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FULL SETS OF STATES ON A C^*-ALGEBRA

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§ 1. Introduction.

This note contains a discussion of necessary and sufficient conditions on a subset S_0 of the set S of states on a C^*-algebra A, for its convex hull $\langle S_0 \rangle$ to be w^*-dense in S. In the case where A has an identity, results to this effect have been known for some time [1], [2], [3]. If A does not have an identity, the situation is a little more complicated, and will be dealt with on the following pages.

The set of states on a C^*-algebra A is the family of all positive linear functionals on A of norm equal to one. If A has an identity, S is convex and compact in the w^*-topology as a subset of A^*. If A has no identity, S is still convex, which may be proved by means of an approximate identity for A, but S is no longer compact. In this case it is therefore often convenient to introduce the set N of all positive linear functionals on A with norm $\leqq 1$. Then N is compact, and S is a subset of N which is a face, i.e., if $\varrho_1, \varrho_2 \in N$ and $\varrho = \lambda\varrho_1 + (1-\lambda)\varrho_2 \in S$, $0 < \lambda < 1$, then ϱ_1 and ϱ_2 both belong to S. Let $\tilde{A}$ denote the C^*-algebra obtained by adjoining an identity to A, and let $\tilde{S}$ be the set of states on $\tilde{A}$. For $\varrho \in N$ let $\tilde{\varrho}(a+\lambda 1) = \varrho(a) + \lambda$, $a \in A$, $\lambda \in \mathsf{C}$. The map $\varrho \to \tilde{\varrho}$ is a bicontinuous, affine isomorphism of N onto $\tilde{S}$, and we have $\tilde{\varrho} \mid A = \varrho$.

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§ 2. Full sets of states.

For the case where A has an identity, the facts are recorded in Theorem 1 below. The essential condition, introduced by Kadison [3], is the following: A family S_0 of positive linear functionals on a C^*-algebra is *full* if, for $a \in A$, $\varrho(a) \geqq 0$ for all $\varrho \in S$ implies that $a \geqq 0$.

In Theorem 1, the equivalence of conditions (i), (ii) and (iii) is known,

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see [1] or [3]. That condition (iv) is equivalent to the others may also be known, but this is not to the author's knowledge.

THEOREM 1. *Let A have an identity. The following conditions on a subset S_0 of S are equivalent:*

(i) S_0 *is full.*

(ii) $\overline{S_0} \supseteqq \partial_e S$ *(the set of extreme points of S).*

(iii) $\overline{\langle S_0 \rangle} = S$.

(iv) $\|a\| = \sup_{\varrho \in S_0} \varrho(a)$ *for any* $a \geqq 0$, $a \in A$.

PROOF. For (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) we refer to [1, 2.6.2 and 3.4.1]. If $a \geqq 0$, then $\|a\| = \varrho(a)$ for some $\varrho \in \partial_e S$, so (ii) $\Rightarrow$ (iv). To finish the proof, we show that (iv) $\Rightarrow$ (i). We do this through two lemmas.

LEMMA 1. *Let A be a C^*-algebra. Suppose $S_0 \subseteqq N$ and that S_0 satisfies* (iv) *above. Let $a \in A_h$, $a^+ \neq 0$, be given. Then, for any $\varepsilon > 0$, there is $\varrho \in S_0$ such that*

$$\varrho(a^+) \geqq \|a^+\| - \varepsilon, \quad \varrho(a^-) \leqq \varepsilon .$$

PROOF. Let $\varepsilon > 0$ be given. We have $a = a^+ - a^-$, and $\|a\| = \max(\|a^+\|, \|a^-\|)$. Suppose first $\|a\| = \|a^+\|$, and choose $\varrho \in S_0$ such that $\varrho(a^+) \geqq \|a^+\| - \varepsilon$. Then

$$\|a^+\| = \|a^+ + a^-\| \geqq \varrho(a^+ + a^-) = \varrho(a^+) + \varrho(a^-) \geqq \|a^+\| - \varepsilon + \varrho(a^-) .$$

