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ON NARROW SPECTRAL ANALYSIS

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1. Introduction.

The object of this paper is to generalize Beurling's theorem in [1] on the spectral analysis of uniformly continuous bounded functions on $\mathbf{R}$. In [4] Domar introduced and studied the narrow spectrum of bounded linear functionals on commutative Banach algebras with identity. We shall follow his line, but we will consider algebras without identity, too. As regards the general Banach algebra theory used in the sequel, we refer once for all to [8], especially chapters II and III.

Let B be a commutative Banach algebra, normed so that $\|fg\| \leqq \|f\|\,\|g\|$ for every $f,g \in B$. Whenever B has an identity e we suppose $\|e\|=1$. Denote by B^* its dual space of bounded linear functionals and by $\mathscr{M}$ the space of all nontrivial multiplicative linear functionals on B. Since we can identify the maximal regular ideals of B and the functionals of $\mathscr{M}$, we shall not distinguish between them and we shall use the same symbol M to denote a multiplicative linear functional and its kernel. We denote by $f(M)$ the image of an element $f \in B$ under an $M \in \mathscr{M}$.

We provide B^* with the weak-star topology, denoting by $U(F;A;\varepsilon)$ the neighborhood of $F \in B^*$ defined by the number $\varepsilon > 0$ and the finite subset A of B. Let $\mathscr{M}$ carry the induced topology; then $\mathscr{M} \cup \{0\}$ is a compact subset of the unit ball $S=\{F \in B^*;\ \|F\| \leqq 1\}$. $\mathscr{M}$ can be empty. If B has an identity, then $\mathscr{M}$ is a nonempty compact subset of the unit sphere $S_1=\{F \in B^*;\ \|F\|=1\}$.

For any $F \in B^*$ and any $f \in B$ we define the functional $F \circ f$ by the relation $(F \circ f)(g)=F(fg)$ for every $g \in B$. It is immediately seen that $F \circ f \in B^*$ and that $\|F \circ f\| \leqq \|F\|\,\|f\|$. With every $F \in B^*$ we associate the linear subspace $L_F=\{F \circ f;\ f \in B\}$.

We now define the two sets

$$\Lambda_F = \overline{L_F} \cap \mathscr{M} \quad \text{and} \quad \Lambda_F' = \overline{L_F \cap S} \cap \mathscr{M}\,,$$

and we call Λ_F *the spectrum of* F and Λ_F' *the narrow spectrum of* F. The reason for this terminology will become clear in Section 4. Let us remark that if $\mathscr{M} \subset S_1$, then $\Lambda_F' = \overline{L_F \cap S_1} \cap \mathscr{M}$; thus, our definition

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agrees with the definition of Λ_F' in [4] in the case when B has an identity. Indeed, this follows easily from the fact that if $\{G_\alpha\}$ is a net (directed set) in B^*, which converges to $G \in B^*$ in the weak-star topology, then $\underline{\lim}\, \|G_\alpha\| \geqq \|G\|$.

Obviously, $\Lambda_F' \subset \Lambda_F$. The following problem has been treated by Domar [4]:

(i) *Is* $\Lambda_F' = \Lambda_F$ *for every* $F \in B^*$?

We shall give a counter-example when B is the disk algebra (section 6). However, if B is regular and semisimple, then the answer of (i) is in the affirmative (Theorem 4). In particular, this result contains Beurling's theorem (Theorem 5). We shall also prove that $\Lambda_F' = \Lambda_F$ whenever Λ_F is totally disconnected (Theorem 3).

When B is an algebra with identity, Λ_F is nonempty if and only if $F \neq 0$ (see below, section 2). (Of course, this is not true in general for the simple reason that $\mathcal{M}$ can be empty.) This suggests the following question:

(ii) *Does* $F \neq 0$ *imply* $\Lambda_F' = \emptyset$ *when* B *is an algebra with identity*?

Problem (ii) will be treated for algebras generated by one element f and its inverse, and we shall show that the answer is "yes" at least in the case when the spectrum of f is an annulus (Theorem 6). This extends an analogous result of Domar [4] for algebras with one generator whose spectrum is a disk. Domar has also proved that the answer of (ii) is affirmative when the subclass of all real-valued Gelfand transforms of B strongly separates the points in $\mathcal{M}$. The question whether (ii) always holds remains open.

Finally, let us mention that instead of L_F we could consider any linear subspace V of B^* which is *invariant* in the sense that $F \in V$ implies $F \circ f \in V$ for every $f \in B$, defining $\Lambda_V = \overline{V} \cap \mathcal{M}$ and $\Lambda_V' = \overline{V \cap S} \cap \mathcal{M}$. Then all our results concerning Λ_F and Λ_F' are true for Λ_V and Λ_V' and our proofs apply with some minor modifications. However, we shall not insist on this point.

2. Preliminaries.

Let I_F be the annihilator of $\overline{L_F}$ in B. Then $I_F = \{f \in B;\, F \circ f = 0\}$ and it is a closed ideal of B. According to a general property of the weak-star topology, $\overline{L_F}$ is the annihilator of I_F in B^*. Thus,

$$\Lambda_F = \{M \in \mathcal{M};\, I_F \subset M\}. \tag{1}$$

If $F \in \overline{L_F}$ — and this is certainly the case whenever B has an approximating identity — then $f \in I_F$ implies $F(f) = 0$, and it follows that I_F is a proper ideal for $F \neq 0$. So in that case we may conclude that $F \neq 0$ implies $\Lambda_F \neq \emptyset$ if every closed proper ideal of B is contained in a maximal regular ideal. In particular, this is true for algebras with identity.

