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ON HOMOTOPY INVARIANCE OF THE TANGENT BUNDLE II

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1. Introduction.

This paper is a subsequence of the paper [5], in which the following problem is considered.

Let M and M' be oriented compact, differentiable manifolds, let $f\colon M \to M'$ be a homotopy equivalence preserving orientation, and denote the tangent sphere bundles τ and τ' respectively. Is it true then, that τ and $f^*\tau'$ are fibre homotopy equivalent?

This is actually shown by R. Benlian and J. Wagoner [3]; but here we will prove it by the simple method developed in [5]. As kindly pointed out to me by C. T. C. Wall, this method also applies to define the unstable tangent sphere fibration for a Poincaré complex which is necessary for developing a theory for embedding and surgery of Poincaré complexes.

Finally I also want to thank M. F. Atiyah, W. Browder and W. Sutherland for interesting remarks on the note [5] which made this paper possible.

2. Sphere fibrations.

In this section we will study more closely the "action" defined in [5, § 2]. The results of this section are closely related to the work of James and Thomas [7], Rutter [8] and Barcus and Barratt [2]. In particular our Corollary 2.3 and Proposition 2.7 are reformulations of Theorem 1.8 in James and Thomas [7]. (Compare the remark following our Definition 4.6.)

As usual $H(n)$ denotes the space of homotopy equivalences of S^{n-1}, $SH(n)$ denotes the component of $H(n)$ consisting of maps of degree $+1$, and $F(n)$ denotes the subspace of $SH(n+1)$ consisting of basepoint preserving maps. There is a natural inclusion of $SH(n)$ in $F(n)$ by means of unreduced suspension.

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A Hurewicz fibration with fibre a homotopy $n-1$ sphere is called a $n-1$ sphere fibration. According to J. Stasheff [12] there is a space $B_n = BSH(n)$ classifying oriented $n-1$ sphere fibrations over CW-complexes, such that homotopy classes of the classifying maps are in one-to-one correspondence with equivalence classes of sphere fibrations under orientation preserving fibre homotopy equivalence.

If ξ_1 and ξ_2 are sphere fibrations over a space X, $\xi_1 + \xi_2$ denotes the fibrewise join of ξ_1 and ξ_2. If ξ_1 and ξ_2 are sphere fibrations over X_1 and X_2 respectively and $p_i\colon X_1 \times X_2 \to X_i$ are the projections, we put

$$\xi_1 \times \xi_2 = (p_1{}^* \xi_1) + (p_2{}^* \xi_2) .$$

The trivial $k-1$ sphere fibration is simply denoted by k.

Analogously there are spaces $BF(n)$ classifying pairs (ξ, s) consisting of an oriented n sphere fibration ξ and a section s. Homotopy classes of the classifying maps are in one-to-one correspondence with equivalence classes of pairs under section and orientation preserving fibre homotopy.

If ξ is an oriented sphere fibration and s and s' are homotopic sections, then the pairs (ξ, s) and (ξ, s') are clearly equivalent.

For any CW-complex X the natural map

$$[X, BF(n)] \to [X, BSH(n+1)]$$

corresponds to forgetting the section, and the map

$$[X, BSH(n)] \to [X, BF(n)]$$

corresponds to the map sending ξ to the pair $(\xi + 1, s_1)$, where s_1 is the section which is constantly 1.

Lemma 2.1. *Let ξ be a q sphere fibration over a q-dimensional finite CW-complex X.*

Any section s of ξ gives rise to a $q-1$ sphere fibration ξ', such that $(\xi' + 1, s_1)$ and (ξ, s) are equivalent pairs. The equivalence class of ξ' only depends on the homotopy class of s.

Proof. According to James [6], the map

$$j_*\colon \pi_i(SH(q)) \to \pi_i(F(q))$$

is an isomorphism for $i < 2(q-2)$ and an epimorphism for $i = 2(q-2)$. This, together with an easy calculation for $q = 2, 3$, implies that j_* is an isomorphism for $i \leqq q-1$. Hence the map

$$[X, BSH(q)] \to [X, BF(q)]$$

is bijective for X at most q-dimensional. This proves the lemma.

Especially consider $X = L \cup e^q$, where L is a $(q-1)$-dimensional complex (according to Wall [15] this is the case for a q-dimensional Poincaré complex), and let $\xi = \xi_0 + 1$, where ξ_0 is a $q-1$ sphere fibration.

By obstruction theory any section s of ξ is homotopic over L to the trivial section s_1 which is constantly 1. Extending this homotopy to X (Strøm [13]) we conclude that any homotopy class of sections of $\xi_0 + 1$ is representable by a section which is trivial over L. Trivializing ξ_0 over e^q, s defines a map

$$(e^q, S^{q-1}) \to (S^q, *)$$

of a certain degree $d(s)$.

Later in this section we will see that for ξ_0 oriented, $d(s)$ depends only on the homotopy class of s, and thus $d(s)$ determines this uniquely. (For ξ_0 non-orientable the homotopy class of s is determined by the mod 2 degree.)

For any integer d let g_d denote the composite map

$$X \xrightarrow{c} X \vee S^q \xrightarrow{1 \vee f_d} X \vee S^q,$$

where c is the pinching map and $f_d \colon S^q \to S^q$ is of degree d. Further let ξ_d denote the fibration

$$\xi_d = g_d^* (\xi_0 \vee \tau_q),$$

where τ_q is the tangent sphere bundle of S^q. If $v_0 \colon X \to B_q$ is classifying for ξ_0 and $\mu_0 \colon S^q \to B_q$ is classifying for τ_q, then in the notation of [5, Section 2] $v_0{}^{d\mu_0}$ is classifying for ξ_d. Clearly there is a natural equivalence

$$\xi_d + 1 = g_d^*((\xi_0 + 1) \vee (\tau_q + 1)) \cong g_d^*((\xi_0 + 1) \vee (q+1)) \cong \xi_0 + 1$$

which we denote by γ_d. Under this the constant section of $\xi_d + 1$ defines a section $\sigma(d)$ of $\xi_0 + 1$ of degree d. In fact the constant section of $\tau_q + 1$ over S^q has degree one with respect to the obvious trivialization.