Hence $\varrho(a^-) \leqq \varepsilon$. If $\|a\| = \|a^-\|$, let $b = a^+ - (\|a^+\|/\|a^-\|)a^-$, so $\|b\| = \|a^+\|$. Then choose $\varrho \in S_0$ such that

$$\varrho(b^+) = \varrho(a^+) \geqq \|a^+\| - (\|a^+\|/\|a^-\|)\varepsilon \geqq \|a^+\| - \varepsilon .$$

Then, by the first result

$$\varrho(b^-) = (\|a^+\|/\|a^-\|)\varrho(a^-) \leqq (\|a^+\|/\|a^-\|)\varepsilon ,$$

so $\varrho(a^-) \leqq \varepsilon$. The proof is complete.

LEMMA 2. *Let A be a C^*-algebra. If $S_0 \subseteqq N$ and $\|a\| = \sup_{\varrho \in S_0} \varrho(a)$ for any $a \geqq 0$ in A, then S_0 is full.*

PROOF. First, let $a \in A_h$ and suppose $\varrho(a) \geqq 0$ for all $\varrho \in S_0$. Suppose $a^- \neq 0$, and apply Lemma 1 to $b = -a$. We can find $\varrho \in S_0$ such that

$$\varrho(a^-) \geqq \tfrac{3}{4}\|a^-\|, \qquad \varrho(a^+) \leqq \tfrac{1}{4}\|a^-\| .$$

But then $\varrho(a)=\varrho(a^+)-\varrho(a^-)\leqq\frac{1}{4}\|a^-\|-\frac{3}{4}\|a^-\|=-\frac{1}{2}\|a^-\|<0$, a contradiction. Hence $a\geqq 0$.

Now let $a\in A$ be arbitrary, with $\varrho(a)\geqq 0$ for all $\varrho\in S_0$. Then $\varrho(a^*)=\overline{\varrho(a)}=\varrho(a)\geqq 0$, so $\varrho(\mathrm{Im}\,a)=0$, and $\varrho(a)=\varrho(\mathrm{Re}\,a)\geqq 0$ for all $\varrho\ \varepsilon\ S_0$. Hence, by the first part of the proof $\mathrm{Re}\,a\geqq 0$ and both $\mathrm{Im}\,a\geqq 0$ and $-\mathrm{Im}\,a\geqq 0$ so $\mathrm{Im}\,a=0$. Consequently $a=\mathrm{Re}\,a\geqq 0$, and the proof of Lemma 2 and Theorem 1 is complete.

§ 3. Algebras without identity.

We now assume specifically that A has no identity. Let P be the pure states on A, that is, a state ϱ on A is in P if $0\leqq\varphi\leqq\varrho$, $\varphi\in N$, implies $\varphi=\lambda\varrho$ for some λ between 0 and 1.

Lemma 3. (a) $P=\partial_e S$.

(b) $S\subseteqq\overline{\langle P\rangle}$ (*closure taken in* N).

Proof. (a). We know that $\partial_e N=\{0\}\cup P$ [1, 2.5.5]. Hence each pure state of A is an extreme point of S, since $P\subseteqq S\subseteqq N$. Conversely, each extreme point of S is also an extreme point of N, since S is a face of N. Since these extreme points have norm equal to one, they must belong to P by the fact quoted above. Hence $P=\partial_e S$.

(b). $\langle\{0\},P\rangle=\langle\partial_e N\rangle$ is dense in N by the Krein–Milman theorem. Hence $\langle\{0\},\overline{\langle P\rangle}\rangle$, which is compact, must be equal to N. Now S consists of the elements of N with norm equal to one, hence $S\subseteqq\overline{\langle P\rangle}$.

Lemma 4. (Glimm [1, ex. 2.12.13]). $0\in\overline{P}$.