We shall consider the quotient Banach algebra $\tilde{B}=B/I_F$ of cosets $\tilde{f}=f+I_F$, endowed with the norm $\|\tilde{f}\|=\inf\{\|f\|; f\in\tilde{f}\}$. According to a duality theorem for arbitrary normed spaces, the dual $\tilde{B}^*$ of $\tilde{B}$ is congruent to the annihilator of I_F in B^*, that is, to $\overline{L_F}$. Indeed, this congruence is established by the mapping $G \to \tilde{G}$ of $\overline{L_F}$ onto $\tilde{B}^*$, defined by the relation

$$(2) \qquad \tilde{G}(\tilde{f}) = G(f) \quad \text{for every } f\in B .$$

Since $\overline{L_F}\cap U(G;A;\varepsilon)$ is mapped onto $U(\tilde{G};\tilde{A};\varepsilon)$, it is also a weak-star homeomorphism. Obviously, G is multiplicative if and only if $\tilde{G}$ is multiplicative. Hence $\Lambda_F=\overline{L_F}\cap\mathscr{M}$ is mapped onto $\tilde{\mathscr{M}}$, the space of maximal regular ideals of $\tilde{B}$. Let $G\in\overline{L_F}$; then $\Lambda_G\subset\Lambda_F$, because $\overline{L_G}\subset\overline{L_F}$. Since $(G\circ f)^\sim=\tilde{G}\circ\tilde{f}$ for every $f\in B$, L_G is mapped onto $L_{\tilde{G}}$. Consequently, $\Lambda_G=\overline{L_G}\cap\mathscr{M}=\overline{L_G}\cap\Lambda_F$ is mapped onto $\overline{L_{\tilde{G}}}\cap\tilde{\mathscr{M}}=\Lambda_{\tilde{G}}$. The isometry gives that $L_G\cap S$ is mapped onto $L_{\tilde{G}}\cap\tilde{S}$, $\tilde{S}$ denoting the unit ball in $\tilde{B}^*$. Thus, $\Lambda_G'=\overline{L_G\cap S}\cap\Lambda_F$ is mapped onto $\overline{L_{\tilde{G}}\cap\tilde{S}}\cap\tilde{\mathscr{M}}=\Lambda_{\tilde{G}}'$. Let us summarize:

LEMMA 1. *The map $G\to\tilde{G}$ of $\overline{L_F}$ onto $\tilde{B}^*$, defined by* (2), *is a congruence and a weak-star homeomorphism, mapping Λ_G onto $\Lambda_{\tilde{G}}$ and Λ_G' onto $\Lambda_{\tilde{G}}'$. Moreover, Λ_F is mapped onto $\tilde{\mathscr{M}}$.*

By Lemma 1, we may identify $\tilde{\mathscr{M}}$ with Λ_F. Henceforth we shall do so, writing $\tilde{f}(M)$ instead of $\tilde{f}(\tilde{M})$. Then $\Lambda_{\tilde{G}}=\Lambda_G$ and $\Lambda_{\tilde{G}}'=\Lambda_G'$.

When B is an algebra without identity, it will be convenient to consider the Banach algebra B_1, obtained from B upon adjunction of the identity e and by introducing the norm $\|f+\lambda e\|=\|f\|+|\lambda|$, $f\in B$, $\lambda\in \mathbf{C}$. We extend the functionals $M\in\mathscr{M}$ to multiplicative linear functionals on B_1 by defining $(f+\lambda e)(M)=f(M)+\lambda$. Then the maximal ideal space of B_1 is $\mathscr{M}_1=\mathscr{M}\cup\{M_\infty\}$, where $M_\infty=B$. For an arbitrary $f_1=f+\lambda e\in B_1$ we define $F\circ f_1$ by the relation $F\circ(f+\lambda e)=F\circ f+\lambda F$. It is then clear that $F\circ f_1\in B^*$ and that $\|F\circ f_1\|\leq\|F\|\,\|f_1\|$ for every $f_1\in B_1$. Finally, for any $f\in B$ and any $f_1\in B_1$ we define $(F\circ f)(f_1)=F(ff_1)$.

If B is a Banach algebra with identity, we set $B_1=B$ and $\mathscr{M}_1=\mathscr{M}$ in order to simplify our notation. Finally, we define $\tilde{B}_1$ to be the Banach algebra associated with $\tilde{B}$ in the same manner as B_1 is associated with B, and we extend the natural homomorphism $B\to\tilde{B}$ to a homomorphism $B_1\to\tilde{B}_1$ by mapping the identity of B_1 onto the identity of $\tilde{B}_1$. Then $(G\circ f)^\sim=\tilde{G}\circ\tilde{f}$ for every $G\in\overline{L_F}$ and every $f\in B_1$.

It follows easily from (1) that

$$\Lambda_F \cap \{M \in \mathcal{M}; f(M) \neq 0\} \subset \Lambda_{F\circ f} \subset \Lambda_F \tag{3}$$

for every $f \in B_1$.

Before proceeding further we need the following definition, given in [4] in a different but equivalent formulation (in the case $B_1 = B$).

DEFINITION 1. Let $F \in B^*$. A closed nonempty subset E of $\mathcal{M}$ is called an *F-determining set* if for every finite subset A of B_1

$$\overline{\lim}_{n\to\infty} \sup \{\|F\circ\textstyle\prod_1^n f_\nu\|^{1/n}; f_\nu \in A\} \leqq \sup \{|f(M)|; f \in A, M \in E\}\,. \tag{4}$$

It is useful to note that the left side of (4), here denoted provisionally by $l(A)$, depends continuously on A. In fact, if A_ε is the set of elements in B_1 whose distance to A is less than ε, then $l(A_\varepsilon) \leqq l(A) + \varepsilon$. To see this, let $\{g_\nu\}$ be an arbitrary sequence of elements in A_ε and choose for each ν an $h_\nu \in A$ such that $\|g_\nu - h_\nu\| < \varepsilon$. For every $\eta > 0$ there exists a constant C such that

$$\sup \{\|F\circ\textstyle\prod_1^n f_\nu\|; f_\nu \in A\} \leqq C\big(l(A)+\eta\big)^n \quad \text{for every } n\,.$$

Consequently,

$$\begin{aligned}\|F\circ\textstyle\prod_1^n g_\nu\| &= \|F\circ\textstyle\prod_1^n\big(h_\nu + (g_\nu - h_\nu)\big)\| \\ &\leqq C\sum_{k=0}^{n}\binom{n}{k}\big(l(A)+\eta\big)^k \varepsilon^{n-k} = C\big(l(A)+\eta+\varepsilon\big)^n\,.\end{aligned}$$

It follows that $l(A_\varepsilon) \leqq l(A) + \varepsilon + \eta$, and since η was arbitrary, $l(A_\varepsilon) \leqq l(A) + \varepsilon$.

In virtue of the inequality $|f(M)| \leqq \|f\|$, the right side of (4) also depends continuously on A. It follows that for E to be F-determining it suffices to have (4) for every finite subset A of a dense set in B_1. We may also conclude, using a standard covering argument, that if E is F-determining, then (4) holds for every compact subset A of B_1.

