p^m/p^{m(1+\varepsilon)} = 1/(p^m)^\varepsilon \to 0 \quad \text{as } p^m \to \infty\,.$$

If $m \geqq k$,

$$f(p^m) = p^m(1+p^{-k})/p^{m(1+\varepsilon)} = (1+p^{-k})/p^{m\varepsilon} \leqq (1+2^{-k})/(p^m)^\varepsilon \to 0 \quad \text{as } p^m \to \infty\,.$$

Thus in any case, $f(p^m) \to 0$ as $p^m \to \infty$. Hence $f(n) \to 0$ as $n \to \infty$ (cf. [6, Theorem 316]), so that (4.2) follows.

Now, (4.1) is an immediate consequence of Corollary 3.2.1 and (4.2).

To prove (4.3), we write $g(n) = \psi_{(k)}(n)/n^{1+\varepsilon}$. This function $g(n)$ is multiplicative, and by Theorem 3.3,

$$g(p^m) = \frac{p^m + \binom{k}{1} p^{m-1} + \ldots + \binom{k}{m}}{p^{m(1+\varepsilon)}} = \frac{1 + \binom{k}{1} p^{-1} + \ldots + \binom{k}{m} p^{-m}}{p^{m\varepsilon}}$$

$$\leqq \frac{(1+p^{-1})^k}{p^{m\varepsilon}} \quad \text{(equality if } m \geqq k \text{ and inequality if } m < k)$$

$$\leqq \frac{(1+2^{-1})^k}{(p^m)^\varepsilon} \to 0 \quad \text{as } p^m \to \infty .$$

Hence $g(n) \to 0$ as $n \to \infty$ (cf. [6, Theorem 316]), so that (4.3) follows. Thus the theorem is proved.

Throughout the following x denotes a real variable $\geqq 2$ and $\zeta(s)$ denote the usual Riemann Zeta function. We need the well-known (cf. [6, Theorem 302]) result, viz.,

$$\text{(4.4)} \qquad \sum_{n=1}^{\infty} \mu^2(n) n^{-s} = \zeta(s)/\zeta(2s) \quad \text{for } s > 1 .$$

THEOREM 4.2. *The average order* (cf. [6, § 18.2]) *of* $\psi_k(n)$ *is* $n^k \zeta(k+1)/\zeta(2k+2)$. *More precisely,*

$$\text{(4.5)} \qquad \sum_{n \leqq x} \psi_k(n) = \frac{\zeta(2) x^2}{2\zeta(4)} + O(x \log x) \qquad \text{if } k=1,$$

$$= \frac{\zeta(k+1) x^{k+1}}{(k+1)\zeta(2k+2)} + O(x^k) \quad \text{if } k \geqq 2.$$

PROOF. By Theorem 3.4,

$$\sum_{n \leqq x} \psi_k(n) = \sum_{n \leqq x} \sum_{d\delta = n} \mu^2(d) \delta^k = \sum_{d\delta \leqq x} \mu^2(d) \delta^k$$

$$= \sum_{d \leqq x} \mu^2(d) \sum_{\delta \leqq x/d} \delta^k$$

$$= \sum_{d \leqq x} \mu^2(d) \left\{ \int_1^{x/d} t^k \, dt + O(x^k/d^k) \right\}$$

$$= \sum_{d \leqq x} \mu^2(d) \left\{ \frac{x^{k+1}}{(k+1) d^{k+1}} + O(x^k/d^k) \right\}$$

$$= \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^{k+1}} + O\left(x^{k+1} \sum_{d > x} d^{-(k+1)}\right) + O\left(x^k \sum_{d \leqq x} d^{-k}\right) .$$

The first O-term is $O(x)$ and the second O-term is $O(x \log x)$ or $O(x^k)$ according as $k=1$ or $k \geqq 2$.

Hence (4.5) follows by (4.4).

