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DUALITY OVER GORENSTEIN RINGS

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1.

The principal purpose of this note is to state and prove a result for Gorenstein rings which Roos [10] has shown for regular rings. For a commutative noetherian ring A, and an A-module M of finite type, with $\mathrm{Ext}_A{}^i(M,A)=0$ for $0 \leqq i < n$, there is a natural homomorphism $M \to \mathrm{Ext}_A{}^n(\mathrm{Ext}_A{}^n(M,A),A)$ which is a generalization of the natural homomorphism $M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A)$. In this first section the existence of this homomorphism is established using a spectral sequence of the type considered by Roos [10]. The main result, Roos' theorem for Gorenstein rings, appears in Section 2 along with a relation to a result of Auslander, the spherical filtration theorem (Proposition 8, page 39 of [11]). There is also a generalization of another result of Auslander and Buchsbaum [3] which appears as Proposition 10.

These results perhaps follow from results in Hartshorne's book [8], but I am not competent to elaborate on this point.

Throughout this note, A denotes a commutative noetherian ring with unit. All A-modules are to be of finite type. $\mathrm{Ann}_A M$ denotes the annihilator of the A-module M. An element $f \in A$ is *regular* on M provided $fx=0$, $x \in M$, implies $x=0$. An ordered set of elements of $A, f_1,\ldots,f_n$ is a *regular M-sequence* of length n provided f_1 is regular on M and f_i is regular on $M/(f_1M+\ldots+f_{i-1}M)$ for $1<i\leqq n$.

PROPOSITION 1. ([9, Proposition 3.3]). *If I is an ideal in A and M is an A-module, the following are equivalent*:

i) $\mathrm{Ext}_A{}^i(N,M)=0$ *for all A-modules N of finite type such that* $\mathrm{Supp}\,N \subset \mathrm{Supp}\,A/I$ *and all integers $i<n$.*

ii) $\mathrm{Ext}_A{}^i(A/I,M)=0$ *for all $i<n$.*

iii) *There is a regular M-sequence $f_1,\ldots,f_n$ with each $f_j \in I$.*

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COROLLARY AND DEFINITION. *Let* $g = \max\{i+1 : \mathrm{Ext}_A^j(A/I, M) = 0$ *for* $0 \leqq j \leqq i\}$. *Then* g *is the maximal length of a regular* M*-sequence contained in* I. *This integer is called the* I*-depth of* M ($\mathrm{depth}_I M$). *If* A *is a local ring with maximal ideal* $\mathfrak{m}$, *then* $\mathrm{depth}\, M = \mathrm{depth}_{\mathfrak{m}} M$.

PROPOSITION 2 ([11, Chap. 0, Proposition 2]). *Let* $f \in A$ *be regular on both* A *and* M. *Then, on the category of* A/fA*-modules, there are isomorphisms of functors*

$$\begin{aligned} \mathrm{Tor}^{A/fA}{}_n(M/fM, \cdot) &\to \mathrm{Tor}^A{}_n(M, \cdot)\,, \\ \mathrm{Ext}_{A/fA}{}^n(M/fM, \cdot) &\to \mathrm{Ext}_A{}^n(M, \cdot)\,, \\ \mathrm{Ext}_{A/fA}{}^n(\cdot, M/fM) &\to \mathrm{Ext}_A{}^{n+1}(\cdot, M) \end{aligned}$$

for all n.

Another integer, related to the depth, is defined by $\mathrm{grade}_A M = \min\{i : \mathrm{Ext}_A{}^i(M, A) \neq 0\}$. It follows that $\mathrm{grade}_A M = \mathrm{depth}_{\mathrm{Ann}\, M} A$. This leads to a result which will be needed later.

PROPOSITION 3. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of* A*-modules of finite type.*

a) $\mathrm{grade}\, M' \geqq \mathrm{grade}\, M$.
   $\mathrm{grade}\, M'' \geqq \mathrm{grade}\, M$.
   $\mathrm{grade}\, M = \min(\mathrm{grade}\, M', \mathrm{grade}\, M'')$.

b) *There exists a unique submodule* $M(i)$ *of* M *with the properties*:
   i) $\mathrm{grade}\, M(i) \geqq i$,
   ii) *If* $M_1 \subset M$ *with* $\mathrm{grade}\, M_1 \geqq i$, *then* $M_1 \subset M(i)$.