LEMMA 2. *Suppose $G \in \overline{L_F}$. Then a subset E of Λ_F is G-determining if and only if it is $\tilde{G}$-determining.*

PROOF. Let A be a finite subset of B_1 and let $\tilde{A}$ be the corresponding subset of $\tilde{B}_1$. Then, by virtue of Lemma 1, (4), with F replaced by G, will hold if and only if

$$\overline{\lim}_{n\to\infty} \sup \{\|\tilde{G}\circ\textstyle\prod_1^n \tilde{f}_\nu\|^{1/n}; \tilde{f}_\nu \in \tilde{A}\} \leqq \sup \{|\tilde{f}(M)|; \tilde{f} \in \tilde{A}, M \in E\}\,.$$

This proves the lemma.

For arbitrary $f \in B_1$, $\lim_{n\to\infty} \|f^n\|^{1/n} = \sup\{|f(M)|;\ M \in \mathcal{M}_1\}$. In this formula we want to replace $\mathcal{M}_1$ by $\mathcal{M}$. This can trivially be done when

$B_1 = B$. In view of the fact that f is continuous on $\mathcal{M}_1$, this can also be done in the case $B_1 \neq B$ provided M_∞ is not an isolated point of $\mathcal{M}_1$, that is, provided $\mathcal{M}$ is non-compact. So suppose that M_∞ is isolated. If we exclude the trivial case $\mathcal{M} = \emptyset$, then $\mathcal{M}_1$ is disconnected, $\mathcal{M}$ and $\{M_\infty\}$ being disjoint closed subsets of $\mathcal{M}_1$. According to a theorem on the decomposition of a Banach algebra into the direct sum of ideals (see e.g. [5, p. 96]), there exists a unique element $u \in B$ such that $u^2 = u$, $u(M_\infty) = 0$ and $u(M) = 1$ on $\mathcal{M}$. Thus, since $uf^n = (uf)^n$ and

$$\sup\{|u(M)f(M)|;\ M \in \mathcal{M}_1\} = \sup\{|f(M)|;\ M \in \mathcal{M}\},$$

we obtain the following modified spectral radius formula:

$$\text{(5)} \qquad \lim_{n\to\infty} \|uf^n\|^{1/n} = \sup\{|f(M)|;\ M \in \mathcal{M}\} \quad \text{for every } f \in B_1 .$$

If B has an identity or if $\mathcal{M}$ is non-compact, we shall set $u = e$. Then, by what we have said above, (5) remains valid, and we always have $u(M) = 1$ on $\mathcal{M}$. From (5) it follows easily (compare [4, p. 7]) that

$$\overline{\lim}_{n\to\infty} \sup\{\|u \cdot \textstyle\prod_1^n f_\nu\|^{1/n};\ f_\nu \in A\} = \sup\{|f(M)|;\ f \in A,\ M \in \mathcal{M}\}$$

for every finite subset A of B_1. This gives

Lemma 3. *If $\mathcal{M}$ is nonempty, then $\mathcal{M}$ is $F \circ u$-determining for every $F \in B^*$.*

Lemma 4. *Suppose Λ_F is nonempty. There exists an element $v \in B_1$ such that, for every $G \in \overline{L_F}$, the spectrum Λ_F is $G \circ v$-determining and $\Lambda_{G \circ v} = \Lambda_G$.*

Proof. By Lemma 1 and by Lemma 3 applied to $\tilde{B}$, there is an element $\tilde{v} \in \tilde{B}_1$ with $\tilde{v}(M) = 1$ on Λ_F, such that Λ_F is $\tilde{G} \circ \tilde{v}$-determining for every $G \in \overline{L_F}$. It follows from Lemma 2 that Λ_F is $G \circ v$-determining. Since $\Lambda_G \subset \Lambda_F$ and $v(M) = \tilde{v}(M) = 1$ on Λ_F, (3) gives $\Lambda_{G \circ v} = \Lambda_G$. We note that if B is an algebra with identity, we may take $v = e$.

The following definition from [4] will be useful in section 5. Again we choose a different but equivalent formulation.

Definition 2. Let E be an F-determining subset of $\mathcal{M}$. We shall say that a point $M_0 \in E$ has *the property $A(F)$ with respect to E*, if for every neighborhood U of M_0 there exists an element $f \in B$ such that

$$\sup\{|f(M)|;\ M \in E\} \leqq 1 ,$$

$$\sup\{|f(M)|;\ M \in E - U\} < 1 ,$$

$$\overline{\lim}_{n\to\infty} \|F \circ f^n\|^{1/n} = 1 .$$

We note that, given $\varepsilon > 0$, we may suppose that $|f(M)| < \varepsilon$ on $E-U$, replacing f by f^N, if necessary. That still

$$\overline{\lim}_{n\to\infty} \|F \circ (f^N)^n\|^{1/n} = 1$$

is a consequence of the inequality

$$\|F \circ f^m\| \leqq \|F \circ f^{Nn}\| \max(1, \|f\|^N) \quad \text{for } Nn \leqq m \leqq N(n+1) .$$

3. Main results.

THEOREM 1. *Let $F \in B^*$ and $M_0 \in \mathscr{M}$. Suppose that for every neighborhood U of M_0 there exists an element $f \in B$ with $F \circ f \neq 0$ and an $F \circ f$-determining subset E_f of U. Then $M_0 \in \Lambda_F'$.*

COROLLARY. *If for every neighborhood U of $M_0 \in \Lambda_F$ there exists an element $f \in B$ such that $\Lambda_{F\circ f}$ is a nonempty subset of U, then $M_0 \in \Lambda_F'$.*

PROOF OF THEOREM 1. Let $U(M_0;A;\delta)$ be an arbitrary neighborhood of M_0. We have to show that $L_F \cap S \cap U(M_0;A;\delta)$ is nonempty. Set

$$A_1 = \{g - g(M_0)e;\ g \in A\};$$

then A_1 is a finite subset of B_1 and the elements of A_1 vanish at M_0. If

$$\varepsilon \max\{1 + |g(M_0)|;\ g \in A\} < \delta ,$$

then $L_F \cap U(M_0;A_1 \cup \{e\};\varepsilon)$ is a subset of $L_F \cap U(M_0;A;\delta)$. Therefore, it is sufficient to show that $L_F \cap S \cap U(M_0;A_1 \cup \{e\};\varepsilon)$ is nonempty, i.e. that there exists a $G \in L_F \cap S$ such that

$$|G(e) - 1| < \varepsilon \tag{6}$$

and

$$|G(g)| < \varepsilon \quad \text{for every } g \in A_1 . \tag{7}$$

Let us assume the contrary and let $H \in L_F$ be arbitrary $\neq 0$. There is an $h \in B$ with $\|h\| = 1$ such that $(1-\varepsilon)\|H\| < H(h) \leqq \|H\|$. Thus, since $\|H \circ h\| \leqq \|H\|$, (6) will be satisfied if we take $G = (\|H\|)^{-1}(H \circ h)$. Consequently, there is an element in A_1, say g_1, such that (7) does not hold, i.e.