Proof. There is no loss in generality by assuming that A is faithfully represented on a Hilbert space X, and that A operates non-degenerately on X. A is an ideal in $\tilde{A}$, so if $0\leqq x\in A$, then x is not invertible in $\tilde{A}$ since 1, the identity operator on X, does not belong to A. Hence x is not invertible in $\mathscr{L}(X)$, so

$$\inf_{\|\xi\|=1}\omega_\xi(x)\;=\;\inf_{\|\xi\|=1}(x\xi,\xi)\;=\;0,\qquad \xi\in X\,.$$

If $\{e_\nu\}$ is an approximate identity for A, then $\{e_\nu\}$ converges strongly to 1. If $\|\omega_\xi\|$ denotes the norm of ω_ξ as a linear functional on A, we have $\|\omega_\xi\|=\lim_\nu\omega_\xi(e_\nu)=\|\xi\|^2$, so $\omega_\xi\in S$ when $\|\xi\|=1$. This means that $0\in\overline{S}$, since sets

$$U(x,\varepsilon)\;=\;\{\varrho\in A^*:\;|\varrho(x)|\leqq\varepsilon;\;0\leqq x\in A,\;\varepsilon>0\}$$

form a basis for the neighbourhoods of 0 in A^*. By Lemma 3 it then follows that $0 \in \overline{\langle P \rangle}$, which clearly implies that $0 \in \overline{P}$. The proof is complete.

PROPOSITION 1. $\bar{S} = \overline{\langle P \rangle} = N$.

PROOF. Since $\langle P \rangle \subseteqq S$, the first equality follows by Lemma 3. $\bar{S}$ is closed, convex and contains $\partial_e N$ by Lemma 4, so $\bar{S} = N$.

LEMMA 5. *Let S_0 be a full subset of N. Let $a \in A_h$, $\lambda \in \mathsf{R}$, and suppose $0 \leqq \varrho(a) + \lambda$ for all $\varrho \in S$. Then $\lambda \geqq 0$.*

PROOF. Assume that $\lambda < 0$, and let $\{e_\nu\}$ be an approximate identity for A, $0 < e_\nu$, $\|e_\nu\| = 1$ for all ν. Then

$$\varrho(a + \lambda e_\nu) = \varrho(a) + \lambda \varrho(e_\nu) \geqq \varrho(a) + \lambda \geqq 0$$

for all $\varrho \in S_0$, so $a + \lambda e_\nu \geqq 0$ for all ν. Hence, for any $\varrho \in P$,

$$\varrho(a) + \lambda = \lim_\nu \varrho(a) + \lambda \varrho(e_\nu) \geqq 0 .$$

Since $0 \in \bar{P}$, it follows that $\lambda \geqq 0$, a contradiction. Hence $\lambda \geqq 0$ and the proof is complete.

COROLLARY. *If S_0 is a full subset of N, then $0 \in \bar{S}_0$.*

PROOF. If $0 \notin \bar{S}_0$, then there is an element $0 \leqq a \in A$ and a real number $\lambda > 0$ such that $\inf_{\varrho \in S_0} \varrho(a) \geqq \lambda > 0$. But then $\varrho(a) - \lambda \geqq 0$ for all $\varrho \in S_0$ which contradicts Lemma 5. The proof is complete.

It is fairly easy to see that if S_0 is a full subset of N, then $\langle S_0 \rangle$ need not be dense in N. However, with the strengthened assumption that $S_0 \subseteqq S$ one may ask whether $\langle S_0 \rangle$ will be dense in S (and then automatically in N, by Proposition 1). The following example, due to R. V. Kadison (oral communication), shows that this need not be so.