Since this result does not seem to be available, a proof is provided. Proposition 7, Chapt. 0 of [11] states that

$$\mathrm{grade}\, M = \min\{\mathrm{depth}\, A_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{Supp}\, M\}\,.$$

Since $\mathrm{Supp}\, M = \mathrm{Supp}\, M' \cup \mathrm{Supp}\, M''$ part a) follows. Since M is of finite type, there is a submodule $M(i)$ of M maximal with respect to the property that $\mathrm{grade}\, M(i) \geqq i$. If $M_1 \subset M$ and $\mathrm{grade}\, M_1 \geqq i$, then

$$\mathrm{grade}(M_1 + M(i)) \geqq i$$

by part a). Hence $M_1 + M(i) = M(i)$, and part b) follows.

Let $\mathcal{G}_i$ denote the class of A-modules M of finite type with $\mathrm{grade}\, M \geqq i$. Proposition 3 shows that each A-module of finite type admits a filtration $M = M(0) \supset M(1) \supset \ldots$ such that $M(i)$ is the unique maximal submodule of M in $\mathcal{G}_i$, and it then follows that $\mathrm{grade}\, M = i$ if and only if $M = M(i)$ and $M \neq M(i+1)$.

Let $\mathcal{N}_i$ denote the class of those A-modules M of finite type such that $M_{\mathfrak{p}}=0$ for all prime ideals $\mathfrak{p}$ of A with $\operatorname{ht}\mathfrak{p}<i$. (See [7] for other properties of $\mathcal{N}_i$.) It is clear that each A-module M admits a unique maximal submodule $M[i]\in\mathcal{N}_i$ and that $\mathcal{N}_{i+1}\subset\mathcal{N}_i$.

PROPOSITION 4. a) $\mathcal{G}_i\subseteqq\mathcal{N}_i$ *for all* i.

b) $\mathcal{G}_i=\mathcal{N}_i$ *for* $0\leqq i\leqq n$ *if and only if* $\operatorname{depth}A_{\mathfrak{p}}\geqq\inf(n,\operatorname{ht}\mathfrak{p})$ *for all prime ideals* $\mathfrak{p}$ *of* A.

REMARK. This last condition is condition S_n of Serre for the ring A. See [12] as a reference.

PROOF OF PROPOSITION 4. Since $\operatorname{grade}M=\inf_{\mathfrak{p}\in\operatorname{Supp}M}\operatorname{depth}A_{\mathfrak{p}}$ it follows that $M\in\mathcal{G}_i$ and $\mathfrak{p}\in\operatorname{Supp}M$ implies $\operatorname{ht}\mathfrak{p}\geqq\operatorname{depth}A_{\mathfrak{p}}\geqq i$. Hence $M\in\mathcal{N}_i$.

b) Suppose $\mathcal{G}_i=\mathcal{N}_i$ for $0\leqq i\leqq n$. Let $\mathfrak{p}$ be a prime ideal of A. Then by hypothesis $A/\mathfrak{p}\in\mathcal{N}_{\operatorname{ht}\mathfrak{p}}\subseteqq\mathcal{G}_{\inf(n,\operatorname{ht}\mathfrak{p})}$. Hence

$$\inf(n,\operatorname{ht}\mathfrak{p})\leqq\operatorname{grade}A/\mathfrak{p}\leqq\inf_{\mathfrak{q}\supseteqq\mathfrak{p}}\operatorname{depth}A_{\mathfrak{q}}.$$

In particular, $\operatorname{depth}A_{\mathfrak{p}}\geqq\inf(n,\operatorname{ht}\mathfrak{p})$. On the other hand, if this last condition is satisfied for each prime ideal in A and $M\in\mathcal{N}_j$, then

$$\operatorname{grade}M=\inf_{\mathfrak{p}\in\operatorname{Supp}M}\operatorname{depth}A_{\mathfrak{p}}\geqq\inf_{\mathfrak{p}}\inf(n,\operatorname{ht}\mathfrak{p})\geqq\inf(n,j).$$

Hence $M\in\mathcal{G}_{\inf(n,j)}$.

