MATH. SCAND. 26 (1970), 241–254

ON THE STRUCTURE OF THE SPACES $\mathscr{L}_k^{p,\lambda}$

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Introduction.

Under special conditions on the subset Ω in R^n, S. Campanato [3] proved that the spaces $\mathscr{L}_k^{p,\lambda}(\Omega)$ are isomorphic to the Lipschitz spaces $C^{h,\varepsilon}(\Omega)$ (see definitions in section 1), where $h+\varepsilon=(\lambda-n)/p>0$ and $0<\varepsilon<1$, h integer $<k$.

With another method, based on the theory of interpolation spaces, we intend to prove that $\mathscr{L}_k^{p,\lambda}(\Omega)$ is equal to the Besov space $B^\alpha(\Omega)$, where $0<(\lambda-n)/p=\alpha<k$, even when $(\lambda-n)/p$ is an integer and with other conditions on Ω.

The plan of this article is as follows. In section 1 we give the definition of $\mathscr{L}_k^{p,\lambda}(\Omega)$, $C^{h,\varepsilon}(\Omega)$ and $B^\alpha(\Omega)$. Section 2 contains alternative definitions of $B^\alpha(\Omega)$, when $\Omega=\mathsf{R}^n$. In section 3 we prove

$$B^\alpha(\Omega) = \mathscr{L}_k^{p,\lambda}(\Omega), \quad 0<\alpha<(\lambda-n)/p<k ,$$

if $\Omega=\mathsf{R}^n$ (theorem 3.1). Section 4 treats the corresponding result for an open, bounded subset Ω of R^n, subject to certain restrictions (theorem 4.1).

The subject of this paper was suggested to me by professor Jaak Peetre. I thank him for valuable criticism and great interest in my work. I also thank Jörgen Löfström, who read the manuscript and gave much advice.

1. Definition of $\mathscr{L}_k^{p,\lambda}(\Omega)$, $C^{h,\varepsilon}(\Omega)$ and $B^\alpha(\Omega)$.

Let Ω be an open subset of R^n and $p\geqq 1$. Write

$$I_{x_0,r} = \{x\in \mathsf{R}^n \mid |x-x_0|\leqq r\}$$

and $\Omega_{x_0,r}=\Omega\cap I_{x_0,r}$.

DEFINITION 1.1. For k integer $\geqq 0$ and $\lambda\geqq 0$ we say that $f\in\mathscr{L}_k^{p,\lambda}(\Omega)$ if $f\in L^p_{loc}(\bar\Omega)$ and for every $r>0$ and $x_0\in\bar\Omega$ there exists a polynomial $q_k(x)$

Received June 7, 1969.

of degree $<k$, depending on x_0, r and f, and a constant C, depending on f, such that

$$(1.1)\qquad \left(\int_{\Omega_{x_0,r}} |f(x)-q_k(x)|^p\,dx\right)^{1/p} \leqq C r^{\lambda/p} .$$

The infimum over all constants C in (1.1) is a semi-norm on the space $\mathscr{L}_k^{p,\lambda}(\Omega)$, and it will be denoted $|f|_{\mathscr{L}_k^{p,\lambda}(\Omega)}$. We decide to identify functions whose difference is a polynomial of degree $<k$. Then we can use $|f|_{\mathscr{L}_k^{p,\lambda}}$ as a norm and $\mathscr{L}_k^{p,\lambda}(\Omega)$ is a Banach space.

REMARK 1.1. The spaces $\mathscr{L}_k^{p,\lambda}(\Omega)$ introduced in S.Campanato [3] are not quite the same as our spaces $\mathscr{L}_k^{p,\lambda}(\Omega)$. Campanato works with the norm

$$\|f\|_{\mathscr{L}_k^{p,\lambda}(\Omega)} = \left(|f|^p_{L^p(\Omega)} + \sup_{\substack{x_0\in\bar\Omega\\ 0<r<\operatorname{diam}\bar\Omega}} \left[r^{-\lambda}\inf_{q_k}\int_{\Omega_{x_0,r}} |f(x)-q_k(x)|^p\,dx\right]\right)^{1/p} .$$

Note also that Campanato uses the parameter $k'=k-1$ in place of k, so that our $\mathscr{L}_k^{p,\lambda}$ is the space $\mathscr{L}_{k'}^{p,\lambda}$ in the sense of Campanato.