$$|H \circ g_1(h)| = |H \circ h(g_1)| \geqq \varepsilon \|H\| .$$

It follows that $\|H \circ g_1\| \geqq \varepsilon \|H\|$. Repeating the above procedure with $H \circ g_1$ instead of H, we obtain inductively a sequence $\{g_\nu\}$ of elements of A_1 such that

$$\|H \circ \textstyle\prod_1^n g_\nu\| \geqq \varepsilon^n \|H\|, \quad n = 1, 2, \ldots . \tag{8}$$

Set $U = \{M \in \mathscr{M};\ |g(M)| < \frac{1}{2}\varepsilon$ for every $g \in A_1\}$. Then U is a neighborhood of M_0 and, by assumption, U contains an $F\circ f$-determining subset E_f for some $f \in B$ with $F\circ f \neq 0$. Now, take $H = F\circ f$ in (8). Since

$$\overline{\lim}_{n\to\infty} \|F\circ f\circ \textstyle\prod_1^n g_\nu\|^{1/n} \leqq \sup\{|g(M)|;\ g\in A_1, M\in E_f\} \leqq \tfrac{1}{2}\varepsilon ,$$

we obtain a contradiction, which proves Theorem 1.

PROOF OF THE COROLLARY. Let $\Lambda_{F\circ f} \subset U$ and $M_1 \in \Lambda_{F\circ f}$. Choose $g \in B$ such that $g(M_1) \neq 0$; then, by (3), $M_1 \in \Lambda_{F\circ fg}$. By Lemma 4, with F replaced by $F\circ f$, $\Lambda_{F\circ f}$ is $F\circ fgv$-determining and $\Lambda_{F\circ fgv} = \Lambda_{F\circ fg}$, which is nonempty. Consequently, $F\circ fgv \neq 0$ and the corollary now follows from Theorem 1.

Theorem 1 should be compared with the following result in [4], which will be used in Section 5.

THEOREM 2. *If $M_0 \in \mathcal{M}$ has the property $A(F)$ with respect to some F-determining set E, then $M_0 \in \Lambda_F'$.*

PROOF. It suffices to modify the end of the proof of Theorem 1 in the following way. Let $f \in B$ fulfill

$$|f(M)| \leqq 1 \text{ on } E, \quad |f(M)| < \varepsilon/D \text{ on } E - U , \quad \overline{\lim}_{n\to\infty}\|F\circ f^n\|^{1/n} = 1,$$

where $D = 2\max\{\|g\|;\ g\in A_1\}$. Then $|(fg)(M)| \leqq \frac{1}{2}\varepsilon$ on E for every $\in A_1$, so we can find an integer N such that

$$\sup\{\|F\circ \textstyle\prod_1^N (fg_i)\|;\ g_i \in A_1\} < (\tfrac{2}{3}\varepsilon)^N \quad \text{and} \quad \|F\circ f^N\| > (\tfrac{3}{4})^N .$$

We now obtain the desired contradiction by taking $H = F\circ f^N$ and $n = N$ in (8).

THEOREM 3. *Suppose $M_0 \in \Lambda_F$ has the following property: In the relative topology of Λ_F there exists a basis at M_0, the sets of which are both open and compact. Then $M_0 \in \Lambda_F'$.*

COROLLARY. *If Λ_F is totally disconnected, then $\Lambda_F' = \Lambda_F$.*

PROOF. Choose $g \in B$ such that $g(M_0) \neq 0$ and set $G = F\circ g$. Then $G \in L_F$ and it will clearly suffice to prove that $M_0 \in \Lambda_G'$. By Lemma 1, this is equivalent to showing that $M_0 \in \Lambda_{\tilde G}'$; therefore, without loss of generality, we may suppose that $\Lambda_F = \mathcal{M}$. Let U be a set of the above-mentioned basis. According to the already cited theorem on the decomposition of a Banach algebra into the direct sum of ideals, there exists an element $e_1 \in B$ such that $e_1^2 = e_1$, $e_1(M) = 1$ on U and $e_1(M) = 0$ on $\mathcal{M} - U$. Let u be the element of Lemma 3. By (3), $M_0 \in \Lambda_{G\circ ue_1}$. Thus, $G\circ ue_1 \neq 0$, so if we show that U is $G\circ ue_1$-determining, then Theorem 3 will follow from Theorem 1. But this is an immediate consequence of the relations

$$G \circ u e_1 \circ \prod_1^n f_\nu = G \circ u \circ \prod_1^n (e_1 f_\nu)$$

and

$$\sup\{|(e_1 f)(M)|;\ M \in \mathscr{M}\} = \sup\{|f(M)|;\ M \in U\}$$

and the fact that $\mathscr{M}$ is $G \circ u$-determining. We note that we could also have used Theorem 2, because $\lim_{n\to\infty} \|G \circ u \circ e_1{}^n\|^{1/n} = 1$ shows that M_0 has the property $A(G \circ u)$ with respect to $\mathscr{M}$.

Since Λ_F is a locally compact Hausdorff space, there exists a basis at M_0 of open compact sets if and only if the component of M_0 in Λ_F equals $\{M_0\}$ (see [6, p. 75]). Therefore, if the component of M_0 in Λ_F reduces to $\{M_0\}$, then $M_0 \in \Lambda_F'$. This proves the corollary.

For any $f \in B$, let supp$f$ denote the closure in $\mathscr{M}$ of the set $\{M \in \mathscr{M}; f(M) \neq 0\}$. We have the following converse of (3).

Lemma 5. *If B is regular and semisimple, then $\Lambda_{F \circ f} \subset \Lambda_F \cap \operatorname{supp} f$ for every $f \in B$ and $F \in B^*$.*

Proof. Suppose $M_0 \notin \operatorname{supp} f$ and let U be a neighborhood of M_0 which does not intersect supp$f$. There is a $g \in B$ such that $g(M_0) \neq 0$ and $g(M) = 0$ on $\mathscr{M} - U$. Thus, $g(M)f(M) = 0$ on $\mathscr{M}$ and by the semi-simplicity, $fg = 0$. This gives $g \in I_{F \circ f}$ and since $g(M_0) \neq 0$, $M_0 \notin \Lambda_{F \circ f}$. The semisimplicity hypothesis can not be omitted in Lemma 5.