This completes the proof.

It is possible to give a characterization of $\mathcal{G}_i$ in terms similar to the above for $\mathcal{N}_i$. In fact let

$$Y_i=\{\mathfrak{p}\in\operatorname{Spec}A:\ A/\mathfrak{p}\in\mathcal{G}_i\}.$$

Then $\mathcal{G}_i$ is the class of A-modules of finite type M such that $\operatorname{Supp}M\subset Y_i$. Let $\mathcal{G}'_i$ denote the class of all A-modules N such that $\operatorname{Supp}N\subset Y_i$. Let $E(M)$ be the injective envelope of M. Then

$$E(M)=\coprod_{\mathfrak{p}\in\operatorname{Ass}M}\mu_0(\mathfrak{p},M)E(A/\mathfrak{p}).$$

Now $\operatorname{Supp}E(A/\mathfrak{p})=V(\mathfrak{p})=\{\mathfrak{q}\in\operatorname{Spec}A:\mathfrak{q}\supseteqq\mathfrak{p}\}$. Hence $M\in\mathcal{G}_i$ if and only if $E(M)\in\mathcal{G}'_i$.

PROPOSITION 5. *Let M be an A-module of finite type. Let*

$$E(M)=\coprod_{\mathfrak{p}\in\operatorname{Ass}M}\mu_0(\mathfrak{p},M)E(A/\mathfrak{p}),$$

and let

$$E(i)(M) = \coprod_{\mathfrak{p}\in \mathrm{Ass} M \cap Y_i} \mu_0(\mathfrak{p}, M) E(A/\mathfrak{p}) .$$

Then $E(i)(M) = E(M(i))$ *and* $E(M)/E(i)(M) = E(M/M(i))$.

PROOF. $E(i)(M)$ is the maximal submodule of $E(M)$ in $\mathcal{G}'_i$. It is then standard that $E(i)(M) = E(M(i))$. From the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & M(i) & \to & M & \to & M/M(i) & \to & 0 \\ & & \psi\downarrow & & \downarrow & & \downarrow\varphi & & \\ 0 & \to & E(M(i)) & \to & E(M) & \to & E(M)/E(i)(M) & \to & 0 \end{array}$$

it is seen that $\operatorname{Ker}\varphi$ is isomorphic to a submodule of $\operatorname{Coker}\psi$. As $\operatorname{Coker}\psi \in \mathcal{G}'_i$ and as $(M/M(i))(i) = 0$, it follows that $\operatorname{Ker}\varphi = 0$, and hence that $E(M)/E(i)(M) = E(M/M(i))$.

The next proposition is a partial generalization of a result due to Auslander (cf. Proposition 1, Chap. 3 of [11]).

PROPOSITION 6. *Let M be an A-module with* $\operatorname{grade}_A M \geqq n$. *Then there is an A-module D and exact sequences of functors*

$$\begin{aligned} &0 \to \operatorname{Ext}_A^1(D, \cdot) \to \operatorname{Tor}^A_n(M, \cdot) \to \operatorname{Ext}_A^0(\operatorname{Ext}_A^n(M,A), \cdot) \to \dots \\ &\dots \to \operatorname{Ext}_A^j(D, \cdot) \to \operatorname{Tor}^A_{n+1-j}(M, \cdot) \to \operatorname{Ext}_A^{j-1}(\operatorname{Ext}_A^n(M,A), \cdot) \to \dots \\ &\dots \to \operatorname{Ext}_A^{n+1}(D, \cdot) \to M \otimes_A \cdot \to \operatorname{Ext}_A^n(\operatorname{Ext}_A^n(M,A), \cdot) \to \\ &\to \operatorname{Ext}_A^{n+2}(D, \cdot) \to 0 , \end{aligned}$$