Let A be the abelian C^*-algebra of all complex sequences $\{a_n\}$ such that $a_n \to 0$ as $n \to \infty$, equipped with the norm $\|a\| = \sup_n |a_n|$, $a = \{a_n\}$. For $n \neq m$ let $\varrho_{n,m}(a) = \frac{1}{2}(a_n + a_m)$. Clearly $\varrho_{n,m} \in S$, and we take $S_0 = \{\varrho_{n,m} : n \neq m\}$. S_0 is full: Let $a \in A_h$ and suppose $a^- \neq 0$. There is an $a_m < 0$, such that

$$\lim_n \varrho_{n,m}(a) = \lim_n \tfrac{1}{2}(a_n + a_m) = \tfrac{1}{2} a_m < 0 .$$

Hence $\varrho_{n,m}(a) < 0$ for some n. Now let $\varphi \in S$ be given by $\varphi(a) = a_1$, and let $b=\{1/n\}$. Then $\varrho_{n,m}(b)=\frac{1}{2}(1/n+1/m)\leqq\frac{3}{4}$ for all $n\neq m$, so $\varrho(b)\leqq\frac{3}{4}$ for all $\varrho\in\langle S_0\rangle$. But $\varphi(b)=1$, so $|\varphi(b)-\varrho(b)|\geqq\frac{1}{4}$ for all $\varrho\in\langle S_0\rangle$, which shows that $\langle S_0\rangle$ is not dense in S.

If, however, S_0 is a full set of *pure* states on an abelian C^*-algebra A, then $\langle S_0\rangle$ is dense in S. Indeed, we have $A=\mathscr{C}_0(X)=$ the family of continuous, complex functions vanishing at infinity, on a locally compact Hausdorff space X. We may identify X with P, and if S_0 is a full subset of X, this means that a function $f\in\mathscr{C}_0(X)$ is everywhere non-negative if it is non-negative on S_0. But then clearly S_0 is dense in X, and consequently $\langle S_0\rangle$ is dense in $\overline{\langle P\rangle}\supseteqq S$.

It is interesting to note that this is not true in general. That is, even if S_0 is a full subset of P, $\langle S_0\rangle$ need not be dense in S. The following example is due to G. Kjærgård Pedersen (private communication).

Let A be the C^*-algebra of compact operators on an infinite dimensional Hilbert space X. Take an arbitrary unit vector $\eta_0\in X$, and let

$$K = \{\xi\in X:\ \|\xi\|=1, |(\xi,\eta_0)|\leqq\tfrac{1}{2}\}\ .$$

Now take S_0 to be the set of pure states ω_ξ on A associated with vectors $\xi\in K$. We claim that S_0 is full, but $\langle S_0\rangle$ not dense in S. Indeed, let p be the one-dimensional projection determined by η_0, that is, $p\xi=(\xi,\eta_0)\eta_0$, $\xi\in X$. Then, for $\xi\in K$,

$$\omega_\xi(p) = (p\xi,\xi) = |(\xi,\eta_0)|^2 \leqq \tfrac{1}{4} < \|p\|\,,$$

which shows that $\langle S_0\rangle$ is not dense in S.

Now let $x\in A_h$, and suppose that $\omega_\xi(x)\geqq 0$ for all $\xi\in K$. Let $\varepsilon>0$ be arbitrary, and let x_1 be the restriction of x to $(1-p)X$. Then x_1: $(1-p)X\to X$ has no bounded inverse y. For if such a y existed, then $1=x_1y=xy$, making 1 compact which is impossible since X is infinite dimensional. Hence there is a unit vector $\xi_0\in(1-p)X$ such that $\|x\xi_0\|\leqq\varepsilon$. Now take any unit vector $\eta\in X$ and define $\xi=\frac{1}{2}\eta+\alpha\xi_0$ with α, $|\alpha|\leqq 1$, chosen so as to make $\|\xi\|=1$. Clearly $\xi\in K$, so by assumption

$$0 \leqq (x\xi,\xi) \leqq \tfrac{1}{4}(x\eta,\eta)+2\|x\xi_0\| \leqq \tfrac{1}{4}(x\eta,\eta)+2\varepsilon\ .$$

Hence $(x\eta,\eta)\geqq -8\varepsilon$ which, by the arbitrariness of ε and η, implies that $x\geqq 0$. So S_0 is full and we have our example.