Theorem 4. *Suppose B is regular and semisimple. Then $\Lambda_F' = \Lambda_F$ for every $F \in B^*$.*

Proof. We have to show that $\Lambda_F \subset \Lambda_F'$. Suppose $M_0 \in \Lambda_F$ and let U be a neighborhood of M_0. Let V be another neighborhood such that $\bar{V} \subset U$ and choose $f \in B$ such that $f(M_0) \neq 0$ and $f(M) = 0$ on $\mathscr{M} - V$. By (3), $M_0 \in \Lambda_{F \circ f}$ and by Lemma 5, $\Lambda_{F \circ f} \subset \bar{V} \subset U$. Thus, the corollary of Theorem 1 gives that $M_0 \in \Lambda_F'$.

In [4] a weaker form of Theorem 4 is obtained from Theorem 2. The above result will become more interesting if we impose some condition on B, which guarantees that $F \neq 0$ implies $\Lambda_F \neq \emptyset$. If B is regular and semisimple, then the condition that the set $\{f \in B;\ \operatorname{supp} f \text{ is compact}\}$ is dense in B, implies that every closed proper ideal is included in some maximal regular ideal. In particular, the group algebras, which we shall consider in the next section, satisfy this condition, and since they have also approximating identities, it follows, in view of the discussion at the beginning of Section 2, that Theorem 4 may be applied to them, with the additional information that Λ_F is nonempty whenever $F \neq 0$.

4. Beurling's theorem.

For each $F \in B^*$, $\varepsilon > 0$ and each compact subset C of B, define

$$[F;C;\varepsilon] = \{G \in B^*;\ \sup_{f\in C}|G(f)-F(f)| + |\,\|G\| - \|F\|\,| < \varepsilon\}\,.$$

The topology in B^*, having for basis all sets $[F;C;\varepsilon]$, is called *the narrow topology.*

Suppose the multiplicative linear functionals have norm 1. Then, combining the definition of Λ_F' with an easy covering argument, we obtain

(9) $$\Lambda_F' = (\text{the narrow closure of } L_F) \cap \mathscr{M}\,.$$

We shall give a more concrete interpretation of (9) when $B=L^1(\mathsf{G})$, where G is a locally compact abelian group with Haar measure dx. We identify the functions $F \in L^\infty(\mathsf{G})$ with the bounded linear functionals on $L^1(\mathsf{G})$ by means of the duality relation

$$F(f) = \int_{\mathsf{G}} F(-x)\,f(x)\,dx\,.$$

Then $F \circ f$ is the ordinary convolution $F*f$. To avoid confusion, we denote the Fourier–Gelfand transform of $f \in L^1(\mathsf{G})$ by $\hat{f}(M)$; in other words,

$$\hat{f}(M) = \int_{\mathsf{G}} M(-x)\,f(x)\,dx\,,$$

where $M(\cdot)$ is a character of G.

Suppose now that $F \in L^\infty(\mathsf{G})$ and that $M_0 \in \Lambda_F$. Choose $g \in L^1(\mathsf{G})$ such that $\hat{g}(M_0) \neq 0$ and such that $\operatorname{supp}\hat{g}$ is compact and set $G=F*g$. By (3), $M_0 \in \Lambda_G$. Choose $h \in L^1(\mathsf{G})$ such that $\hat{h}(M)=1$ on $\operatorname{supp}\hat{g}$; then $\hat{h}(M)\hat{g}(M)=\hat{g}(M)$ on $\mathscr{M}$ and it follows that $g*h=g$. Consequently, $G*h=G$. Set $h_y(x)=h(x+y)$.

Let K be an arbitrary compact subset of G. Then $C=\{h_y;\ y \in K\}$ is a compact subset of $L^1(\mathsf{G})$. By Theorem 4, $M_0 \in \Lambda_G'$; therefore, in virtue of (9), given $\varepsilon > 0$, there exists an $f \in L^1(\mathsf{G})$ such that

(10) $$G*f \in [M_0;C;\varepsilon]\,.$$

Now,

$$G*f(h_y) = G(f*h_y) = G(h*f_y) = G*h(f_y)$$
$$= \int G(-x)\,f(x+y)\,dx = (G*f)(y)\,,$$

$$\hat{h}_y(M_0) = \int M_0(-x)\,h(x+y)\,dx = M_0(y)\hat{h}(M_0) = M_0(y)\,.$$

Thus, (10) means that

$$\sup_{y \in K} |(G * f)(y) - M_0(y)| \; + \; | \, ||G * f||_{L^\infty} - 1| \; < \; \varepsilon \, .$$

Since $G * f = F * (g * f)$, this proves the following theorem

THEOREM 5. *Let $F \in L^\infty(\mathsf{G})$. If $M_0 \in \Lambda_F$, then there exists a net $\{F * f_\alpha\}$, $f_\alpha \in L^1(\mathsf{G})$, such that*

$$||F * f_\alpha||_{L^\infty} = 1 \quad \text{and} \quad F * f_\alpha \to M_0 \, ,$$

uniformly on every compact subset of G.

If $F \in L^\infty(\mathsf{G})$ is uniformly continuous, then we can replace the net $\{F * f_\alpha\}$ by a net of finite linear combinations of translates of F in the statement of the preceding theorem; this is easily seen by approximating the integral involved in $F * f_\alpha$. This is the original version of Beurling's theorem [1], except that it was stated and proved for $\mathsf{G} = \mathsf{R}$. Beurling used analytic function theory in his proof. Domar proved the theorem for arbitrary groups using Banach algebra theory in [2] and purely by Fourier analysis arguments in [3]. A short proof can also be found in [7]. In this connection we should mention that the net in Theorem 5 can not be replaced by a net of linear combinations of translates of F for an arbitrary function F. Indeed, Koosis [7] has constructed a continuous function $F \in L^\infty(\mathsf{R})$ such that $1 \in \Lambda_F$ but such that 1 does not belong to the weak closure of any norm-bounded set of finite linear combinations of translates of F.

5. Algebras with one or two generators of special type.

In this section we suppose B is an algebra with identity.

For any $f \in B$ we set $\operatorname{sp} f = \{f(M);\ M \in \mathcal{M}\}$ and we recall that $\operatorname{sp} f$ coincides with the set of complex numbers λ for which $f - \lambda e$ is not invertible, commonly called the spectrum of f. Denote by C_r the circle $\{z \in \mathsf{C};\ |z| = r\}$ and by $A(r_1; r_2)$ the closed annulus $\{z \in \mathsf{C};\ r_1 \leqq |z| \leqq r_2\}$, with the tacit agreement that $0 < r_1 \leqq r_2 < \infty$.