DEFINITION 1.2. Let h be an integer $\geqq 0$ and let $C^h(\Omega)$ be the space of all h times continuously differentiable functions in $\bar\Omega$.

Then $C^h(\Omega)$ is a Banach space with the graph-norm

$$|f|_{C^h(\Omega)} = \textstyle\sum_{|l|\leqq h} \sup_{x\in\bar\Omega} |D^l f(x)| .$$

Here $l=(l_1,l_2,\ldots,l_n)$ is an n-tuple, $|l|=l_1+l_2+\ldots+l_n$, and

$$D^l f(x) = D_1^{l_1} D_2^{l_2}\ldots D_n^{l_n} f(x), \quad \text{where } D_\nu = \partial/\partial x_\nu .$$

DEFINITION 1.3. For $0<\varepsilon\leqq 1$ we say that $f\in C^{h,\varepsilon}(\Omega)$ if $f\in C^h(\Omega)$ and the derivatives of order h are Lipschitz continuous in $\bar\Omega$ with exponent ε. Take as a norm in $C^{h,\varepsilon}(\Omega)$

$$|f|_{C^{h,\varepsilon}(\Omega)} = |f|_{C^h(\Omega)} + \sup_{|p|=h}\ \sup_{\substack{x,y\in\bar\Omega\\ x\neq y}} \frac{|D^p f(x)-D^p f(y)|}{|x-y|^\varepsilon} .$$

Campanato ([2] and [3]) has given the following characterizations of $\mathscr{L}_k^{p,\lambda}(\Omega)$, $1\leqq p<\infty$, for Ω open, bounded, and of "type $\mathscr{A}$" (see Campanato [3, p. 138]).

$\mathscr{L}_k^{p,\lambda}(\Omega) = L^p(\Omega)$ if $\lambda=0$;
$\mathscr{L}_k^{p,\lambda}(\Omega) = L^{p,\lambda}(\Omega)$, Morrey space, if $0\leqq\lambda<n$ (for definition of $L^{p,\lambda}(\Omega)$ see Campanato [3, p. 157]);

$\mathscr{L}_k^{p,\lambda}(\Omega) = C^{h,\varepsilon}(\Omega)$ if $n<\lambda\leqq n+k\cdot p$, h integer $\leqq k-1$, $(\lambda-n)/p=h+\varepsilon$, $0<\varepsilon<1$ and Ω convex;

$\mathscr{L}_k^{p,\lambda}(\Omega) = \mathscr{E}_h(\Omega) = \mathscr{L}_h^{1,n+h}(\Omega)$ if $(\lambda-n)/p=h$ integer $\leqq k-1$ and Ω convex;

$\mathscr{L}_k^{p,\lambda}(\Omega) = P_k(\Omega) = \{$polynomials of degree $\leqq k-1\}$ if $\lambda>n+kp$.

REMARK 1.2. From the definition of $\mathscr{L}_k^{p,\lambda}(\Omega)$ it follows that $\mathscr{L}_j^{p,\lambda}(\Omega)\subset\mathscr{L}_k^{p,\lambda}(\Omega)$, $j\leqq k$.

We are now going to give another characterization of $\mathscr{L}_k^{p,\lambda}(\Omega)$ for $0<(\lambda-n)/p<k$ with aid of interpolation spaces (see J. Peetre [6]).

Let A_0 and A_1 be Banach spaces with norms $|\cdot|_{A_0}$ and $|\cdot|_{A_1}$, respectively. Put

$$K_\nu(a) = \inf_{a=a_0+a_1}(|a_0|_{A_0}+m^\nu|a_1|_{A_1}),$$

$$J_\nu(a) = \max(|a|_{A_0}, m^\nu|a|_{A_1}), \quad m\neq 1.$$

The interpolation space $(A_0,A_1)_{\theta,q}$, $0<\theta<1$, $1\leqq q\leqq\infty$, is then defined by each one of the equivalent norms

$$\left(\sum_{\nu=-\infty}^{\infty}(m^{-\nu\theta}K_\nu(a))^q\right)^{1/q}, \tag{1.2}$$

$$\inf\left(\sum_{\nu=-\infty}^{\infty}(m^{-\nu\theta}J_\nu(u_\nu))^q\right)^{1/q}, \tag{1.3}$$

where infimum is to be taken over all u_ν such that $a=\sum_{-\infty}^{\infty}u_\nu$ in A_0+A_1. (If $q=\infty$ we take as usual the supremum norm.)

