MATH. SCAND. 26 (1970), 233–240

ON FINITELY GENERATED FLAT MODULES

S. JØNDRUP

0. Introduction.

In this note we shall consider rings A with the property that any flat and finitely generated A-module is projective, and we will prove that this class of rings is rather big. If we require that any flat left A-module is projective, we get the class of left perfect rings (cf. [1]), which is a small class of rings. For instance, a (commutative) integral domain D is left perfect if and only if D is a field, but any finitely generated flat module over an integral domain is projective (cf. [5]).

1. General remarks.

In this section A denotes a ring with an identity, and all modules considered are unitary left modules.

Definition. We say that a ring A has property P (we write $A \in P$), if every finitely generated flat A-module is projective.

A very useful tool in the study of flat modules is the following lemma.

Lemma 1.1. *Let*

$$0 \to K \to F \to M \to 0$$

be an exact sequence of A-modules, where F is A-free, then the following statements are equivalent:

i) *M is A-flat.*
ii) *Given any $k \in K$, there exists a homomorphism $u_k \colon F \to K$, such that $u_k(k) = k$.*
iii) *Given any $(k_i)_{1 \leqq i \leqq n}$, there exists a homomorphism $u_k \colon F \to K$, such that $u_k(k_i) = k_i$ for every i, $1 \leqq i \leqq n$.*

Proof. See S. U. Chase [3, prop. 2.2].

As applications of this lemma we get the following two propositions.

Received March 5, 1969.

Proposition 1.2. *Let A and B be any two rings, and let $\varphi: A \to B$ ($\varphi(1_A) = 1_B$) be a ring-homomorphism. If B viewed as a left A-module is flat and finitely generated, then $A \in P$ implies $B \in P$.*

Proof. If N is any flat and finitely generated B-module, we have to prove that N is B-projective. If we consider N as an A-module, N is finitely generated and flat (cf. N. Bourbaki [2, Chap. 1, § 2, no. 7, prop. 8, cor. 3]), and hence N is A-projective.

We have an exact sequence

$$(1) \qquad 0 \to L \to B^r \to N \to (0)$$

of B-modules, which is also an exact sequence of A-modules, and therefore it is split exact over the ring A. Since B^r is a finitely generated A-module, L is a finitely generated A-module too, in particular finitely generated as a B-module. From this we get that (1) is split exact over B (lemma 1.1). Hence N is B-projective.

Corollary 1.3. *If $A \in P$ and G is a finite group, then the group ring $A[G] \in P$.*

Proposition 1.4. *Let A and B be rings and $\varphi: A \to B$ a ring-homomorphism. If $B \in P$ and B is a faithfully flat right A-module, then $A \in P$.*

Proof. Let M be any flat and finitely generated A-module, and let

$$(2) \qquad 0 \to K \to F \to M \to 0$$

be an exact sequence, where F is free and finitely generated. From (2) we derive the exact sequence

$$(3) \qquad 0 = \mathrm{Tor}_1^A(B, M) \to B \otimes_A K \to B \otimes_A F \to B \otimes_A M \to (0)$$

of B-modules. $B \otimes_A M$ is B-flat [2, Chap. 1, § 2, no. 7, prop. 8, cor. 2] and B-finitely generated. Hence $B \otimes_A M$ is B-projective, and we get that $B \otimes_A K$ is a finitely generated B-module. Since B is faithfully flat, it is readily checked that K is a finitely generated A-module, and proposition 1.4 follows from lemma 1.1

Corollary 1.5. *If A is a semilocal (commutative) ring, then $A \in P$.*

Proof. If B is a quasilocal ring (that is, a commutative ring with a unique maximal ideal), then $B \in P$. This follows from [2, Chap. 1, § 2, exerc. 23] or from [5]. If A is semilocal, then $\prod_{m \in \Omega} A_m \in P$ (see [2, Chap. 2, § 3, no. 3] for notation). The corollary follows now from proposition 1.4. and [2, Chap. 2, § 3, no. 3, prop. 10].

Corollary 1.5. is well known (cf. S. Endo [5]).