Using Lemma 2.1 we clearly have

PROPOSITION 2.2. *For any section s of $\xi_0 + 1$,*

$$\xi' = \xi_{d(s)} = g_{d(s)}^* (\xi_0 \vee \tau_q)$$

is the unique fibration such that

$$(\xi' + 1, s_1) \quad \textit{and} \quad (\xi_0 + 1, s)$$

are equivalent pairs.

PROOF. In fact $\sigma(d(s))$ and s are homotopic sections of ξ_0+1.

An equivalence α of a fibration ξ with itself is called an *automorphism* of ξ. For any automorphism α of the fibration ξ_0+1, where ξ_0 is a sphere fibration over an arbitrary space X, we define the section $s_\alpha=\alpha\circ s_1$ of ξ_0+1. Here again s_1 denotes the trivial section, and clearly $s_{id}=s_1$.

We now obtain in the special case of $X=L\cup e^q$:

COROLLARY 2.3. *For q odd, we have ${v_0}^{\mu_0}=v_0$ iff there is an automorphism α of ξ_0+1 such that $d(s_\alpha)$ is odd.*

PROOF. According to [5, Proposition 2.2], we have ${v_0}^{\mu_0}={v_0}^{d\mu_0}$ for d odd. Hence $v_0={v_0}^{\mu_0}$ iff $v_0={v_0}^{d\mu_0}$ or equivalently $\xi_0\simeq\xi_d$ for some odd integer d.

If $\beta\colon \xi_0\to\xi_d$ is an equivalence, then the composite equivalence

$$\gamma_d\circ(\beta+1)\colon\ \xi_0+1\to\xi_0+1$$

defines the section $s_{\gamma_d\circ(\beta+1)}=\sigma(d)$ of degree d.

Conversely, if $\alpha\colon \xi_0+1\to\xi_0+1$ has $d(s_\alpha)=d$, then (ξ_0+1,s_1) and (ξ_0+1,s_α) are equivalent pairs, and hence we conclude from Proposition 2.2 that

$$\xi_0 \cong \xi_{d(s_\alpha)} = \xi_d\,.$$

Turning to the general case of a $q-1$ sphere fibration ξ_0 over an arbitrary space X, we consider the Thom complex $T(\xi_0)$. This is defined as the mapping cone on the projection map, and it is easily seen to be homeomorphic to the space $\xi_0+1/s_1(X)$, in such a way that the inclusion $X\to T(\xi_0)$ in the mapping cone corresponds to the section s_{-1} of ξ_0+1 which is constantly -1.

When ξ_0 is oriented, the Thom class

$$U_{\xi_0}\in H^q(\xi_0+1,\mathrm{Z})$$

is the unique class which restricted to the fibre is the generator and which satisfies ${s_1}^*U_{\xi_0}=0$.

DEFINITION 2.4. For any section s of ξ_0+1, put

$$d(s) = s^*U_{\xi_0}\in H^q(X,\mathrm{Z})$$

and for α an automorphism, put

$$\chi(\alpha) = d(s_\alpha)\in H^q(X,\mathrm{Z})\,.$$

As an example the equivalence α induced by multiplication by -1 in the trivial part 1 of ξ_0+1, has $\chi(\alpha)=e(\xi_0)$, the Euler class of ξ_0.

PROPOSITION 2.5. *For orientation preserving automorphisms α and β of ξ_0+1 we have*

$$\chi(\alpha\circ\beta) = \chi(\alpha)+\chi(\beta) .$$

PROOF. Put $u=U_{\xi_0}$ for short and denote the projection for ξ_0+1 by p. Obviously

$$\alpha^* u = u + p^*(d(s_\alpha)) .$$

Hence

$$\begin{aligned} s^*_{\alpha\circ\beta}u = (\alpha\circ s_\beta)^* u &= s_\beta{}^* u + s_\beta{}^* p^*(d(s_\alpha)) \\ &= d(s_\beta) + d(s_\alpha) . \end{aligned}$$

For X a q-dimensional Poincaré complex Definition 2.4 agrees with the previously defined degree. In fact for any integer d, the degree of $\sigma(d)$ is d.

Notice that we could also have defined d and χ mod 2 for any sphere fibration. Then Proposition 2.5 is valid for all automorphisms.

In view of Corollary 2.3 only the mod 2 degree is essential for our purpose. We will thus restrict to $\mathbf{Z}_2$ coefficients in all cohomology groups for the rest of this paper, unless otherwise specified.

DEFINITION 2.6. Let ξ be a sphere fibration over a space X with base point x_0, and consider an automorphism α of ξ. Denoting the unit interval by I, consider $\xi\times I$ with the identifications

$$\begin{aligned} (x,1) &\sim (\alpha x,0) \quad \text{for} \quad x\in\xi , \\ (x,t) &\sim (x,t') \quad \text{for} \quad x\in\xi_{x_0} \text{ and } t,t'\in I . \end{aligned}$$

This defines a fibration denoted ξ_α over $X\times S^1/x_0\times S^1$.

Denote the Euler class by e, the suspension of X by ΣX, the suspension homomorphism by Σ, and the natural map of $X\times S^1/x_0\times S^1$ onto ΣX by j. We then have

PROPOSITION 2.7. *For any automorphism α of $\xi=\xi_0+1$, where ξ_0 is a $q-1$ sphere fibration, we have*

$$e(\xi_\alpha) = j^*\Sigma(\chi(\alpha)) .$$

PROOF. The Euler class of ξ_α is the image under the transgression of the generator of $H^q(S^q,+)$. The transgression is the additive relation

$$H^q(S^q, +) \xrightarrow{\delta} H^{q+1}(\xi_\alpha, S^q) \xleftarrow{p^*} H^{q+1}(X \times S^1, x_0 \times S^1) ,$$

where p is the projection.

Obviously $e(\xi_0+1)=0$, so $e(\xi_\alpha)$ is in the image of j^*. Consider the commutative diagram with exact columns:

$$\begin{array}{ccccc}
H^q(S^q, +) & \xrightarrow{\delta} & H^{q+1}(\xi_\alpha, S^q) & \xleftarrow{p^*} & H^{q+1}(X \times S^1, x_0 \times S^1) \\
\uparrow{\scriptstyle i^*} & & \uparrow & & \uparrow{\scriptstyle j^*} \\
H^q((\xi_0+1) \times 0, s_1(X) \times 0) & \xrightarrow{\delta} & H^{q+1}(\xi_\alpha, (\xi_0+1) \times 0) & \xleftarrow{p^*} & H^{q+1}(\Sigma X) \\
 & & \uparrow{\scriptstyle \delta'} & & \uparrow{\scriptstyle 0} \\
 & & H^q(\xi_0+1, S^q) & \xleftarrow{p^*} & H^q(X) .
\end{array}$$

It is easy to see that the lower p^* is an isomorphism, and hence $\delta'=0$.