We will work with the space $C^h(\Omega)$, but not with the same norm as Campanato used. We identify functions whose difference is a polynomial of degree $<h$ and take as a norm

$$|f|_{C^h(\Omega)} = \sup_{x\in\bar\Omega,|l|=h}|D^lf(x)|.$$

From now on the notation $C^h(\Omega)$ will refer to this definition.

DEFINITION 1.4. The Besov space $B^\alpha(\Omega)$ is defined by

$$B^\alpha(\Omega) = (C^0(\Omega),C^k(\Omega))_{\alpha/k,\infty}, \quad\text{where}\quad 0<\alpha<k.$$

(In the sequel we let k be the same integer as we used in the definition of $\mathscr{L}_k^{p,\lambda}(\Omega)$.)

The norm in $B^\alpha(\Omega)$ is any one of the above mentioned interpolation norms. Also in $B^\alpha(\Omega)$ we identify functions whose difference is a polynomial of degree $<k$.

We need the following interpolation theorem.

THEOREM 1.1. *Let A_0, A_1, B_0, B_1 be Banach spaces and T a linear operator such that*

$$T\colon A_0 \to B_0, \qquad T\colon A_1 \to B_1 .$$

Then

$$T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q} \quad \text{for } 0<\theta<1 \text{ and } q \geqq 1 .$$

For the corresponding operator norms M_0, M_1 and M, respectively, we have

$$M \leqq M_0^{\theta} M_1^{1-\theta} .$$

The sign $\to$ stands for linear continuous mapping.

We also need

THEOREM 1.2 (S. Spanne [9]). *Let $0<\theta<1$. Then*

$$(\mathscr{L}_k^{p,\lambda_0}(\Omega), \mathscr{L}_k^{p,\lambda_1}(\Omega))_{\theta,\infty} \subset \mathscr{L}_k^{p,\lambda}(\Omega) \quad \text{with } \lambda = (1-\theta)\lambda_0 + \theta\lambda_1 .$$

2. Alternative definitions of $B^\alpha(\mathsf{R}^n)$.

We shall now give two alternative definitions of the space $B^\alpha(\Omega)$ in the case $\Omega = \mathsf{R}^n$. The first one characterizes $B^\alpha(\Omega)$ by means of the modulus of continuity. Write

$$\Delta_{ty} f(x) = f(x+ty) - f(x) ,$$
$$\Delta_{ty}^l f(x) = \Delta_{ty}(\Delta_{ty}^{l-1} f(x)), \quad l = 2,3,\ldots .$$

Let k be the integer in the definition of $\mathscr{L}_k^{p,\lambda}$ and suppose $\alpha<k$. We consider the norm

$$\sup_{0<t<\infty, |y|\leqq 1} t^{-\alpha} |\Delta_{ty}^k f|_{L^\infty(\mathsf{R}^n)} . \tag{2.1}$$

(Again we identify functions whose difference is a polynomial of degree less than k.) We shall prove that (2.1) is then an equivalent norm on $B^\alpha(\mathsf{R}^n)$.

Our second alternative definition of $B^\alpha(\mathsf{R}^n)$ is the following one. Let φ be a function in the Schwartz class S and write

$$\varphi_\nu(x) = 2^{-\nu n} \varphi(2^{-\nu} x), \quad \nu \text{ integer} .$$

Let $\hat\varphi$ be the Fourier transform of φ and suppose that $\hat\varphi(\xi)$ is not zero on the annulus $2^{-1} < |\xi| < 2$ and vanishes outside it. We then consider the norm

$$\sup_\nu 2^{-\nu\alpha} |\varphi_\nu * f|_{L^\infty(\mathsf{R}^n)} . \tag{2.2}$$

We identify functions whose difference is a polynomial of degree less than k, and we exclude all polynomials of degree higher than or equal k.