THEOREM 6. *Suppose there exists an element $f_0 \in B$ such that f_0 and f_0^{-1} generate B and such that $\operatorname{sp} f_0 = A(r; R)$. Then Λ_F' is nonempty for every $F \neq 0$.*

The proof is patterned after the proof of Theorem 3 in [4]. The following two lemmas replace Lemma 5 in [4].

LEMMA 6. *Let B be an arbitrary Banach algebra with identity. If $\operatorname{sp} f \subset A(r_1; r_2)$, then*

$$r_1 \leqq \varliminf_{n\to\infty} ||F \circ f^n||^{1/n} \leqq \varlimsup_{n\to\infty} ||F \circ f^n||^{1/n} \leqq r_2$$

for every $F \neq 0$.

PROOF. Since $0 \notin \mathrm{sp} f$, f^{-1} exists. Choose g, $\|g\|=1$, such that $F(g) \neq 0$. Combining the inequality

$$|F(g)| = |(F\circ f^n)(f^{-n}g)| \leqq \|F\circ f^n\| \, \|f^{-n}\|$$

with the spectral radius formula

$$\lim_{n\to\infty} \|f^{-n}\|^{1/n} = \sup\{|f(M)|^{-1};\, M \in \mathscr{M}\} \leqq r_1^{-1},$$

we obtain $\underline{\lim}_{n\to\infty} \|F\circ f^n\|^{1/n} \geqq r_1$. The other inequality follows from $\|F\circ f^n\| \leqq \|F\| \, \|f^n\|$ and the spectral radius formula.

If B fulfills the assumptions of Theorem 6, then f_0 induces a homeomorphism of $\mathscr{M}$ onto $A(r;R)$. Hence we may assume that $\mathscr{M} = A(r;R)$ and that $f_0(z) = z$ for every $z \in A(r;R)$. Set

$$a = \overline{\lim}_{n\to\infty} \|F\circ f_0^{\,n}\|^{1/n} \quad \text{and} \quad b^{-1} = \overline{\lim}_{n\to\infty} \|F\circ f_0^{\,-n}\|^{1/n}.$$

Applying Lemma 6 to f_0 and f_0^{-1}, we obtain $r \leqq a \leqq R$ and $r \leqq b \leqq R$.

LEMMA 7. *With the assumptions of Theorem* 6:

(i) *if* $b \leqq a$, *then* $C_b \cup C_a$ *is F-determining,*

(ii) *if* $a < b$, *then* C_a *and* C_b *are F-determining.*

PROOF. Let α and β be positive numbers such that $a < \alpha$, $\beta < b$ and $\beta \leqq \alpha$, and consider the auxiliary Banach algebra B' of all power series

$$f(z) = \sum_{-\infty}^{\infty} a_\nu z^\nu,$$

with the norm

$$\|f(z)\|' = \sum_{0}^{\infty} |a_\nu| \alpha^\nu + \sum_{-\infty}^{-1} |a_\nu| \beta^\nu < \infty.$$

The maximal ideal space of B' can be identified with $A(\beta;\alpha)$ (see e.g. [5, p. 118]). With every polynomial $P(z) = \sum_{-m}^{n} a_\nu z^\nu$ we associate the polynomial $P = P(f_0) = \sum_{-m}^{n} a_\nu f_0^{\,\nu}$ in B. It follows from the choice of α and β that there exists a constant D such that $\|F\circ f_0^{\,n}\| \leqq D\alpha^n$ and $\|F\circ f_0^{\,-n}\| \leqq D\beta^{-n}$ for every natural n. Hence

$$\|F\circ P\| \leqq \sum_{-m}^{n} |a_\nu| \, \|F\circ f_0^{\,\nu}\| \leqq D\|P(z)\|'. \tag{11}$$

Let $P_1, \ldots, P_N$ be polynomials and set

$$d = \sup\{|P_i(z)|;\, 1 \leqq i \leqq N, z \in A(\beta;\alpha)\}.$$

Let $\{Q_\nu\}$ be an arbitrary sequence of polynomials chosen among the P_i, $1 \leqq i \leqq N$. According to the spectral radius formula, applied to $P_i(z) \in B'$, there exists, given $\varepsilon > 0$, a constant E such that $\|(P_i(z))^n\|' \leqq E(d+\varepsilon)^n$ for $1 \leqq i \leqq N$ and $n = 0, 1, \ldots$. Then, using (11), we obtain

$$\|F \circ \textstyle\prod_1^n Q_\nu\| \leqq D \|\prod_1^n Q_\nu(z)\|' \leqq D E^N (d+\varepsilon)^n .$$

Hence

$$\overline{\lim}_{n\to\infty} \|F \circ \textstyle\prod_1^n Q_\nu\|^{1/n} \leqq d+\varepsilon .$$

Since this is true for every $\varepsilon > 0$ and for every α, β, subject to the initial restrictions $a < \alpha$, $\beta < b$ and $\beta \leqq \alpha$, we conclude that

$$\overline{\lim}_{n\to\infty} \|F \circ \textstyle\prod_1^n Q_\nu\|^{1/n} \leqq \sup\{|P_i(z)|\,;\ 1 \leqq i \leqq N, z \in E\} ,$$

where $E = A(b;a)$ if $b \leqq a$ (case (i)), and $E = C_a$ or $E = C_b$ if $a < b$ (case (ii)). In case (i) we may replace $A(b;a)$ by $C_b \cup C_a$, because the maximum of each $|P_i(z)|$ is attained on the boundary of $A(b;a)$. Since the polynomials form a dense subset of B, Lemma 7 now follows in virtue of the remark following Definition 1.