$$\begin{aligned} &0 \leftarrow \operatorname{Tor}^A_1(D, \cdot) \leftarrow \operatorname{Ext}_A^n(M, \cdot) \leftarrow \operatorname{Tor}^A_0(\operatorname{Ext}_A^n(M,A), \cdot) \leftarrow \dots \\ &\dots \leftarrow \operatorname{Tor}^A_j(D, \cdot) \leftarrow \operatorname{Ext}_A^{n+1-j}(M, \cdot) \leftarrow \operatorname{Tor}^A_{j-1}(\operatorname{Ext}_A^n(M,A), \cdot) \leftarrow \dots \\ &\dots \leftarrow \operatorname{Tor}^A_{n+1}(D, \cdot) \leftarrow \operatorname{Hom}_A(M, \cdot) \leftarrow \operatorname{Tor}^A_n(\operatorname{Ext}_A^n(M,a), \cdot) \leftarrow \\ &\leftarrow \operatorname{Tor}^A_{n+2}(D, \cdot) \leftarrow 0 \end{aligned}$$

and isomorphisms of functors

$$\begin{aligned} &\operatorname{Ext}_A^{n+j}(\operatorname{Ext}_A^n(M,A), \cdot) \to \operatorname{Ext}_A^{n+j+2}(D, \cdot) , \\ &\operatorname{Tor}^A_{n+j}(\operatorname{Ext}_A^n(M,A), \cdot) \leftarrow \operatorname{Tor}^A_{n+j+2}(D, \cdot) \quad \textit{for } j \geqq 1 . \end{aligned}$$

PROOF. Consider the transformation of functors

$$t \colon M \otimes_A \operatorname{Hom}_A(N, L) \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, N), L)$$

which is natural in all the variables. For M a projective A-module this is an isomorphism. For $M = \mathrm{Hom}_A(P,A)$ where P is a projective A-module and $N = A$, this map becomes the isomorphism $\mathrm{Hom}_A(P,A) \otimes_A L \to \mathrm{Hom}_A(P,L)$, since $P \to \mathrm{Hom}_A(\mathrm{Hom}_A(P,A),A)$ is an isomorphism. Now let M be an A-module with $\mathrm{grade}_A M \geqq n$. Let

$$P_{n+1} \to P_n \to \ldots \to P_0 \to M \to 0$$

be an exact sequence with each P_j a projective A-module and let $\boldsymbol{P}$ denote the complex of projective A-modules

$$\ldots \to 0 \to \ldots \to 0 \to P_{n+1} \to \ldots \to P_1 \to 0 .$$

Suppose N is an A-module and a complex concentrated at 0. The two natural isomorphisms given by t above for the choices $(M,N,L) = (P,A,N)$ and $(\mathrm{Hom}_A(P,A),A,N)$ respectively induce the two Künneth spectral sequences

$$\mathrm{Ext}_A{}^p(H^q(\boldsymbol{P}^*),N) \Rightarrow H_n(\boldsymbol{P} \otimes_A N)$$

and

$$\mathrm{Tor}^A{}_p(H^q(\boldsymbol{P}^*),N) \Rightarrow H^n(\mathrm{Hom}_A(\boldsymbol{P},N)) ,$$

where $\boldsymbol{P}^* = \mathrm{Hom}_A(\boldsymbol{P},A)$. Since $H^q(\boldsymbol{P}^*) = 0$ for $q \neq n, n+1$, the spectral sequences degenerate to the exact sequences and isomorphisms of the proposition, where D is taken to be $D = H^{n+1}(\boldsymbol{P}^*)$ and $H^n(\boldsymbol{P}^*) = \mathrm{Ext}_A{}^n(M,A)$.

Remark. In the notation of [11], $D = D(\Omega^n M)$, so the first four terms of the exact sequences are just those of Proposition 5.8c [ibid.] and Theorem 7.5 of [1] and Theorem 2.8 of [2].

In particular, there results from these exact sequences natural transformations

$$M \otimes_A \cdot \to \mathrm{Ext}_A{}^n(\mathrm{Ext}_A{}^n(M,A), \cdot)$$

and

$$\mathrm{Hom}_A(M, \cdot) \to \mathrm{Tor}^A{}_n(\mathrm{Ext}_A{}^n(M,A), \cdot) .$$

It is possible to deduce the existence of these homomorphisms from Proposition 2 and the results quoted in the Remark above, but the long exact sequences are given because they seem to be of some independent interest.