By definition $i^*U_{\xi_0}$ is the generator of $H^q(S^q, +)$. Hence $(j^*)^{-1}e(\xi_\alpha)$ is the image of U_{ξ_0} under the additive relation on the middle row. Now

$$p\colon\ \xi_\alpha/(\xi_0+1) \times 0 \to \Sigma X$$

has a right inverse s_0 defined by

$$s_1 \times \mathrm{id}\colon\ X \times I \to (\xi_0+1) \times I .$$

That is, $(j^*)^{-1}e(\xi_\alpha)$ is the image of U_{ξ_0} under the map

$$H^q((\xi_0+1) \times 0,\ s_1(X) \times 0) \to H^{q+1}(\xi_\alpha/(\xi_0+1) \times 0) \xrightarrow{s_0^*} H^{q+1}(\Sigma X) .$$

Define a space F as the quotient space of $(X \times I) \cup (\xi_0+1)$ with the identifications

$$(x, 1) \sim s_\alpha(x) \quad \text{for } x \in X .$$

There is a map of triples

$$(F, (\xi_0+1) \cup X \times 0, X \times 0) \to (\xi_\alpha, (\xi_0+1) \times 0, s_1(X) \times 0)$$

defined by sending (x,t) to $(s_1 x, t)$. Hence we have the commutative diagram

$$\begin{array}{ccc}
H^q(\xi_0+1, s_1(X)) & \xrightarrow{\delta} & H^{q+1}(\xi_\alpha, (\xi_0+1) \times 0) \\
\downarrow & & \downarrow{\scriptstyle s_0^*} \\
H^q(\xi_0+1) & \xrightarrow{\delta} & H^{q+1}(\Sigma X) .
\end{array}$$

Here the lower δ is the connecting homomorphism for the pair (C_{s_α}, ξ_0+1), where $C_{s_\alpha}=F/X \times 0$ is the mapping cone on s_α. This proves Proposition 2.7.

We conclude this section with a lemma concerning homotopy of automorphisms. If ξ is a $n-1$ sphere fibration over X with base point x_0 and α is an automorphism of ξ, we have defined the fibration ξ_α over $X \times S^1/x_0 \times S^1$, the restriction of which to $X \times 0$ is ξ. Hence fibre homotopy classes of automorphisms of ξ are in one-to-one correspondence with homotopy classes of maps

$$X \times S^1/x_0 \times S^1 \to BH(n)$$

the restriction of which to $X \times 0$ is the classifying map for ξ.

LEMMA 2.8. *Let ξ be a q sphere fibration over a finite q-dimensional CW-complex, and α an automorphism of $\xi + k$, $k > 0$.*

Then α is homotopic to an automorphism of the form $\alpha' + id$, where α' is an automorphism of ξ.

PROOF. The map $BH(q+1) \to BH(q+k+1)$ is a $q+1$ equivalence according to James [6]. Hence the lemma follows from Spanier [9, Chapter 7, § 6, Theorem 22].

3. S-duality.

We shall need some simple lemmas concerning S-duality of Thom complexes. We refer to the papers of Atiyah [1], Spivak [11] and Wall [15] for the following fact:

Let M denote a q-dimensional Poincaré complex, with $(k-1)$-dimensional normal sphere fibration ν. If ξ and η are $n-1$ and $m-1$ sphere fibrations over M such that $\xi + \eta$ is trivial, then the diagonal $\Delta: M \to M \times M$ induces a map of Thom complexes

$$T(\nu+n+m) \to T(\nu+\eta) \wedge T(\xi) .$$

The composite with a Thom map

$$S^{q+k+n+m} \to T(\nu+n+m) \to T(\nu+\eta) \wedge T(\xi)$$

is a S-duality for $T(\nu+\eta)$ and $T(\xi)$.

PROPOSITION 3.1. *Let $i: M_1 \hookrightarrow M_2$ be an embedding of a closed manifold in another. Denote the normal bundle of M_1 and M_2 by ν_1 and ν_2 respectively, and the normal bundle of i by ν_0. Then the dual map of*

$$T(i^*\nu_2) \to T(\nu_2)$$

is the map

$$(M_2)_+ \to T(\nu_0)$$

which collapses everything outside a tubular neighbourhood of M_1 *in* M_2.

COROLLARY 3.2. *Let* M *be a closed manifold with normal bundle* ν *and angent bund le* τ. *Then the map*

$$T(\nu+\nu) \to T(\nu\times\nu)$$

induced by the diagonal $M \to M\times M$ *is the dual of the map*

$$(M\times M)_+ \to T(\tau)$$

which collapses everything outside a tubular neighbourhood of the diagonal.

PROOF OF PROPOSITION 3.1. Let N be a tubular neighbourhood of M_1 in M_2 with boundary $\dot N$. Clearly

$$T(\nu_1) = T(\nu_{2|N})/T(\nu_{2|\dot N}) .$$

Embedding M_2 in S^n, for n large, the proposition follows from the commutative diagram

$$\begin{array}{ccccc}
 & \nearrow & T(\nu_1) & \xrightarrow{f_1} & T(\nu_0)\wedge T(\nu_{2|M_1}) \\
 & & \| & & \downarrow \\
S^n & \longrightarrow & T(\nu_{2|N})/T(\nu_{2|\dot N}) & \xrightarrow{f_2} & T(\nu_0)\wedge T(\nu_2) \\
 & & & & \uparrow \\
 & \searrow & T(\nu_2) & \xrightarrow{f_3} & ((M_2)_+)\wedge T(\nu_2) .
\end{array}$$

Here f_1, f_2 and f_3 are induced by the diagonals $M_1 \to M_1\times M_1$, $N \to N\times M_2$ and $M_2 \to M_2\times M_2$ respectively.