Proof of Theorem 6. By Theorem 2, it will suffice to find a point which has the property $A(F)$ with respect to one of the respective F-determining sets of Lemma 7. To this end, we proceed as in [4], forming the polynomials $g_\theta(z) = \frac{1}{4}\exp(i\theta) + \frac{1}{2}z + \frac{1}{4}z^2 \exp(-i\theta)$ for $0 \leqq \theta \leqq 2\pi$. Using the identity $g_\theta(e^{i\varphi}) = e^{i\varphi}\big(\frac{1}{2} + \frac{1}{2}\cos(\theta - \varphi)\big)$, it is easy to see that

$$\frac{1}{2\pi}\int_0^{2\pi} [g_\theta(z)]^n \, d\theta = A_n z^n ,$$

where A_n is a constant such that $\lim_{n\to\infty} A_n^{1/n} = 1$. Setting $h_\theta = g_\theta(a^{-1}f_0)$, we obtain an element of B, and it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} (F \circ h_\theta^n) \, d\theta = A_n a^{-n} (F \circ f_0^n) .$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \|F \circ h_\theta^n\| \, d\theta \geqq \left\| \frac{1}{2\pi}\int_0^{2\pi} (F \circ h_\theta^n) \, d\theta \right\| = A_n a^{-n} \|F \circ f_0^n\| ,$$

and we conclude that there exists a number θ_n such that

$$\|F \circ (h_{\theta_n})^n\| \geqq A_n a^{-n} \|F \circ f_0^n\| .$$

By the definition of a,

$$\overline{\lim}_{n\to\infty} \|F \circ (h_{\theta_n})^n\|^{1/n} \geqq 1 .$$

Since h_θ depends continuously on θ, it follows, using an argument similar to that following Definition 1, that for a suitable limit point θ_0 of $\{\theta_n\}$

$$\overline{\lim}_{n\to\infty} \|F\circ(h_{\theta_0})^n\|^{1/n} \geqq 1 . \tag{12}$$

Since $|h_{\theta_0}(z)| \leqq 1$ on the F-determining set $C_b \cup C_a$ if $b \leqq a$ (C_a if $a<b$), we have actually equality in (12). Moreover, on $C_b \cup C_a$ (on C_a), $|h_{\theta_0}(z)|=1$ at $z=a\exp(i\theta_0)$, only. Thus, $a\exp(i\theta_0)$ has the property $A(F)$ with respect to $C_b \cup C_a$ (with respect to C_a).

A similar consideration of $g_\theta(bf_0^{-1})$ would produce a point $b\exp(i\theta_1)$ having the property $A(F)$ with respect to $C_b \cup C_a$ (with respect to C_b).

Remark 1. Suppose $\Lambda_F \neq A(r;R)$; then the part of Λ_F, which is situated in the interior of $A(r;R)$, consists of isolated points, while the intersection of Λ_F with the boundary of $A(r;R)$ is a set of (linear) Lebesgue measure zero. Indeed, $f \in I_F$ implies that $f(z)$ vanishes on Λ_F and since $f(z)$ is analytic in the interior of $A(r;R)$, Λ_F can not be "bigger" unless $f(z)\equiv 0$, that is, unless $\Lambda_F=A(r;R)$. But for sets of the type described above the condition of Theorem 3 is satisfied at every point. Therefore, we may conclude that $\Lambda_F'=\Lambda_F$ provided $\Lambda_F \neq \mathscr{M}$; it is only in the case $\Lambda_F=\mathscr{M}$ that Theorem 6 gives something new.

Remark 2. The conclusion of Theorem 6 is still true if we assume that $\mathrm{sp} f_0$ is homeomorphic to an annulus $A(r;R)$ and that the boundary of $\mathrm{sp} f_0$ is analytic.

To see this, let φ be a conformal mapping of the interior of $A(r;R)$ onto the interior of $\mathrm{sp} f_0$. The analyticity of the boundary means that φ can be extended to a univalent analytic function on some open annulus Ω containing $A(r;R)$. The inverse function φ^{-1} is defined on the open region $\varphi(\Omega)$ containing $\mathrm{sp} f_0$, so we can form $g_0=\varphi^{-1}(f_0)$, obtaining an element in B with spectrum

$$\mathrm{sp}\, g_0 \;=\; \varphi^{-1}(\mathrm{sp} f_0) \;=\; A(r;R) .$$

It is easy to verify that $f_0=\varphi(g_0)$. Obviously, there exist polynomials P_n in z and z^{-1} such that $P_n \to \varphi$ uniformly on compact subsets of Ω. Hence

$$\|P_n(g_0)-f_0\| \;=\; \|P_n(g_0)-\varphi(g_0)\| \to 0 \quad \text{as } n\to\infty .$$

Similarly, $P_n(g_0^{-1}) \to f_0^{-1}$, and it follows that g_0 and g_0^{-1} generate B. Therefore, we may apply Theorem 6. Of course, a similar extension holds for Theorem 3 in [4].

6. A counter-example.

We shall give an example which shows that the narrow spectrum can be a proper subset of the spectrum. Let D be the unit disk $\{z \in \mathbf{C};\ |z| \leqq 1\}$ and let A be the disk algebra that is the subalgebra of $C(D)$ consisting of all functions which are holomorphic in the interior of D. We identify the maximal ideal space of A with D. Let $\{z_\nu\}_1^\infty$ be a sequence of points on

$$\Gamma = \{z \in \mathbf{C};\ |z|=1\}$$

such that

$$E = \overline{\{z_\nu;\ \nu=1,2,\ldots\}}$$

is a proper subset of Γ of positive Lebesgue measure and let $\{a_\nu\}_1^\infty$ be a sequence of nonzero complex numbers such that $\sum_1^\infty |a_\nu| < \infty$. Define $F \in A^*$ by $F(f) = \sum_1^\infty a_\nu f(z_\nu)$. Then $\Lambda_F{}' = E$ whereas $\Lambda_F = D$.

To prove this we first of all note that $\|F\circ f\| = \sum_1^\infty |a_\nu f(z_\nu)|$ for every $f \in A$. This follows from the fact that, given $\varepsilon > 0$ and a complex sequence $\{w_\nu\}_1^N$, we can find a $g \in A$ such that

$$\|g\| < \max_{1\leq\nu\leqq N} |w_\nu| + \varepsilon$$

and

$$g(z_\nu) = w_\nu, \quad 1 \leqq \nu \leqq N .$$

We also have that $f \in I_F$ implies $f(z_\nu) = 0$ for every ν. But then $f(z) = 0$ on E which implies that $f = 0$. Hence $I_F = (0)$, that is $\Lambda_F = D$. Taking an $f \in A$, $\|f\| = 1$, which "peaks" at the point z_ν only, we see that z_ν has the property $A(F)$ with respect to E. By Theorem 2, this yields $z_\nu \in \Lambda_F{}'$ and since $\Lambda_F{}'$ is closed, $E \subset \Lambda_F{}'$. We conclude the proof by showing that if $z_0 \notin E$, then z_0 does not belong to the weak-star closure of any norm-bounded subset of L_F. Let $\varepsilon > 0$ and choose a $g \in A$ such that $g(z_0) = 0$ and $|g(z) - 1| < \varepsilon$ on E. If $F\circ f \in U(z_0; 1, g; \varepsilon)$ then

$$\left|\textstyle\sum_1^\infty a_\nu f(z_\nu) - 1\right| < \varepsilon \quad \text{and} \quad \left|\textstyle\sum_1^\infty a_\nu f(z_\nu)\, g(z_\nu)\right| < \varepsilon .$$