Finally, recall that A is a Gorenstein ring if $\mathrm{grade}\,\mathrm{Ext}_A{}^j(M,A) \geqq j$ for all j and all A-modules M of finite type.

2.

Roos, in [10], uses a similar spectral sequence to obtain the results in his Theorem 1. In this section is a statement and proof of his theorem for Gorenstein rings. His condition (α) is, in fact, the definition of a Gorenstein ring. The condition (δ) is false in general as an easy example shows. However it is possible to prove

PROPOSITION 7. *Let A be a Gorenstein ring, M an A-module of finite type. Let $E_A{}^i(M)=\mathrm{Ext}_A{}^i(M,A)$ and $L^A{}_i(M)=E_A{}^i(E_A{}^i(M))$. (The A will usually be omitted.)*

a) *$L_i(L_i(M))=L_i(M)$, equality being induced by the natural homomorphism of the second module into the first.* (Condition (η) of Roos.)

b) *If* $\mathrm{grade}\, M \geqq i$, *then* $M(i+1)=\mathrm{Ker}(M \to L_i(M))$ (part of condition (ε) of Roos).

c) $\mathrm{grade}\, E^j(L_i(M)) \geqq j+2$ *for* $j>i$ (condition (θ) of Roos).

d) $L_i(M(i))=L_i(M)$.

e) $\mathrm{grade}\,\mathrm{Coker}(M(i) \to L_i(M)) \geqq i+2$ (remainder of condition (ε)).

REMARK. Chr. U. Jensen kindly pointed out to the author that the results of Roos in [10, pages 1720–1722] concerning the derived functors, $\varprojlim^{(i)}$, of $\varprojlim$ can be deduced from Proposition 7 for any quotient ring of a Gorenstein ring. In particular, it follows that for any quotient ring B of a Gorenstein ring, $\varprojlim^{(i)} M_\alpha=0$ when each M_α is of finite type and $i>\mathrm{Krull\,dim}\, B$. (It has been conjectured that this holds for any noetherian ring B.)

PROOF OF PROPOSITION 7. Let $f_1,\ldots,f_i$ be an A-sequence in $\mathrm{Ann}\, E^i(M)$ and set $B=A/\mathfrak{f}$ where $\mathfrak{f}$ is the ideal generated by the A-sequence. From Proposition 2 it follows that $L_i(M)=\mathrm{Hom}_B(E_A{}^i(M),B)$. Since B is Gorenstein and $L^A{}_i(M)$ is the dual of a B-module, it follows that it is reflexive (Proposition 6.1 (6) of [5]). Hence

$$\begin{aligned} L^A{}_i(L^A{}_i(M)) &= L^B{}_0\big(\mathrm{Hom}_B(E_A{}^i(M),B)\big) \\ &= \mathrm{Hom}_B(E_A{}^i(M),B) = L^A{}_i(M)\,. \end{aligned}$$

This proves a).

To see b), note that the kernel of the map $M \to L_i(M)$, which is a module of the form $E_A{}^{i+1}(D)$ from Proposition 6, has grade greater than i. Let N be a submodule of M with $\mathrm{grade}\, N>i$. Let $M''=M/N$. Then $E^i(M)=E^i(M'')$ and so $L_i(M)=L_i(M'')$. From the commutative diagram

$$\begin{array}{ccccc} 0 \to N \to & M & \to & M'' & \to 0 \\ & \downarrow & & \downarrow & \\ & L_i(M) & = & L_i(M'') & \end{array}$$

it is seen that N is in the kernel. Hence $E^{i+1}(D)$ is the maximal submodule of M of grade more than i. So b) is established.

The proof of c) goes by induction on i. If $i=0$, then $N=M^{**}$ is a reflexive A-module. By Theorem 8.2 of [3] it follows that there is an exact sequence of A-modules

$$0 \to N \to P_1 \to P_0 \to L \to 0$$

with P_0 and P_1 projective. Hence $E^j(N)=E^{j+2}(L)$ for $j \geqq 1$. Hence $\operatorname{grade} E^j(N) \geqq j+2$.