We now give the characterization of dense subsets of S, similar to Theorem 1.

THEOREM 2. *Let A be a C^*-algebra without identity, and let S_0 be a subset of N. The following statements are equivalent:*

(ii)′ $\bar{S}_0 \geqq P$.

(iii)′ $\overline{\langle S_0 \rangle} \geqq S$ (*or* $\overline{\langle S_0 \rangle} = N$).

(iv)′ $\|a\| = \sup_{\varrho \in S_0} \varrho(a)$ *for any* $a \geqq 0$, $a \in A$.

Each of these conditions implies

(i)′ S_0 *is full.*

PROOF. (iii)′ $\Rightarrow$ (ii)′ by Proposition 1 and the partial converse of the Krein–Milman theorem. (ii)′ $\Rightarrow$ (iv)′ as in the proof of Theorem 1. (iv)′ $\Rightarrow$ (i)′ is covered by Lemma 2, so we are left to prove (iv)′ $\Rightarrow$ (iii)′. Clearly (iii)′ is equivalent to (iii) of Theorem 1 for $\tilde{A}$ by the map $N \to \tilde{S}$, and it is therefore, by Theorem 1, sufficient to show that S_0 is full as a subset of $\tilde{S}$. Now S_0 is full as a subset of N, so by Lemma 5 it suffices to show that if $a \in A_h$, $\lambda \geqq 0$, and $\varrho(a) + \lambda \geqq 0$ for all $\varrho \in S_0$, then $a + \lambda 1 \geqq 0$. If $a \geqq 0$ there is nothing to prove, so we may assume that $a^- \neq 0$. Then, by Lemma 1, for any $\varepsilon > 0$ there is a $\varrho \in S_0$ such that $\varrho(a^+) \leqq \varepsilon$ and $\varrho(a^-) \geqq \|a^-\| - \varepsilon$. Now

$$0 \leqq \varrho(a) + \lambda = \varrho(a^+) - \varrho(a^-) + \lambda ,$$

so

$$\|a^-\| - \varepsilon \leqq \varrho(a^+) + \lambda \leqq \varepsilon + \lambda ,$$

so

$$\|a^-\| \leqq \lambda + 2\varepsilon.$$

Hence $\|a^-\| \leqq \lambda$ since $\varepsilon > 0$ was arbitrary, which implies that $a + \lambda 1 \geqq 0$. The proof is complete.

We will give another result in the same direction. A C^*-subalgebra B of a C^*-algebra A is called *facial* (order-related, see [4]) if B^+ is an order-ideal of A^+. For $y \in A$, let $A[y]$ denote the smallest facial C^*-subalgebra of A containing y.

THEOREM 3. *Let* $S_0 \subseteqq N$. *Then* $\langle S_0 \rangle$ *is dense in* N *if and only if for each* $0 \leqq x \in A$ *and each* $\varepsilon > 0$ *there is* $\varrho \in S_0$ *such that*

$$\|\varrho \mid A[x]\| > 1 - \varepsilon .$$

PROOF. To prove the "if" part it is, by Theorem 2 (iv)′, enough to show that $\sup_{\varrho \in S_0} \varrho(x) = 1$ for an arbitrary $x \in A^+$, $\|x\| = 1$. So let $\varepsilon > 0$ together with such an x be given. Let f be the real continuous function on $[0,1]$ given by

$$\begin{aligned} f(\lambda) &= 0 && \text{for } 0 \leqq \lambda \leqq 1 - \varepsilon , \\ &= \varepsilon^{-1}(\lambda - 1) + 1 && \text{for } 1 - \varepsilon \leqq \lambda \leqq 1 . \end{aligned}$$

Put $y=f(x)$. By assumption there is a $\varrho \in S_0$ such that $\|\varrho \mid A[y]\| > 1-\varepsilon$. Let μ_ϱ be the measure ϱ assigns to $\sigma(x) \subseteqq [0,1]$. Let $E=\sigma(x) \cap (1-\varepsilon, 1]$. Then clearly $\mu_\varrho(E)=\lim_n \varrho(y^{1/n})$.