This norm will depend on the function φ, but two different φ will give rise to equivalent norms. See J. Peetre [8] and J. Löfström [5].

LEMMA 2.1. *The norms* (2.1) *and* (2.2) *are equivalent on* $B^\alpha(\mathsf{R}^n)$.

PROOF. We let $\varphi \in S$ and $\hat{\varphi}(\xi) \neq 0$ in $\frac{1}{2} < |\xi| < 2$, $\operatorname{supp}\hat{\varphi} = \{\xi \mid \frac{1}{2} \leqq |\xi| \leqq 2\}$, $\varphi_\nu(x) = 2^{-\nu n}\varphi(2^{-\nu}x)$. We can take φ such that

$$\sum_{\nu=-\infty}^{\infty} \hat{\varphi}_\nu(\xi) = 1 \quad \text{for } \xi \neq 0 .$$

See Hörmander [4, p. 121]. Now take f such that

$$|\varphi_\nu * f|_{L^\infty} \leqq C\, 2^{\nu\alpha} .$$

It suffices to show that

$$|\Delta_{te_1}^k f|_{L^\infty} \leqq C t^\alpha .$$

Form the function

$$\hat{\psi}_\nu(\xi) = (e^{it\xi_1} - 1)^k \hat{\varphi}_\nu(\xi) .$$

We get at once

$$|\psi_\nu * f|_{L^\infty} \leqq 2^k |\varphi_\nu * f|_{L^\infty} \leqq 2^k C\, 2^{\nu\alpha} . \tag{2.3}$$

Further we have

$$\hat{\psi}_\nu(\xi) = (e^{it\xi_1} - 1)^k\, \hat{\varphi}(2^\nu \xi) = (t2^{-\nu})^k \left(\frac{e^{it\xi_1} - 1}{t\xi_1}\right)^k (\xi_1 2^\nu)^k\, \hat{\omega}(2^\nu \xi)\, \hat{\varphi}(2^\nu \xi) ,$$

if $\hat{\omega} \in S$ and $\hat{\omega}(\xi) = 1$, when $\hat{\varphi}(\xi) \neq 0$.

Let M_n be the space of Fourier transforms of bounded measures on R^n, normed by

$$|\hat{\mu}|_{M_n} = \int_{\mathsf{R}^n} |d\mu| .$$

Then it is easy to see that

$$\left| \left(\frac{e^{it\xi_1} - 1}{t\xi_1}\right)^k \right|_{M_n} \leqq C \quad \text{and} \quad |(t\xi)^k\, \hat{\omega}(t\xi)|_{M_n} \leqq C$$

for $0 < t < \infty$ (see L. Hörmander [4]). Thus we get the estimate

$$|\psi_\nu * f|_{L^\infty} \leqq C (t2^{-\nu})^k\, |\varphi_\nu * f|_{L^\infty} \leqq (t2^{-\nu})^k C\, 2^{\nu\alpha} . \tag{2.4}$$

From (2.3) and (2.4) we get

$$|\psi_\nu * f|_{L^\infty} \leqq C \min\left(1, (t2^{-\nu})^k\right) 2^{\nu\alpha} .$$

However $\Delta_{te_1}^k f(x) = \sum_{-\infty}^{\infty} \psi_\nu * f(x)$, so we have

$$|\Delta_{te_1}^k f|_{L^\infty} \leqq C \sum_{\nu=-\infty}^{\infty} \min\left(1, (t2^{-\nu})^k\right) 2^{\nu\alpha} \leqq C \int_0^\infty x^\alpha \min\left(1, (tx^{-1})^k\right) \frac{dx}{x} \leqq C t^\alpha .$$

From the above calculation we obtain the desired norm inequality.