Suppose the statement has been established for all integers k, $0 \leqq k \leqq i-1$. Let $f \in \operatorname{Ann} L_i(M)$ be regular on A. Let $B=A/fA$. Then $E_A{}^j(L^A{}_i(M))=E_B{}^{j-1}(L^A{}_i(M))$ by Proposition 2. Since $L_i(L_i(M))=L_i(M)$ by a), it follows that $L^B{}_{i-1}(L^A{}_i(M))=L^A{}_i(M)$. Hence, by the induction hypothesis, $\operatorname{grade}_B E_B{}^{j-1}(L_i(M)) \geqq j+1$ for $j-1>i-1$. But $\operatorname{grade}_A N=1+\operatorname{grade}_B N$ for a B-module N. Thus the statement is proved.

To show d), the isomorphism is shown first for $i=1$. There is the exact sequence

$$0 \to M(1) \to M \to M^{**} \to Q \to 0$$

from Proposition 6, where $\operatorname{grade} Q \geqq 2$. Let $M''=M/M(1)$ and consider the resulting exact sequence

$$\ldots \to E^j(Q) \to E^j(M^{**}) \to E^j(M'') \to E^{j+1}(Q) \to \ldots .$$

Since $\operatorname{grade} E^j(M^{**}) \geqq j+2$ for $j \geqq 1$ and $\operatorname{grade} E^{j+1}(Q) \geqq j+1$, it follows that $\operatorname{grade} E^j(M'') \geqq j+1$ by Proposition 3. Consider also the exact sequence

$$0 \to E^1(M'') \to E^1(M) \to E^1(M(1)) \to E^2(M'') \to \ldots ,$$

For each j, let

$$\begin{aligned} K_j &= \operatorname{Ker}\big(E^j(M) \to E^j(M(1))\big) , \\ W_j &= \operatorname{Coker}\big(E^j(M) \to E^j(M(1))\big) , \\ U_j &= \operatorname{Coker}\big(E^j(M'') \to E^j(M)\big) . \end{aligned}$$

Then there are the corresponding short exact sequences for each j. Since K_j is a homomorphic image of $E^j(M'')$ it has grade greater than j.

Since W_j is a submodule of $E^{j+1}(M'')$ it has grade greater than j. Hence $L_j(M)=E^j(U_j)=L_j(M(1))$.

Now let $f \in \operatorname{Ann} M(1)$ be regular on A. Let $B=A/fA$. Let N denote the B-module $M(1)$. Then it follows that $N(j)=M(j+1)$. Now it may be assumed that the statement has been established for B-modules, so that $L^B{}_{j-1}(N(i-1))=L^B{}_{j-1}(N)$ for $j-1 \geqq i-1$. But then, by Proposition 2, $L^A{}_j(M(i))=L^A{}_j(M(1))$ and d) is established.

To see e), note that the cokernel of the homomorphism $M(i) \to L_i(M(i))=L_i(M)$ is of the form $E^{i+2}(D)$ for some A-module D, and hence this statement is established.

The proof of the proposition is now complete.

The chain $M=M(0) \supseteqq M(1) \supseteqq \ldots$ is related to Auslander's spherical filtration. A chain $M=M_0 \supseteqq M_1 \supseteqq \ldots \supseteqq M_n$ is a spherical filtration of length n provided

a) $\operatorname{pd}_A(M_{i-1}/M_i) \leqq i$ and $E_A{}^j(M_{i-1}/M_i)=0$ for $1 \leqq j \leqq i-1$,

b) For each i, $M_{i-1} \to M_{i-1}/M_i$ induces an isomorphism $E_A{}^j(M_{i-1}/M_i) \to E_A{}^j(M_{i-1})$ for $1 \leqq j \leqq i$.

According to Proposition 8, Chap. 2 of [11] (also found in [2]) each module M of finite type over the Gorenstein ring A admits a free module F such that $F \oplus M$ has a spherical filtration of length n for any fixed n. The next two propositions relate the two chains.