On the other hand, y is strictly positive in $A[y]$ in the sense of [5]. Indeed, if γ is a state on $A[y]$ and $\gamma(y)=0$, then $\gamma \equiv 0$ on the set $\{z \in A[y] : \exists r \in \mathrm{R}^+,\ 0 \leqq z \leqq ry\}$. This set is dense in $A[y]^+$ by [4, Lemma 1.1], hence $\gamma \equiv 0$ on $A[y]$, a contradiction. But then $\{y^{1/n}\}$ is an approximate identity for $A[y]$ by [5, Theorem 1]. Hence

$$\|\varrho \mid A[y]\| = \lim_n \varrho(y^{1/n}) ,$$

so $\mu_\varrho(E) > 1-\varepsilon$. It follows that

$$\varrho(x) = \int_{\varrho(x)} \lambda \, d\mu_\varrho(\lambda) \geqq \int_E \lambda \, d\mu_\varrho(\lambda) \geqq (1-\varepsilon)\mu_\varrho(E) > (1-\varepsilon)^2 .$$

Since $\varepsilon > 0$ was arbitrary, we get $\sup_{\varrho \in S_0} \varrho(x) = 1$. The proof of the converse is straight forward and is left to the reader.

In conclusion we wish to remark that the property of a set S_0 to be dense in the set of states S has an obvious probabilistic interpretation. Let a be a self-adjoint element of a C^*-algebra A with identity, and let $\sigma(a)$ be the spectrum of a. Each $\varrho \in S$ induces a probability measure μ_ϱ on $\sigma(a)$ by

$$\varrho(f(a)) = \int_{\varrho(a)} f(\lambda) \, d\mu_\varrho(\lambda), \quad \lambda \in \sigma(a) ,$$

where $f \to f(a)$ is the Gelfand-transformation of $\mathscr{C}(\sigma(a))$ onto the C^*-subalgebra $A(a)$ generated by a and 1. Let λ_0 be an element of $\sigma(a)$, and let $\varepsilon > 0$ be arbitrary. We claim that if S_0 is dense in S, then for each $\delta > 0$ there is a $\varrho_0 \in S_0$ such that

$$\mu_{\varrho_0}([\lambda_0-\varepsilon, \lambda_0+\varepsilon]) > 1-\delta .$$

Indeed, let $f \colon \sigma(a) \to [0,1]$ be a continuous function such that $f(\lambda_0)=1$, and $f \equiv 0$ outside the interval $[\lambda_0-\varepsilon, \lambda_0+\varepsilon]$. Then $\|f(a)\|=1$ so there is a $\varrho_0 \in S_0$ such that $\varrho_0(f(a)) > 1-\delta$. Hence

$$1-\delta < \int_{\sigma(a)} f(\lambda) \, d\mu_{\varrho_0}(\lambda) = \int_{\lambda_0-\varepsilon}^{\lambda_0+\varepsilon} f(\lambda) \, d\mu_{\varrho_0}(\lambda) \leqq \mu_{\varrho_0}([\lambda_0-\varepsilon,\ \lambda_0+\varepsilon]) .$$

In the context of quantum mechanics, this may be formulated as follows. If S_0 is a dense family of states, then it is sufficiently rich in the following sense: If a is an observable, λ_0 a point in the spectrum of a

and E an arbitrary open interval in $\sigma(a)$ containing λ_0, then there is a state $\varrho_0 \in S_0$ such that the probability for a measurement of a in the state ϱ_0 to fall in E can be made arbitrarily close to one.

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