THEOREM 4.3. *The average order of* $\Psi_k(n)$ *is* $n\zeta(2k)/\zeta(4k)$. *More precisely,*

$$(4.6) \qquad \sum_{n \leqq x} \Psi_k(n) = \tfrac{1}{2}x^2 \zeta(2)/\zeta(4) + O(x \log x) \quad \text{if } k=1\,,$$
$$= \tfrac{1}{2}x^2 \zeta(2k)/\zeta(4k) + O(x) \quad \text{if } k \geqq 2\,.$$

PROOF. By Theorem 3.7,

$$\begin{aligned} \sum_{n \leqq x} \Psi_k(n) &= \sum_{n \leqq x} \sum_{d^k \delta = n} \mu^2(d)\,\delta \\ &= \sum_{d^k \delta \leqq x} \mu^2(d)\,\delta \\ &= \sum_{d^k \leqq x} \mu^2(d) \sum_{\delta \leqq x/d^k} \delta \\ &= \sum_{d \leqq x^{1/k}} \mu^2(d) \{\tfrac{1}{2}([x/d^k]^2 + [x/d^k])\} \\ &= \tfrac{1}{2} \sum_{d \leqq x^{1/k}} \mu^2(d) \{x^2/d^{2k} + O(x/d^k)\} \\ &= \tfrac{1}{2}x^2 \sum_{d=1}^{\infty} \mu^2(d)/d^{2k} + O(x^2 \sum_{d > x^{1/k}} d^{-2k}) + O(x \sum_{d \leqq x^{1/k}} d^{-k})\,. \end{aligned}$$

The first O-term is $O(x^{1/k})$ and the second O-term is $O(x \log x)$ or $O(x)$ according as $k=1$ or $k \geqq 2$.

Hence (4.6) follows by (4.4).

THEOREM 4.4. *The average order of* $\psi_{(k)}(n)$ *is* $n\zeta^k(2)/\zeta^k(4) = n(15/\pi^2)^k$. *More precisely,*

$$(4.7) \qquad \sum_{n \leqq x} \psi_{(k)}(n) = \tfrac{1}{2}x^2 \zeta^k(2)/\zeta^k(4) + O(x \log^k x)\,.$$

To prove this theorem we need the following

LEMMA. *For* $h > 0$,

$$\sum_{n \leqq x} xn^{-1} \log^h (xn^{-1}) = O(x \log^{h+1} x)\,.$$

PROOF. Since $t \log^h t$ is an increasing function of t, we have, for $n \geqq 2$,

$$xn^{-1} \log^h (xn^{-1}) \leqq \int_{n-1}^{n} x t^{-1} \log^h (xt^{-1})\, dt\,.$$

Hence

$$\sum_{2\leqq n\leqq x} xn^{-1}\log^h(xn^{-1}) \leqq \int_1^x xt^{-1}\log^h(xt^{-1})\,dt$$
$$= x\int_1^x u^{-1}\log^h u\,du$$
$$= O(x\log^{h+1}x)\,.$$

Hence the lemma follows.

PROOF OF (4.7). We prove this by induction on k. It follows by (4.5) that (4.7) is true for $k=1$. Let $k\geqq 2$. Assuming (4.7) for $k-1$, we prove it for k. By the induction assumption, we have

$$\sum_{n\leqq x}\psi_{(k-1)}(n) = \tfrac{1}{2}x^2\zeta^{k-1}(2)/\zeta^{k-1}(4) + O(x\log^{k-1}x)\,. \tag{4.8}$$