PROPOSITION 8. *Suppose A is Gorenstein and M is an A-module with $E_A{}^i(M)=0$ for $1 \leqq i < n$. Let M' be a submodule of M satisfying the conditions:*

i) $E_A{}^i(M')=0$ *for* $1 \leqq i \leqq n$.

ii) $M^* \to M'^* \to 0$ *is exact.*

iii) $\operatorname{Ext}_A{}^j(M, \cdot) \to \operatorname{Ext}_A{}^j(M', \cdot)$ *is an isomorphism for* $j > n$.

Then the following hold:

a) $\operatorname{pd}_A(M/M') \leqq n$.

b) $E_A{}^i(M/M')=0$ *for* $1 \leqq i < n$ *and* $E_A{}^n(M/M') \to E_A{}^n(M)$ *is an isomorphism.*

c) $M(1)=M(n)$.

d) $M(n+1)=M'(1)=M'(n+1)$.

PROOF. $M''=M/M'$ satisfies conditions a) and b) by Proposition 9, Chap. 2 of [11]. Since $L^A{}_j(M)=0=L^A{}_i(M')$ for $1 \leqq j < n$ and $1 \leqq i \leqq n$, it follows that

$$M(j) = \operatorname{Ker}(M(j-1) \to L^A{}_{j-1}(M)) = M(j-1) \quad \text{for } 2 \leqq j \leqq n .$$

Hence c) is verified. Likewise $M'(1)=M'(n+1)$.

It remains to show that $M(n+1)=M'(n+1)$. If

$$R_1 \to R_0 \to M \to 0$$

is exact with R_0 and R_1 projective, then, with $D(M)=\mathrm{Coker}(R_0^* \to R_1^*)$, Propositions 6 and 7 show that the natural homomorphism $E_A{}^1(D(M)) \to M$ induces an isomorphism $E_A{}^1(D(M))=M(1)$. Let

$$0 \to P_n \to \ldots \to P_0 \to M'' \to 0$$

be an exact sequence with each P_j projective and let

$$Q_1 \to Q_0 \to M' \to 0$$

be exact with each Q_j projective. Then R_0 and R_1 and homomorphisms can be found making the diagram below commutative with exact rows and columns.

$$\begin{array}{cccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Q_1 & \to & Q_0 & \to & M' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
R_1 & \to & R_0 & \to & M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
P_1 & \to & P_0 & \to & M'' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

When $\mathrm{Hom}_A(\cdot, A)={}^*$ is applied to this diagram, there results, by virtue of condition ii), the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M''^* & \to & P_0^* & \to & P_1^* & \to & D'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M^* & \to & R_0^* & \to & R_1^* & \to & D & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'^* & \to & Q_0^* & \to & Q_1^* & \to & D' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Now $E_A{}^1(D'')=E_A{}^n(E_A{}^n(M''))=L^A{}_n(M)$ since the sequence

$$0 \to D'' \to P_2^* \to \ldots \to P_n^* \to E_A{}^n(M'') \to 0$$

is exact and $M \to M''$ induces an isomorphism $E_A{}^n(M'') = E_A{}^n(M)$. When * is applied to the exact sequence of the D's, there results the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ & E_A{}^1(D') & \to & E_A{}^1(D) & \to & E_A{}^1(D'') & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & M' & \to & M & \to & M'' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & M'^{**} & \to & M^{**} & \to & M''^{**} & \end{array}.$$

Hence $0 \to E^1(D') \to E^1(D) \to L_n(M)$ is exact, and so

$$M'(n+1) = E^1(D') = \operatorname{Ker}(M(n) \to L_n(M)).$$

Thus $M'(n+1) = M(n+1)$ and the proof is complete.

When applied to a spherical filtration of an A-module, this proposition yields

Proposition 9. *Suppose A is a Gorenstein ring, and*

$$M = M_0 \supset M_1 \supset \ldots \supset M_n$$

a spherical filtration of the A-module M. Then

$$M(i) = M_{i-1}(1) = M_{j-1}(i)$$

for all $i \leqq n$ and all $j < i+1$.

Proof. For each $1 \leqq j \leqq n$, the modules $M_{j-1} \supset M_j$ with $M'' = M_{j-1}/M_j$ satisfy the conditions of Proposition 8. Thus the result follows.

3.