Now let M denote an arbitrary Poincaré complex with normal sphere fibration ν, and let ξ and η be sphere fibrations such that $\xi+\eta$ is trivial.

LEMMA 3.3. *If* α *and* β *are automorphisms of* ξ *and* η *respectively, such that the automorphism* $\alpha+\beta$ *of* $\xi+\eta$ *is fibre homotopic to the identity, then*

$$T(1+\beta)\colon\ T(\nu+\eta) \to T(\nu+\eta)$$

is the dual of

$$T(\alpha)\colon\ T(\xi) \to T(\xi) .$$

LEMMA 3.4. *For any automorphism* α *of* ξ, *there is an automorphism* α' *of the trivial* $k-1$ *sphere fibration for some* $k>0$, *such that* $\alpha+\mathrm{id}$ *and* $\mathrm{id}+\alpha'$ *are fibre homotopic automorphisms of* $\xi+k$.

Lemma 3.5. *For any automorphism α of ξ there is an automorphism β of $\eta + k$, for some k, such that $\alpha + \beta$ is fibre homotopic to the identity.*

Proofs. The proof of Lemma 3.3 is trivial. Adding η to ξ it suffices to prove Lemmas 3.4 and 3.5 for ξ trivial.

For ξ trivial the stable fibre homotopy class of α corresponds to a map $\Sigma M \to BH$, where $BH = \lim BH(n)$. Lemma 3.5 now follows by well-known arguments from the fact that $[\Sigma M, BH]$ is a group in one and only one way.

Finally 3.4 follows from 3.5.

For later reference we finally state without proof the following well-known fact.

Lemma 3.6. *For M an n-dimensional Poincaré complex with normal $k-1$ sphere fibration ν, the composite map*

$$H^i(M) \xrightarrow{D} H_{n+k-i}(T(\nu)) \xrightarrow{\Phi} H_{n-i}(M)$$

of the S-duality homomorphism D and the Thom isomorphism Φ equals the Poincaré duality homomorphism. That is, $\Phi \circ D$ is cap product with the orientation class $[M]$.

4. Definition of $b(\xi)$.

We recall the notation of [5, § 4].

Assume q odd. The map

$$v_{q+1}\colon B_n \to K(\mathsf{Z}_2, q+1)$$

represents the Wu class v_{q+1}. Consider the fibration

$$\pi\colon B_n\langle v_{q+1}\rangle \to B_n$$

induced by v_{q+1} from the path fibration over $K(\mathsf{Z}_2, q+1)$ with fibre $\Omega K(\mathsf{Z}_2, q+1) = K(\mathsf{Z}_2, q)$. Put $\bar{\gamma}_n = \pi^*\gamma_n$, where γ_n is the universal $n-1$ sphere fibration over B_n. Then $Y_n = T(\bar{\gamma}_n)$ defines a Wu spectrum in the sense of Browder [4]. $\{X_n\}$ is the dual Wu cospectrum.

Now consider M a q-dimensional compact differentiable oriented manifold with normal bundle ν, and let ξ be any oriented $q-1$ sphere fibration over M. Choose a fibration η such that $\xi + \eta$ is trivial, and choose a lifting φ' through π of the classifying map φ for $\nu + \eta$.

Clearly $\nu + \eta = (\varphi')^*(\bar{\gamma}_n)$. This defines maps

$$T(\nu + \eta) \to Y_n$$

and thus dual maps

$$g_k\colon X_{-2q-k} \to \Sigma^k T(\xi)$$

for k large, such that

$$g_{k*}\colon H_{2q+k}(X_{-2q-k}, \mathsf{Z}) \to H_{2q+k}(\Sigma^k T(\xi), \mathsf{Z})$$

is an isomorphism. We say that g_k has degree one.

A system $g = \{g_k\}$ of maps constructed in this way is called an X-orientation for ξ.

In the following all homology and cohomology have Z_2 coefficients.

Definition 4.1. Let $U_\xi \in H^q(T(\xi))$ be the Thom class. For a fixed orientation g of ξ satisfying

$$g_{k*}(\Sigma^k U_\xi) = 0$$

define the composite map

$$\delta = \Sigma^k h \circ g_k,$$

where $h\colon T(\xi) \to K(\mathsf{Z}_2, q)$ represents U_ξ, and put

$$b_g(\xi) = Sq_\delta^{q+1}(\Sigma^k \iota) \in H^{2q+k}(X_{-2q-k}) = \mathsf{Z}_2 .$$

Here Sq_δ^{q+1} is the functionalized Sq^{q+1} on δ. As in Browder [4] it is clear that the indeterminacy is 0, and that $b_g(\xi)$ is independent of k.

Lemma 4.2. *Let ξ be stably equivalent to a SO sphere bundle. Then*

$$g_k^*(\Sigma^k U_\xi) = 0$$

if

$$w_{i_1}(\nu+\eta) \cup \ldots \cup w_{i_s}(\nu+\eta) = 0 \quad \textit{for} \quad i_1 + \ldots + i_s = q .$$

Proof. Here w_i denotes the ith Stiefel-Whitney class. Since U_ξ is the bottom class of $T(\xi)$, by S-duality

$$g_k^*(\Sigma^k U_\xi) = 0$$

iff

$$T(\varphi')_*\colon H_{q+n}(T(\nu+\eta)) \to H_{q+n}(T(\bar{\gamma}))$$

is zero. Now

$$\pi_*\colon H_q(B_n\langle v_{q+1}\rangle) \to H_q(B_n)$$

is injective. Hence we only need to see that

$$\varphi_*\colon H_q(M) \to H_q(B_n)$$

is zero. When φ factors through $BSO(n)$, this is clearly fulfilled when the Stiefel–Whitney numbers of $\nu+\eta$ are zero.

REMARK. The condition of 4.2 is fulfilled for q odd and ξ stably equivalent to τ, because $w_i(\nu+\nu)\neq 0$ only for i even.

A similar necessary and sufficient criterion in general needs the structure of $H^*(BSH, Z_2)$. This is calculated by J. Milgram.

When ξ is X-orientable, the orientation depends on the following choices:

I a) ν and the Thom map for $T(\nu)$.
  b) η and the trivialization of $\xi+\eta$.

II The lifting φ' of φ.