For the other part of the proof we take f such that $|\Delta_{te_1}^k f|_{L^\infty} \leqq Ct^\alpha$. Take also $\Phi \in S(\mathsf{R})$ such that $\hat{\Phi}(\xi) \neq 0$ exactly for

$$\frac{1}{2+c(n)} < |\xi| < 3, \quad c(n) > 0$$

and set $\Phi_\nu(x) = \Phi(x2^{-\nu}) \cdot 2^{-\nu}$. It is easy to prove that

$$\left| \frac{\hat{\Phi}_\nu(\xi)}{(e^{it\xi}-1)^k} \right|_{M_1} \leqq C \quad \text{for } t = 2^\nu .$$

Now let $\hat{h}^1 \in S(\mathsf{R}^n)$ be such that $\hat{h}^1(\xi) = \hat{\Phi}(\xi_1)\hat{g}(\tilde{\xi})$, where $\hat{\Phi}$ is as above and $\hat{g}(\tilde{\xi}) \neq 0$ for exactly $|\xi_k| < 3$, $k = 2,3,\ldots,n$, where $\tilde{\xi} = (\xi_2, \xi_3, \ldots, \xi_n)$. Then

$$\left| \frac{\hat{h}_\nu^1(\xi)}{(e^{it\xi_1}-1)^k} \right|_{M_n} \leqq C \quad \text{for } t = 2^\nu$$

and we conclude

$$\begin{aligned} |h_\nu^1 * f|_{L^\infty} &= \left| \left(\frac{\hat{h}_\nu^1(\xi)}{(e^{it\xi_1}-1)^k} \right)^v * \Delta_{te_1}^k f \right|_{L^\infty} \\ &\leqq \left| \frac{\hat{h}_\nu^1(\xi)}{(e^{it\xi_1}-1)^k} \right|_{M_n} |\Delta_{te_1}^k f|_{L^\infty} \leqq Ct^\alpha \leqq C 2^{\nu\alpha} . \end{aligned}$$

Here g^v denotes the inverse Fourier transform of g. Repeat the construction for each one of the coordinate axes and add the functions to get $\hat{\psi} = \sum_{k=1}^n \hat{h}^k$. Now take φ as in (2.2). The function $\hat{\psi}(\xi)$ can be chosen such that $\hat{\psi}(\xi) = 1$ for $\frac{1}{2} \leqq |\xi| \leqq 2$. Then $\hat{\psi}(\xi)\hat{\varphi}(\xi) = \hat{\varphi}(\xi)$ (because $\operatorname{supp} \hat{\varphi} = \{\xi \mid \frac{1}{2} \leqq |\xi| \leqq 2\}$). So we have

$$|\varphi_\nu * f|_{L^\infty} = |\psi_\nu * \varphi_\nu * f|_{L^\infty} \leqq |\varphi_\nu|_{L_1} |\psi_\nu * f|_{L^\infty} \leqq C 2^{\nu\alpha} .$$

We also get the desired norm inequality.

Theorem 2.1. *The norms* (2.1) *and* (2.2) *are equivalent norms on* $B^\alpha(\mathsf{R}^n)$.

Proof. In view of lemma 2.1 it suffices to show that (2.2) is equivalent to the norm on $B^\alpha(\mathsf{R}^n)$. We take φ as in (2.2) and f such that

$$|\varphi_\nu * f(x)| \leqq C 2^{\nu\alpha} .$$

As before we can choose φ such that $\sum_{-\infty}^{\infty} \hat{\varphi}_\nu(\xi) = 1$, $\xi \neq 0$. Let $f_\nu(x) = \varphi_\nu * f(x)$. Then

$$f(x) = \sum_{\nu=-\infty}^{\infty} f_\nu(x) \quad \text{(modulo polynomials of degree } < k) ,$$

where $f_\nu(x)$ and its derivatives are continuous functions. For any integer $s \geqq 0$,

$$|f_\nu|_{C^s(\mathsf{R}^n)} = \sup_{|l|=s} |D^l f_\nu|_{C^0(\mathsf{R}^n)} = \sup_{|l|=s} |(D^l\varphi_\nu) * f|_{C^0(\mathsf{R}^n)}$$
$$\leqq 2^{-s\nu} \sup_{|l|=s} |(D^l\varphi)_\nu * f|_{C^0(\mathsf{R}^n)} \leqq C\, 2^{\nu(\alpha-s)} ,$$

because $D^l\varphi$ is a function with essentially the same properties as φ.