In connection with these results, it is possible to give a result, which, for A an integrally closed integral domain, is due to Auslander and Buchsbaum ([3] and [6, p. 52]).

Proposition 10. *Let A be a Gorenstein ring, N an A-module such that* grade $N = i$ *and $N(i+1) = 0$. Let M be a submodule with $Q = N/M$.*

a) *If $M = L_i(M)$, then $Q(i+2) = 0$.*

b) *If $N = L_i(N)$ and $Q(i+2) = 0$, then $M = L_i(M)$.*

Proof. Since $N(i+1) = 0$, N may be considered to be a submodule of $L_i(N)$. If $W = L_i(N)/M$, then Q may be considered to be a submodule of W. Then $Q(i+2)=Q\cap W(i+2)$. So one may assume that $N=L_i(N)$ and show that $Q(i+2)=0$. Now apply the functor * twice to the exact sequence relating M, N, and Q to get the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & L_i(M) & \to & L_i(N) & \to & Q & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E^i(S) & \to & L_i(N) & \to & L_i(Q) & \to & E^{i+1}(S) \to \dots \\ & & \downarrow & & \downarrow & & & & \\ & & L_{i+1}(Q) & & 0 & & & & \end{array}$$

for some A-module S. Now $\operatorname{Ker}(Q\to L_i(Q))=Q(i+1)$. From the serpent lemma, $Q(i+1)$ is a submodule of $L_{i+1}(Q)$ and hence $Q(i+2)=0$. This proves a).

If, on the other hand, $N=L_i(N)$ and $Q(i+2)=0$, then the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & M & \to & N & \to & Q & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_i(M) & \to & L_i(N) & \to & W & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & V & & 0 & & & & \end{array}$$

is obtained where $\operatorname{Ker}(Q\to W)=V$ and grade $V\geqq i+2$ by e) of Proposition 7. Hence $V\subseteqq Q(i+2)=0$. Thus the proof of the proposition is complete.

REMARK. This proposition does not imply the result of Auslander and Buchsbaum. However it is possible to state and prove modified forms of this result under less restrictive hypotheses: For instance one could suppose $A_\mathfrak{p}$ is Gorenstein for height $\mathfrak{p}\leqq n$, and then add some unmixedness properties. As an example, Theorem (1.4) of [12] has as a corollary

PROPOSITION 10′. *Let A be a quasi-normal ring (i.e. $A_\mathfrak{p}$ is Gorenstein for prime ideals $\mathfrak{p}$ with* $\operatorname{ht}\mathfrak{p}\leqq 1$ *and* $\operatorname{depth}A_\mathfrak{p}\geqq\inf(2,\operatorname{ht}\mathfrak{p})$ *for all prime ideals $\mathfrak{p}$ of A). Let $0\to M\to N\to Q\to 0$ be an exact sequence of A-modules with N torsion free.*

a) *If M is reflexive, then $\mathfrak{p}\in\operatorname{Ass}Q$ implies* $\operatorname{ht}\mathfrak{p}\leqq 1$.

b) *If N is reflexive and $\mathfrak{p}\in\operatorname{Ass}Q$ implies* $\operatorname{ht}\mathfrak{p}\leqq 1$, *then M is reflexive.*

PROOF. The proof of a) is the same as the proof of Theorem (1.4) in [12]. To see b), suppose f_1, f_2 is a regular A-sequence. It is required to show that this is a regular M-sequence. f_1 is regular on M since it is regular on N. There results the exact sequence

$$0 \to \mathrm{Ker}_Q f_1 \to M/f_1M \to N/f_1N \to Q/f_1Q \to 0$$

where $\mathrm{Ker}_Q f_1$ denotes the kernel of the homomorphism on Q induced by multiplication by f_1. If $f_2m' = 0$ for $m' \in M/f_1M$, then $m' \in \mathrm{Ker}_Q f_1$ so f_2 is in a prime ideal $\mathfrak{p}$ associated to $\mathrm{Ker}_Q f_1$. But f_1 is also in this prime ideal and hence $\mathrm{ht}\,\mathfrak{p} \geqq 2$, a contradiction. So f_1, f_2 is a regular M-sequence and then M is reflexive. Thus the proof of the proposition is complete.

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