First let us examine the choices according to I:

If ν' is equivalent to ν and η' is equivalent to η, a choice of equivalences β_1 and β_2 respectively defines the S-duality

$$S^N \to T(\nu+\eta) \wedge T(\xi+k) \to T(\nu'+\eta') \wedge T(\xi+k)$$

where the last map is $T(\beta_1+\beta_2)\wedge \mathrm{id}$. With respect to this S-duality an orientation $T(\nu'+\eta') \to T(\bar{\gamma}_n)$ defines the same X-orientation for $T(\xi+k)$ as the composite map

$$T(\nu+\eta) \xrightarrow{T(\beta_1+\beta_2)} T(\nu'+\eta') \to T(\bar{\gamma}_n)$$

does with respect to the original S-duality.

Another choice of ν' and η' thus amounts to a change of the S-duality

$$S^N \to T(\nu+\eta+\xi+k) \to T(\nu+\eta) \wedge T(\xi+k) \tag{4.1}$$

by automorphisms of ν and η.

Also, fixing ν and η, another choice of trivialization of $\xi+\eta$ just changes the S-duality map (4.1) by an automorphism of $\eta+\xi$.

Finally, according to Theorem 3.5 in Wall [15], another choice of Thom map changes the S-duality map (4.1) by an automorphism of ν.

Hence in all cases, a different choice according to I just changes the S-duality map (4.1) by an automorphism of $\nu+\eta+\xi+k$. Choosing η of sufficiently large dimension, it follows from Lemma 3.4 that this automorphism can be assumed to be of the form $id+\beta+\mathrm{id}$, where β is an automorphism of η only.

In this way we conclude from Lemma 3.3 that a different choice according to I is equivalent to

I′ Replace the orientation

$$g_k\colon X_{-2q-k} \to T(\xi+k)$$

by the orientation

$$g_k' = T(\alpha) \circ g_k ,$$

where $T(\alpha): T(\xi + k) \to T(\xi + k)$ is induced by an automorphism α of $\xi + k$.

LEMMA 4.3. *If $b_g(\xi)$ is independent of the choices* I, *it is also independent of the choices* II, *and hence independent of the choice of X-orientation for ξ.*

PROOF. If $\varphi': M \to B_n\langle v_{q+1}\rangle$ is a lifting of φ, the other lifting is homotopic to the composite φ'':

$$M \xrightarrow{c} M \vee S^q \xrightarrow{\varphi' \vee \iota} B_n\langle v_{q+1}\rangle \vee K(\mathsf{Z}_2, q) \xrightarrow{\nabla} B_n\langle v_{q+1}\rangle .$$

Here c is the pinching map, and ∇ the map folding $K(\mathsf{Z}_2, q)$ onto the fibre of π. Since $\nabla^* \bar{\gamma}_n$ is trivial over $K(\mathsf{Z}_2, q)$,

$$T(\nabla^* \bar{\gamma}_n) = T(\bar{\gamma}_n) \vee \Sigma^n(K(\mathsf{Z}_2, q)) .$$

Taking the dual it is clear that $\Sigma^n(K(\mathsf{Z}_2, q))$ gives no contribution to the functionalized Sq^{q+1}.

We now consider the change of orientation originating from I′. According to Lemma 2.8, we can assume that the automorphism α of $\xi + k$ (ξ a $q-1$ sphere fibration) is of the form $\alpha' + \mathrm{id}$, where α' is an automorphism of $\xi + 1$.

THEOREM 4.4. *Let ξ_0 be an X-orientable $q-1$ sphere fibration over M^q, q odd, and let α be an automorphism of $\xi_0 + 1$. Further choose an X-orientation g of ξ_0 and let g' denote the orientation defined by*

$$g_k' = T(\alpha + \mathrm{id}) \circ g_k$$

for k large. Then

$$b_g(\xi_0) - b_{g'}(\xi_0) = \chi(\alpha) .$$

COROLLARY 4.5. *The number $b_g(\xi_0)$ depends on the choice of X-orientation, iff every $q-1$ sphere fibration which is stably equivalent to ξ_0, automatically is equivalent to ξ_0.*

PROOFS. Corollary 4.5 clearly follows from Theorem 4.4, Corollary 2.3 and Definition 2.4.

For the proof of Theorem 4.4 it suffices, according to Proposition 2.7, to show that

$$b_g(\xi_0) - b_{g'}(\xi_0) = e(\xi_\alpha) .$$

In the stable track group $\{T(\xi_0), T(\xi_0)\}$ put

$$\gamma = T(\alpha) - \mathrm{id} .$$

Use the Puppe sequences for the cofibrations

$$S^q \to T(\xi_0) \xrightarrow{j} T(\xi_0)/T(\xi_{0|*})$$

and

$$T(\xi_{0|N}) \xrightarrow{i} T(\xi_0) \to S^{2q} ,$$

where N is homotopy equivalent to a $(q-1)$-dimensional complex, and $*$ is the base point of M. We then get a factorization of γ through j and i, that is, there is a stable element

$$\eta\colon\ T(\xi_0)/T(\xi_{0|*}) \to T(\xi_{0|N})$$

such that $\gamma = i \circ \eta \circ j$. It is easy to see that if γ is represented by the map

$$\gamma_k\colon\ \Sigma^k T(\xi_0) \to \Sigma^k T(\xi_0) ,$$

then

$$Sq^{q+1}_{\gamma_k}(\Sigma^k U_{\xi_0})$$

is well defined with zero indeterminacy, and furthermore

$$Sq^{q+1}_{\gamma_k}(\Sigma^k U_{\xi_0}) = b_g(\xi_0) - b_{g'}(\xi_0) .$$

Put $T = \Sigma^k T(\xi_0)$ and $f = T(\alpha + \mathrm{id})$, where f is a map of ΣT into itself. Define $M_f = \Sigma T \times I$ with identifications

$$(x,1) \sim (f(x),0) \quad \text{and} \quad (*,t) \sim (*,t')$$

for $x \in \Sigma T$ and $t,t' \in I$. Clearly

$$M_f = \Sigma^k T(\xi_\alpha) .$$

On the other hand, f is homotopic to the map

$$\Sigma T \xrightarrow{\Delta} \Sigma T \vee \Sigma T \xrightarrow{\mathrm{id} \vee \Sigma\gamma_k} \Sigma T \vee \Sigma T \xrightarrow{\nabla} \Sigma T ,$$

where Δ is the pinching map and ∇ the folding map. Hence M_f is homotopy equivalent to $\Sigma T \times I$, with the identifications

$$(x,t,1) \sim \begin{cases} (x,2t,0) & \text{for} \quad t \leqq \frac{1}{2} \\ (\gamma_k x, 2t-1, 0) & \text{for} \quad t \geqq \frac{1}{2} \end{cases}$$

and $(*,s) \sim (*,s')$, where $x \in T$, $s,s' \in I$ and t is in the interval defining ΣT.