By Theorem 3.10,

$$\begin{aligned}\sum_{n\leqq x}\psi_{(k)}(n) &= \sum_{n\leqq x}\sum_{d\delta=n}\mu^2(d)\,\psi_{(k-1)}(\delta)\\ &= \sum_{d\delta\leqq x}\mu^2(d)\,\psi_{(k-1)}(\delta)\\ &= \sum_{d\leqq x}\mu^2(d)\sum_{\delta\leqq x/d}\psi_{(k-1)}(\delta)\,,\end{aligned}$$

so that by (4.8),

$$\begin{aligned}\sum_{n\leqq x}\psi_{(k)}(n) &= \sum_{d\leqq x}\mu^2(d)\{\tfrac{1}{2}x^2d^{-2}\zeta^{k-1}(2)/\zeta^{k-1}(4) + O(xd^{-1}\log^{k-1}(xd^{-1}))\}\\ &= (\tfrac{1}{2}x^2\zeta^{k-1}(2)/\zeta^{k-1}(4))\sum_{d=1}^{\infty}d^{-2}\mu^2(d) + O(x^2\sum_{d>x}d^{-2}) +\\ &\qquad + O(\sum_{d\leqq x}xd^{-1}\log^{k-1}(xd^{-1}))\,.\end{aligned}$$

The first O-term is $O(x)$ and the second O-term is $O(x\log^k x)$, by the above lemma. Hence by (4.4), it follows that (4.7) is true for k. Thus (4.7) is proved.

5. Two special properties.

In this section we establish two results of type (1.2).

THEOREM 5.1. $\sum_{n\in L_2}\psi_k(n)^{-1}=\zeta(2k)$, *where* L_2 *is the set of all 2-full integers.*

PROOF. If $a(n)$ is the characteristic function of L_2 (that is, $a(n)=1$ or 0 according as $n\in L_2$ or $n\notin L_2$), then

$$\sum_{n\in L_2}\psi_k(n)^{-1} = \sum_{n=1}^{\infty}a(n)\,\psi_k(n)^{-1}\,.$$

This series is absolutely convergent by (4.1), and hence it may be expanded into an infinite product of Euler type (cf. [6, § 17.4]). By Corollary 3.1.1,

$$\sum_{n=1}^{\infty}\frac{a(n)}{\psi_k(n)}=\prod_p\left\{\sum_{m=0}^{\infty}\frac{a(p^m)}{\psi_k(p^m)}\right\}=\prod_p\left\{1+\sum_{m=2}^{\infty}\frac{p^{-mk}}{1+p^{-k}}\right\}$$

$$=\prod_p\left\{1+\frac{1}{1+p^{-k}}\cdot\frac{p^{-2k}}{1-p^{-k}}\right\}$$

$$=\prod_p\left\{\frac{1}{(1-p^{-2k})}\right\}=\zeta(2k)\,.$$

Hence the theorem follows.

THEOREM 5.2. *$\sum_{n\in L_4}\psi_k(n)^{-1}=\zeta(4k)\zeta(6k)/\zeta(12k)$, where L_4 is the set of all 4-full integers.*

PROOF. If $b(n)$ is the characteristic function of L_4, then

$$\sum_{n\in L_4}\psi_k(n)^{-1}=\sum_{n=1}^{\infty}\frac{b(n)}{\psi_k(n)}.$$

By the same reasoning as in Theorem 5.1,

$$\sum_{n=1}^{\infty}\frac{b(n)}{\psi_k(n)}=\prod_p\left\{\sum_{m=0}^{\infty}\frac{b(p^m)}{\psi_k(p^m)}\right\}$$

$$=\prod_p\left\{1+\sum_{m=4}^{\infty}\frac{p^{-mk}}{1+p^{-k}}\right\}$$

$$=\prod_p\left\{1+\frac{1}{1+p^{-k}}\cdot\frac{p^{-4k}}{1-p^{-k}}\right\}$$

$$=\prod_p\left\{\frac{1-p^{-2k}+p^{-4k}}{1-p^{-2k}}\right\}=\prod_p\left\{\frac{1+p^{-6k}}{1-p^{-4k}}\right\}$$

$$=\prod_p\left\{\frac{1-p^{-12k}}{(1-p^{-4k})(1-p^{-6k})}\right\}=\frac{\zeta(4k)\zeta(6k)}{\zeta(12k)}.$$

Hence the theorem follows.

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