COROLLARY 1.6. *Let A be any ring and G any group. If the group ring $A[G] \in P$, then $A \in P$.*

THEOREM 1.7. *Let A be any ring. Then $A \in P$ if and only if $A[[x]] \in P$.*

PROOF. We need the following ideas.

i) Let B be any ring and x a central non-unit non-zero-divisor in B. If the B-module M is B-flat, then M/xM is B/xB-flat.

For a proof see [2, Chap. 1, § 2, prop. 8].

ii) Let M be any B-modul, then there exists a natural B-isomorphism between $B[[x]] \otimes_B M/x([[x]] \otimes_B M)$ and M.

The proof is trivial.

iii) Let M be any finitely generated flat $B[[x]]$-module. If M/xM is B-free, then M is $B[[x]]$-free.

(This might be well known, but I have not been able to find a complete proof in the literature.)

The statement may be proved as follows. If $(\overline{m}_i)_{i \in I}$ is a finite base for the B-module M/xM, and m_i denotes a representative in M for $\overline{m}_i$, then $(m_i)_{i \in I}$ generate the $B[[x]]$-module M [2, Chap. 2, § 3, no. 2, prop. 4, cor. 2].

From the exact sequence

$$0 \to K \to F \xrightarrow{\varphi} M \to 0$$

of $B[[x]]$-modules, where F is free with base $(e_i)_{i \in I}$ and $\varphi(e_i) = m_i$ for every i, we derive the exact sequence of B-modules

$$0 = \operatorname{Tor}_1^{B[[x]]}(B[[x]] \,|\, (x), M) \to K/xK \to F/xF \xrightarrow{\bar{\varphi}} M/xM \to (0) \, .$$

$\bar{\varphi}$ is a B-isomorphism so $K = xK$, and hence $K = 0$, that is, M is $B[[x]]$-free.

Let us return to the proof of theorem 1.8. We assume $A \in P$ and have to prove that $A[[x]] \in P$. Let M be any flat and finitely generated $A[[x]]$-module, then M/xM is finitely generated and flat (cf. i)) viewed as an A-module, hence there exists a finitely generated projective A-

module N such that $M/xM \oplus N$ is A-free with a finite base. Since $A[[x]] \otimes_A N$ is a finitely generated projective $A[[x]]$-module, $(A[[x]] \otimes_A N) \oplus M$ is a finitely generated flat $A[[x]]$-module. From the isomorphisms

$$(A[[x]] \otimes_A N) \oplus M/x(A[[x]] \otimes_A N) \oplus M)$$
$$\cong A[[x]] \otimes_A N/x([[x]] \otimes_A N) \oplus M/xM$$
$$\cong N \oplus M/xM$$

(cf. ii)) we infer that $(A[[x]] \otimes_A N) \oplus M$ is $A[[x]]$-free (cf. iii)), and hence M is $A[[x]]$-projective.

Conversely, assume that $A[[x]] \in P$. If M is any flat and finitely generated A-module, then $A[[x]] \otimes_A M$ is a flat and finitely generated $A[[x]]$-module [2, Chap. 1, § 2, no. 7, prop. 8, cor. 2], so $A[[x]] \otimes_A M$ is $A[[x]]$-projective. If N is any $A[[x]]$-module, and x is a non-zero-divisor in N, then it is well known that $\mathrm{lhd}_A N/xN \leqq \mathrm{lhd}_{A[[x]]} N$. From this remark we infer that $A[[x]] \otimes_A M/x(A[[x]] \otimes_A M)$ is A-projective, hence M is A-projective (cf. ii)).

For later purposes we need the following proposition, which is due to I. I. Sahaev.

Proposition 1.8. *If every cyclic flat left A-module is projective, then A has no infinite set of orthogonal idempotents.*

A proof may be found in [9].

For a commutative ring A proposition 1.8. is due to Endo [6].

2. On a generalization of a theorem of S. Endo.

In this section A denotes a commutative ring with an identity.

The following theorem which might be known is essential for this section.

Theorem 2.1. *For a commutative ring A the following properties are equivalent:*

i) *Every cyclic flat A-module is projective.*

ii) $A \in P$.

Proof. D. Lazard has proved that i) implies that every D-closed subset of $X = \mathrm{Spec}(A)$ is open (cf. [8] for a proof and definitions), and if this condition is satisfied, then $A \in P$. The last statement follows immediately from [8, corollary 5.2] and [2, Chap. 2, § 5, no. 2, théorème 1].