Let Y be the subspace of points with coordinates $(x,t,0)$ satisfying $t \geqq \frac{1}{2}$ or coordinates $(x,\frac{1}{2},s)$ satisfying $0 \leqq s \leqq 1$. Obviously Y is homeomorphic to ΣT. The image of the set

$$\{(x,t,s) \mid t \leqq \tfrac{1}{2}\}$$

in M_f/Y is homotopy equivalent to the space $\Sigma T \times S^1/(*) \times S^1$ whereas the image of the set

$$\{(x,t,s) \mid t \geqq \tfrac{1}{2}\}$$

is homotopy equivalent to $C_{\Sigma\gamma_k}$, the mapping cone on $\Sigma\gamma_k$.

In this way M_f/Y is homotopy equivalent to the space

$$\Sigma T \times S^1/(*) \times S^1 \cup C_{\Sigma\gamma_k},$$

where the base of the cone is $\Sigma T \times 0$ in $\Sigma T \times S^1$. Denoting the projection $M \times S^1/(*) \times S^1 \to M$ by π, we have

$$\Sigma T \times S^1/(*) \times S^1 = T(\pi^*(\xi_0 + k + 1)) .$$

There is a unique class

$$u \in H^{q+k+1}(M_f/Y)$$

such that the restriction to $\Sigma T \times S^1/(*) \times S^1$ is the bottom class. Let p be the natural map $M_f \to M_f/Y$. Then p^*u is the bottom class of

$$M_f = \Sigma^k(T(\xi_\alpha))$$

and

$$p^*\colon H^{2q+k+2}(M_f/Y) \to H^{2q+k+2}(M_f)$$

is the sum map $Z_2 \oplus Z_2 \to Z_2$. Now

$$Sq^{q+1}(\Sigma^k U_{\xi_\alpha}) = p^* Sq^{q+1} u .$$

In order to calculate $Sq^{q+1}u \in Z_2 \oplus Z_2$ we restrict to $T(\pi^*(\xi_0 + k + 1))$ and $C_{\Sigma\gamma_k}$ respectively.

Clearly Sq^{q+1} is zero in $T(\pi^*(\xi_0 + k + 1))$ so as an element in Z_2

$$\begin{aligned} Sq^{q+1}(\Sigma^k U_{\xi_\alpha}) &= Sq^{q+1}(i^*u) \\ &= Sq^{q+1}_{\Sigma\gamma_k}(\Sigma^{k+1} U_{\xi_0}), \end{aligned}$$

where $i\colon C_{\Sigma\gamma_k} \to M_f/Y$ is the inclusion. On the other hand

$$Sq^{q+1}(\Sigma^k U_{\xi_\alpha}) = \Sigma^k U^2_{\xi_\alpha} = \Sigma^k \Phi(e(\xi_\alpha)) ,$$

where

$$\Phi\colon H^*(M \times S^1/(*) \times S^1) \to H^*(T(\xi_\alpha))$$

is the Thom isomorphism. This ends the proof of Theorem 4.4.

DEFINITION 4.6. If $b_g(\xi_0)$ is independent of the choice of X-orientation, we write $b(\xi) = b_g(\xi)$ for any $q-1$ sphere fibration, which is stably equivalent to ξ_0.

REMARK. Theorem 4.4 shows that $b_g(\xi_0)$ is not independent of the choice of orientation, precisely in case there is an automorphism α of ξ_0+1 satisfying $\chi(\alpha)\neq 0$ or equivalently $e(\xi_\alpha)\neq 0$.

Now $e(\xi_\alpha)=w_{q+1}(\xi_\alpha)$, and the collection of stable fibrations over $M\times S^1/(*)\times S^1$ represented by ξ_α, where α is any automorphism of ξ_0+1, is the same as the collection of stable fibrations of the form $\pi^*\xi_0+\eta$, where π is the projection onto M and η is induced from a fibration over ΣM.

Hence $b(\xi_0)$ is not well defined iff there is a sphere-fibration η over ΣM satisfying

$$w_{q+1}(\pi^*\xi_0+\eta) = \sum_{i=0}^{q+1}\pi^*w_i(\xi_0)\cup w_{q+1-i}(\eta) \neq 0 .$$

This is the criterion of James and Thomas [7] saying that there is only one $q-1$ sphere fibration which is stably equivalent to ξ_0.

5. The invariance theorem.

We are now in the position to prove the following theorem.

THEOREM 5.1. *Let M and M' be closed q-dimensional differentiable manifolds with tangent sphere bundles τ and τ' respectively. If $f\colon M\to M'$ is an orientation preserving homotopy equivalence, then τ and $f^*\tau'$ are fibre homotopy equivalent.*

PROOF. This theorem is proved in [5] for q even and $q=1,3,7$, and according to Atiyah [1], τ and $f^*\tau'$ are at least stably equivalent. We know from Lemma 4.2 that τ is X-orientable in the sense of Definition 4.1. Hence we conclude from Corollary 4.5 that either τ and $f^*\tau'$ are in fact equivalent, or the invariant $b(\xi)$ is well defined for $q-1$ sphere fibrations which are stably equivalent to τ.

The theorem now follows as in [5] from the following two lemmas. Using the notation of [5] we have for q odd different from $1,3,7$:

LEMMA 5.2. *Let ξ_1 and ξ_2 be $q-1$ sphere fibrations over M with classifying maps v_1 and v_2 respectively, and let ζ be a stably trivial $q-1$ sphere fibration over S^q with classifying map μ.*

If $v_2=v_1{}^{\mu}$, then $b(\xi_2)=b(\xi_1)+b(\zeta)$, whenever $b(\xi_1)$ is defined and independent of orientation.