Hence

$$\begin{aligned}\varepsilon &> \left|1 + \left(\textstyle\sum a_\nu f(z_\nu) - 1\right) + \textstyle\sum a_\nu f(z_\nu)\left(g(z_\nu) - 1\right)\right| \\ &\geqq 1 - \varepsilon - \varepsilon \textstyle\sum |a_\nu f(z_\nu)| ,\end{aligned}$$

that is

$$\|F\circ f\| = \textstyle\sum |a_\nu f(z_\nu)| > (1-2\varepsilon)/\varepsilon = K(\varepsilon) ,$$

which shows that z_0 does not belong to the weak-star closure of $L_F \cap \{G \in A^*;\ \|G\| \leqq K(\varepsilon)\}$. Since $K(\varepsilon) \to \infty$ as $\varepsilon \to 0$, this yields our proposition.

7. F-determining sets.

Since F-determining sets have played a decisive role in our investigation, we devote this final section to a study of them. We suppose that B is an algebra with identity; then $F \in \overline{L_F}$ and the Lemmas 3 and 4 are simplified because $u = v = e$.

We shall call an F-determining set *F-minimal,* if it contains no proper F-determining subset.

Every F-determining set contains an F-minimal set. The proof of this fact is a straightforward application of Zorn's lemma; all we have to verify is that, if $\{E_\alpha\}$ is a totally ordered family (under the inclusion ordering) of F-determining sets, then $\bigcap_\alpha E_\alpha$ is also F-determining. $\bigcap_\alpha E_\alpha$ will then be a lower bound of the chain $\{E_\alpha\}$. To this end, let A be a finite subset of B and set

$$a = \overline{\lim}_{n\to\infty} \sup\{\|F\circ\textstyle\prod_1^n f_\nu\|^{1/n}; f_\nu \in A\}$$

and

$$Q = \{M \in \mathscr{M};\ |f(M)| \geqq a \text{ for some } f \in A\}\ .$$

Q is obviously closed. Since E_α is compact, the supremum of the right side of (4), with E_α instead of E, is attained and it follows that $Q\cap E_\alpha$ is nonempty. Consequently, $Q\cap(\bigcap_\alpha E_\alpha)=\bigcap_\alpha(Q\cap E_\alpha)$ is nonempty, i.e.

$$a \leqq \sup\{|f(M)|; f\in A,\ M \in \textstyle\bigcap_\alpha E_\alpha\}\ .$$

Thus, $\bigcap_\alpha E_\alpha$ is F-determining.

The following simple example shows that an F-determining subset can be disjoint from Λ_F, which is F-determining according to Lemma 4.

EXAMPLE 1. Let B be a Banach algebra with one generator f_0 such that $\mathrm{sp} f_0 = \{z \in \mathbf{C};\ |z|\leqq 1\}$. Identify $\mathscr{M}$ with $\mathrm{sp} f_0$ and define $F\in B^*$ by $F(f)=f(0)$. Then $\Lambda_F=\{0\}$ and since $\mathscr{M}$ is F-determining, so is also the Shilov boundary, that is, $\{z\in\mathbf{C};\ |z|=1\}$.

It follows that there is no unique F-minimal set in general. We can not do better by restricting our attention to F-determining subsets of Λ_F only, as shows the next example, suggested by Domar as an illustration of the fact that case (ii) of Lemma 7 can occur.

EXAMPLE 2. Let B be the algebra of Theorem 6 (with $r<R$). Using the notation of section 5, we define $F\in B^*$ by

$$F(f) = \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z}\,dz = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z}\,dz\ ,$$

which is consistent since $f(z)$ is analytic in the interior of $A(r;R)$. By considering $F\circ f(f_0{}^n)$ one easily obtains $I_F=\{f\in B; f(z)\equiv 0\}$. Thus, $\Lambda_F = A(r;R)$. It follows from

$$|F\circ f_0{}^n(g)| = \frac{1}{2\pi}\left|\int_{C_r} g(z)\, z^{n-1}\, dz\right| \leqq r^n \sup|g(z)| \leqq r^n\|g\|$$

that $a \leqq r$, which combined with Lemma 6 gives $a = r$. By using the contour C_R instead of C_r, we obtain $b = R$. In virtue of Lemma 7, C_r and C_R are F-determining subsets of Λ_F.

However, we have the following positive results.

THEOREM 7. (a) *If B is regular and semisimple, then Λ_F is the unique F-minimal subset of $\mathscr{M}$.*

(b) *If E is F-determining and if $M_0 \in E$ has the property $A(F)$ with respect to E, then every F-determining subset of E contains M_0. In particular, if Λ_F is totally disconnected, then Λ_F is F-minimal.*

PROOF. (a): Since Λ_F is F-determining, it will suffice to prove the following assertion: Given $M_0 \in \Lambda_F$ and a neighborhood U of M_0, then $\mathscr{M} - U$ is not F-determining. To this end, choose a neighborhood V of M_0 such that $\overline{V} \subset U$ and choose $f,g \in B$ such that $f(M) = 1$ on $\overline{V}$, $f(M) = 0$ on $\mathscr{M} - U$, $g(M) = 0$ on $\mathscr{M} - V$ and $g(M_0) \neq 0$. Then $(f^n g)(M) = (fg)(M)$ on $\mathscr{M}$ and by the semisimplicity, $f^n g = fg$ for every n. It follows from $f(M_0)g(M_0) \neq 0$ that $F \circ fg \neq 0$. Therefore, the inequality

$$\|F \circ f^n\| \, \|g\| \geqq \|F \circ f^n g\| = \|F \circ fg\|$$

gives

$$\overline{\lim}_{n \to \infty} \|F \circ f^n\|^{1/n} \geqq 1 > \sup \{|f(M)| \, ; \, M \in \mathscr{M} - U\} ,$$

which proves the assertion.

(b): The first assertion is a trivial consequence of the definition of the property $A(F)$. If M_0 satisfies the condition of Theorem 3, then M_0 has the property $A(F)$ with respect to Λ_F; this follows from the proof of Theorem 3 because we may now choose $g = u = e$.

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