Non-commutative rings for which condition i) or condition ii) holds have been studied by I. I. Sahaev [9].

THEOREM 2.2. *Let A be a subring of B (B not necessarily commutative), and suppose A is contained in the center of B. If $B \in P$, then $A \in P$.*

PROOF. Let $A/\mathfrak{a}$ be a flat A-module. Consider the exact sequence

$$(0) \to \mathfrak{a} \to A \to A/\mathfrak{a} \to (0) \tag{5}$$

of A-modules, and we have to prove that $\mathfrak{a}$ is finitely generated. From (5) we derive the exact sequence

$$(0) \to B\mathfrak{a} \to B \to B/B\mathfrak{a} \to (0)$$

of B-modules. $B/B\mathfrak{a}$ is B-flat [2, Chap. 1, § 2, no. 7, prop. 8, cor. 2]. Since $B \in P$, we have $B\mathfrak{a} = Be$, where e is an idempotent in B. Let $e = b_1a_1 + \ldots + b_sa_s$. Since $A/\mathfrak{a}$ is flat, there exists an element $a' \in \mathfrak{a}$ such that $a_ia' = a_i$ for every $i \in \{1,\ldots,s\}$, so we conclude that

$$ea' = b_1a_1 + \ldots + b_sa_s = e\,. \tag{6}$$

Since $a' \in \mathfrak{a}$, we have $a' = be$ for a suitable $b \in B$, and therefore $a'e = a'$. This together with (6) implies that $e = a'$, that is, $e \in \mathfrak{a}$. For any $a \in \mathfrak{a}$, we have $a = ae$, hence $\mathfrak{a} = Ae$ and (5) must be split exact, that is, $A/\mathfrak{a}$ is A-projective.

COROLLARY 2.3. (cf. [6]). *A finitely generated flat module over an integral domain is projective.*

COROLLARY 2.4. (S. Endo, cf. [6]). *Let A be any commutative ring for which there exists a multiplicatively closed set S consisting of non-zero divisors, such that A_S is semilocal or $A_S \in P$. Then $A \in P$.*

COROLLARY 2.5. *$A \in P$ if and only if $A[x] \in P$.*

PROOF. Assume $A \in P$, then $A[[x]] \in P$ (theorem 1.7), so $A[x] \in P$ (theorem 2.2).

Conversely, if $A[x] \in P$, proposition 1.4 implies that $A \in P$.

3. Examples and some remarks.

LEMMA 3.1. *The ring A has property P if and only if any flat, countably related, finitely generated left A-module is projective.*

PROOF. "only if" is obvious.

"if". Suppose $A \notin P$, and let M be a finitely generated, not finitely related flat left A-module. Consider the exact sequence

$$0 \to K \to F \xrightarrow[\varphi]{} M \to 0,$$

where F is a free left A-module with base $(e_1,\ldots,e_n)$, and $K = \ker \varphi$. If $(k_j^0)_{1 \leqq j \leqq n}$ is any set of n elements of K, then $\sum_{1 \leqq j \leqq n} Ak_j^0 = K_0 \subset K$. (Here $\subset$ means "is a proper subset of".) Choose $k_0 \in K$, $k_0 \notin K_0$ and $\theta_1 : F \to K$ such that $\theta_1(k_j^0) = k_j^0$, $1 \leqq j \leqq n$, $\theta_1(k_0) = k_0$ (cf. lemma 1.1). If $\theta_1(e_j) = k_j^1$, $1 \leqq j \leqq n$, then $K_0 \subset \sum_{1 \leqq j \leqq n} k_j^1 \subset K$. If we continue this process, we get modules $(K_i)_{i \in \mathbb{N} \cup \{0\}}$ such that

$$K_0 \subset K_1 \subset \ldots \subset K_n \subset \ldots .$$

Let K^* be equal to $\bigcup_{i=1}^{\infty} K_i$. Then F/K^* is flat (lemma 1.1), countably related, but not finitely related, and the lemma is proved.

COROLLARY 3.2. *If* $\mathrm{lfPD}(A) = 0$, *then* $A \in P$.

PROOF. Let M be any finitely generated, countably related flat A-module. We conclude that $\mathrm{lhd}_A M \leqq 1$ (cf. C. U. Jensen [7, lemma 2]), hence M is A-projective.