PROOF. Let η be a fibration such that $\eta+\xi_1$ is trivial, and choose an X-orientation of ξ_1 originating from a classifying map

$$\varphi\colon M \to B_n\langle v_{q+1}\rangle$$

for $\nu+\eta$.

Consider the commutative diagram

$$\begin{array}{ccc} M \cup (*) \xrightarrow{i} M \cup S^q & \xrightarrow{\varphi \cup j} & B_n\langle v_{q+1}\rangle \cup (*) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & B_n\langle v_{q+1}\rangle , \end{array}$$

where i is the inclusion and j the collapsing map. Taking Thom complexes on the appropriate fibrations, we get the dual homotopy commutative diagram for k large:

$$\begin{array}{ccc} X_{-2q-k} \vee S^{2q+k} \xrightarrow{g_k \vee t} T(\xi_1+k) \vee T(q+k) & \xrightarrow{r_1} & T((\xi_1+k) \vee (q+k)) \\ \uparrow \Delta & & \uparrow c_1 \\ X_{-2q-k} & \xrightarrow{g_k} & T(\xi_1+k) . \end{array}$$

Here Δ is the map which splits the top cell into two, and t is the inclusion of S^{2q+k} in

$$T(q+k) = S^{2q+k} \vee S^{q+k} .$$

The natural map $M \cup S^q \to M \vee S^q$ induces the map r_1 of Thom complexes

$$r_1\colon T(\xi_1+k) \vee T(q+k) \to T((\xi_1+k) \vee (q+k))$$

which identifies the bottom cells. The pinching map $c\colon M \to M \vee S^q$ induces

$$\bar{c}_1\colon T(\xi_1+k) \to T((\xi_1+k) \vee (q+k)) .$$

Analogously there are induced maps

$$r\colon T(\xi_2+k) \vee T(\zeta+k) \to T((\xi_2+k) \vee (\zeta+k)) ,$$
$$\bar{c}\colon T(\xi_2+k) \to T((\xi_1+k) \vee (\zeta+k)) .$$

The fact that ζ is stably trivial, shows that there is an equivalence α between ξ_1+k and ξ_2+k, such that there are the commutative diagrams

$$\begin{array}{ccc} T(\xi_1+k) & \xrightarrow{\bar{c}_1} & T((\xi_1+k) \vee (q+k)) \\ \downarrow T(\alpha) & & \downarrow \\ T(\xi_2+k) & \xrightarrow{\bar{c}} & T((\xi_1+k) \vee (\zeta+k)), \end{array}$$

$$\begin{array}{ccc} T(\xi_1+k) \vee T(q+k) & \to & T(\xi_1+k) \vee T(\zeta+k) \\ \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r} \\ T((\xi_1+k) \vee (q+k)) & \to & T((\xi_1+k) \vee (\zeta+k)) . \end{array}$$

Clearly $g_k' = T(\alpha) \circ g_k$ defines an X-orientation for ξ_2 in such a way that we have the commutative diagram

$$\begin{array}{ccc} X_{-2q-k} \vee S^{2q+k} & \longrightarrow & T(\xi_1+k) \vee T(\zeta+k) \\ \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle r} \\ X_{-2q-k} \xrightarrow{g_k'} T(\xi_2+k) & \xrightarrow{\bar{c}} & T((\xi_1+k) \vee (\zeta+k)) . \end{array}$$

The lemma now follows by an easy calculation as in [5, § 3].

LEMMA 5.3. *Let $f\colon M \to M'$ be an orientation preserving homotopy equivalence of oriented q-manifolds with tangent sphere bundles τ and τ' respectively. If $b(\tau)$ is well defined, we have*

$$b(\tau) = b(\tau') .$$

PROOF. Let $A \in H^q(M \times M)$ denote the element defined in the proof of [5, Proposition 3.4]. Also let

$$j\colon M \times M_+ \to T(\tau)$$

denote the map collapsing everything outside a tubular neighbourhood. Finally consider the twisting map

$$t\colon M \times M \to M \times M .$$

We know that $j^*U = A + t^*A$. The normal bundle $\nu \times \nu$ of $M \times M$ clearly satisfies $v_{q+1}(\nu \times \nu) = 0$. Accordingly we can find a map

$$\varphi\colon M \times M \to B_n\langle v_{q+1}\rangle$$

classifying $\nu \times \nu$. Obviously

$$\varphi \circ \Delta\colon M \to M \times M \to B_n\langle v_{q+1}\rangle$$

classifies $\nu + \nu$ over M. Hence we conclude from Corollary 3.2 that the corresponding X-orientation for τ is the composite $\Sigma^k j \circ g_k$, where

$$g_k\colon X_{-2q-k} \to \Sigma^k(M \times M_+)$$

is an X-orientation for $M \times M$ in the sense of Browder [4, § 1]. Hence $b(\tau)$ is the functionalized Sq^{q+1} on the map $\Sigma^k h \circ g_k$, where

$$h\colon M \times M \to K(\mathbf{Z}_2, q)$$

represents $A+t^*A$. Clearly

$$\Sigma^k(f\times f)\circ g_k\colon X_{-2q-k}\to M'\times M'$$

is an X-orientation for $M'\times M'$. If

$$h'\colon M'\times M'\to K(\mathsf{Z}_2,q)$$

represents the analogous element

$$A'+t^*A'\in H^q(M'\times M'),$$

we obviously have

$$(f\times f)^*(A'+t^*A') = A+t^*A ,$$

and thus

$$h'\circ(f\times f) = h\colon M\times M\to K(\mathsf{Z}_2,q) .$$

Hence $b(\tau')$ is also the functionalized Sq^{q+1} on the map

$$\Sigma^k h'\circ\Sigma^k(f\times f)\circ g_k = \Sigma^k h\circ g_k .$$

This ends the proof of Lemma 5.3 and hence of Theorem 5.1.

Analogously using $BSO(n)$ instead of $BSH(n)$ we have the following theorem.

THEOREM 5.4. *Let $f\colon M\to M'$ be a homotopy equivalence of oriented q-manifolds with tangent q-plane bundles τ and τ' respectively. If $f^*\tau'$ and τ are stably isomorphic (as SO-bundles) then they are automatically isomorphic (as $SO(q)$-bundles).*

As a consequence of Theorem 5.1 we have according to Sutherland [14, Corollary 3.4]:

COROLLARY 5.5. *Let M and M' be oriented q-manifolds which are oriented homotopy equivalent and suppose $k\leqq\frac{1}{2}(q-1)$. Then M admits a k-field iff M' does.*

6. Connection with the semi-characteristic.

In this section we will show that under certain circumstances $b(\tau)=\chi^*(M)$, the semi-characteristic of M. This is defined by the formula

$$\chi^*(M) = \sum_{i=0}^{\frac{1}{2}(q-1)}\dim H^i(M,\mathsf{Z}_2)\bmod 2 .$$

First we use $B_n=BO(n)$ for defining an X-orientation

$$g_k\colon X_{-2q-k}\to\Sigma^k(M\times M_+)$$

for M an arbitrary q-dimensional manifold. We assume q odd. Let ψ denote the operation introduced by Browder [4, § 1],

$$\psi\colon \operatorname{Ker}(g_k{}^*)^{q+k} \to \mathbf{Z}_2 .$$

Using the notation of Lemma 5.3 we have

$$b_g(\tau) = \psi(A+t^*A) .$$

LEMMA 6.1. *If* $[M]=0$ *in the non-oriented bordism ring, then*

$$\Sigma^k A \in \operatorname{Ker}(g_k{}^*)^{q+k} .$$

PROOF. Arguing as in the proof of Lemma 4.2 and using Lemma 3.6, we have to show that $\varphi_*(A\cap[M\times M])=0$, where $\varphi\colon M\times M \to B_n$ is the classifying map for $\nu\times\nu$, and $[M\times M]$ is the orientation class of $M\times M$. This is equivalent to show that

$$A \cup w_{i_1}(\nu\times\nu) \cup \ldots \cup w_{i_s}(\nu\times\nu) = 0$$

for all $i_1,\ldots,i_s$ satisfying $i_1+\ldots+i_s=q$. Here of course w_i denotes the ith Stiefel–Whitney class. Now

$$A = \sum_{i=1}^{d} \alpha_i \otimes \beta_i$$

where $\{\alpha_1,\ldots,\alpha_d,\beta_1,\ldots,\beta_d\}$ is a basis for $H^*(M)$ satisfying

$$\alpha_i \cup \beta_j; = \delta_{ij}\sigma_M$$

for $\deg\alpha_i+\deg\beta_j=q$. Here σ_M denotes the top class of $H^*(M)$.

SUBLEMMA 6.2. *Let* $x,y \in H^*(M)$ *satisfy* $\deg x+\deg y=q$. *Then*

1) $(x\otimes y)\cup A \neq 0 \Leftrightarrow x\cup y \neq 0$ *for* $\deg x>\deg y$,
2) $(x\otimes y)\cup A=0$ *for* $\deg x<\deg y$.

SUBPROOF. 1) Assume $\deg x>\deg y$. Write x and y as a sum of β_i's and α_i's respectively. Then

$$(x\otimes y) \cup A \neq 0$$

iff, for an odd number of times, x contains β_i and y contains α_i, iff $x\cup y\neq 0$.

2) is trivial.

Lemma 6.1 follows from the sublemma and the fact that

$$w_i(\nu\times\nu) = \sum_{j=0}^{i} w_j(\nu)\otimes w_{i-j}(\nu) .$$

We now use Theorem 1.4 in Browder [4] and the fact that

$$A \cup t^*A = \chi^*(M)\sigma_{M \times M} ,$$

where $\sigma_{M \times M}$ is the top class of $M \times M$, to conclude that

$$b_g(\tau) = \psi(A) + \psi(t^* A) + \chi^*(M) .$$

THEOREM 6.3. *If* $[M]=0$ *in the non-oriented bordism ring, then there is an* X*-orientation* g *for* τ, *such that*

$$b_g(\tau) = \chi^*(M) .$$

PROOF. We want to show that for some X-orientation for $M \times M$ it happens that $\psi(A)=\psi(t^*A)$.

Analogously to the construction of $B_n\langle v_{q+1}\rangle$ let

$$B_n' = B_n\langle v_{(q+1)/2}, \ldots, v_{q+1}\rangle$$

denote the total space of the fibration

$$\pi'\colon B_n' \to B_n$$

which kills the Wu classes $v_{(q+1)/2}, \ldots, v_{q+1}$. Put $\gamma_n' = (\pi')^*\gamma_n$, $Y_n' = T(\gamma_n')$, and denote the corresponding dual cospectrum by $X' = \{X_n'\}$. Clearly the Whitney sum map

$$B_n \times B_n \to B_n$$

lifts to a map

$$B_n' \times B_n' \to B_{2n}\langle v_{q+1}\rangle .$$

Hence the corresponding map of Thom complexes gives rise to a dual map of degree one:

$$h_k\colon X_{-2q-2k} \to X'_{-q-k} \wedge X'_{-q-k} .$$

Clearly the normal bundle ν of M^q has a classifying map $\varphi\colon M \to B_n'$. The map induced on Thom complexes defines a map

$$f_k\colon X'_{-q-k} \to \Sigma^k(M_+) .$$

Hence the composite map $(f_k \wedge f_k) \circ h_k$ defines an X-orientation for $M \times M$. We can thus use

$$X'_{-q-k} \wedge X'_{-q-k}$$

for computing the functionalized Sq^{q+1}, just we know that $\Sigma^k A$ (and $\Sigma^k(t^*A)$) goes to zero under $f_k \wedge f_k$. In that case $\psi(A)=\psi(t^*A)$, because the twisting map of $X'_{-q-k} \wedge X'_{-q-k}$ into itself has degree one.

Arguing as in the proof of Lemma 6.1 and Lemma 4.2, we need to require that

$$A \cup \big((w_{i_1}(\nu) \cup \ldots \cup w_{i_l}(\nu))\otimes(w_{i_{l+1}}(\nu) \cup \ldots \cup w_{i_m}(\nu))\big) = 0$$

whenever $i_1 + \ldots + i_m = q$. According to the Sublemma 6.2, this is the case precisely when all Stiefel–Whitney numbers are 0. This ends the proof of Theorem 6.3.

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