REMARK 1. In the special case $\mathrm{lFPD}(A) = 0$, A is left perfect (cf. H. Bass [1, theorem 6.3]), and the corollary follows from [1, theorem P].

From corollary 3.2 and section 1 (remark) we infer that if $\mathrm{lfPD}(A) = 0$, then A has no infinite set of orthogonal idempotents, so we have proved the following (cf. [1, theorems P and 6.3]:

PROPOSITION 3.3. *If* $\mathrm{lfPD}(A) = 0$ *and every nonzero right A-module has nonzero socle, then* $\mathrm{lFPD}(A) = 0$.

In general, $\mathrm{lfPD}(A) = 0$ does not imply that $\mathrm{lFPD}(A) = 0$. Example: $F[[x,y]]/(x^2,xy)$, where F is commutative field.

EXAMPLE 1. Let T be any infinite connected normal topological space (i.e. T satisfies (T_2) and (T_4)), then $A = C(T, +, \cdot, \mathbb{R})$ (the ring of continuous real-valued functions on T) is an example of a commutative indecomposable ring not having property P.

PROOF. An application of Urysohn's lemma enables us to construct functions $(f_i)_{1 \leqq i < \infty}$ such that $f_i \in A$, $f_i f_{i+1} = f_i$ and $Af_i \subset Af_{i+1}$. Let $\mathfrak{a}$ be the ideal generated by the f_i's. $A/\mathfrak{a}$ is A-flat (lemma 1.1), but $A/\mathfrak{a}$ is not A-projective. This example is due to C. U. Jensen.

EXAMPLE 2. The ring A defined below is indecomposable, commutative, and coherent, but $A \notin P$.

Let A be the subring of $C(\mathsf{R}, +, \cdot, \mathsf{R})$ consisting of the functions $f(x)$ of the form

$$f(x) = \begin{cases} \dfrac{\bar{p}(x)}{\bar{q}(x)}, & x \leqq -k_f,\ k_f \in \mathsf{N}, \\ \dfrac{p_i(x)}{q_i(x)}, & x \in [i, i+1],\ i \in \{-k_f, \ldots, k_f - 1\}, \\ \dfrac{\tilde{p}(x)}{\tilde{q}(x)}, & x \geqq k_f, \end{cases}$$

where $\bar{p}(x)$, $\bar{q}(x)$, $p_i(x)$, $q_i(x) \in \mathsf{R}[x]$ for every $i \in \{-k_f, \ldots, k_f - 1\}$ and $q_i(x) \neq 0$ for every $x \in [i, i+1]$, $\bar{q}(x) \neq 0$ for $x \leqq -k_f$, $\tilde{q}(x) \neq 0$ for $x \geqq k_f$.

By a straight-forward, but tedious computation, it can be proved that this ring A has the required properties.

If A satisfies a certain extra condition, then $A \in P$.

THEOREM. *Let A be a commutative ring. If A has no infinite set of orthogonal idempotents, A is coherent and* $\mathrm{whd}_A(Aa) < \infty$ *for every $a \in A$, then $A \in P$.*

PROOF. A is a finite direct sum of integral domains (cf. L. W. Small [10]), hence $A \in P$.

REMARK 2. Professor P. M. Cohn has communicated to me an example of a non-commutative ring A, which is an integral domain, and for which $A \notin P$.

Let A be the K-algebra on the generators $a_{ij}^{(\nu)}$, $i,j = 1, 2$, $\nu = 1, 2, \ldots$, and defining relations

$$\sum_j a_{ij}^{(\nu)} a_{jk}^{(\nu')} = \delta_{\nu,\nu'} a_{ik}^{(\nu)}. \tag{7}$$

A is 1-fir (cf. P. M. Cohn [4]), thus A is an integral domain, hence any cyclic flat left A-module is A-projective. The existence of the relations (7) implies that $A_2 \notin P$. From the Morita-equivalens between A and A_n we get that $A \in P$ if and only if $A_n \in P$ for every n. Therefore $A \notin P$.

Thus the commutativity of the ring A is essential for the validity of theorem 2.1.

The author wishes to thank C. U. Jensen for his helpful comments during the preparation of this paper.

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UNIVERSITY OF COPENHAGEN, DENMARK