Let k be the usual integer $> \alpha$. We have shown that

$$|f_\nu|_{C^0(\mathsf{R}^n)} \leqq C\, 2^{\nu\alpha}, \quad |f_\nu|_{C^k(\mathsf{R}^n)} \leqq C\, 2^{\nu(\alpha-k)} .$$

We get (with $m = 2^k$ in (1.3)) that $(2^{k\nu})^{-\alpha/k} J_\nu(f_\nu) \leqq$ const., that is, $f \in (C^0(\mathsf{R}^n), C^k(\mathsf{R}^n))_{\alpha/k,\infty}$.

Although the other part of the proof follows from section 3, we give a direct proof here. We take $f \in B^\alpha(\mathsf{R}^n) = (C^0(\mathsf{R}^n), C^k(\mathsf{R}^n))_{\alpha/k,\infty}$, k integer $> \alpha$. Equivalently this means that

$$m^{-\nu\alpha} K_\nu(f) < \text{const.} ,$$

where

$$K_\nu(f) = \inf_{f=f_0+f_1} (|f_0|_{C^0(\mathsf{R}^n)} + m^\nu |f_1|_{C^k(\mathsf{R}^n)}) .$$

Now take φ as in (2.2). We let $f = f_0 + f_1$, where $f_0 \in C^0(\mathsf{R}^n)$ and $f_1 \in C^k(\mathsf{R}^n)$. We want to estimate $|\varphi_\nu * f(x)|$. Putting $D^k\Phi(y) = \varphi(y)$, we obtain

$$\left| 2^{-\nu\alpha} \int \varphi_\nu(y) f(x-y)\, dy \right| = \left| 2^{-\nu\alpha} \int \varphi_\nu(y) (f_0(x-y) + f_1(x-y))\, dy \right|$$
$$\leqq 2^{-\nu\alpha} \left(\int |\varphi_\nu(y)|\, dy\, |f_0|_{C^0(\mathsf{R}^n)} + \left| \int \varphi_\nu(y) f_1(x-y)\, dy \right| \right)$$
$$\leqq 2^{-\nu\alpha} \left(C |f_0|_{C^0(\mathsf{R}^n)} + \left| \int \Phi_\nu(y)\, D^k f_1(x-y)\, dy \right| \right)$$
$$\leqq 2^{-\nu\alpha} C (|f_0|_{C^0(\mathsf{R}^n)} + 2^{\nu k} |f_1|_{C^k(\mathsf{R}^n)})$$

(we omit the partition of φ, see lemma 2.1.) and thus the estimation

$$2^{-\nu\alpha} |\varphi_\nu * f(x)| \leqq C\, 2^{-\nu\alpha} \inf_{f=f_0+f_1} (|f_0|_{C^0} + 2^{\nu k} |f_1|_{C^k})$$
$$\leqq C\, (2^{\nu k})^{-\alpha/k} K_\nu(f) \leqq \text{const.}$$

The corresponding norm inequality follows from the calculations.

3. The case $\Omega = \mathsf{R}^n$.

THEOREM 3.1. *$B^\alpha(\mathsf{R}^n) = \mathscr{L}_k^{p,\lambda}(\mathsf{R}^n)$ for $0 < \alpha = (\lambda - n)/p < k$.*

PROOF. First we prove $B^\alpha(\mathsf{R}^n) \subset \mathscr{L}_k^{p,\lambda}(\mathsf{R}^n)$. By theorem 1.2 we get

$$\left(\mathscr{L}_0^{p,n}(\mathsf{R}^n), \mathscr{L}_k^{p,n+kp}(\mathsf{R}^n)\right)_{\alpha/k,\infty} \subset \mathscr{L}_k^{p,\lambda}(\mathsf{R}^n)$$

with

$$\lambda = (1-\alpha k^{-1})n + \alpha k^{-1}(n+kp) = n + \alpha p ,$$

that is, $\alpha = (\lambda - n)/p$. (Here we use $\mathscr{L}_0^{p,n}(\mathsf{R}^n) \subset \mathscr{L}_k^{p,n}(\mathsf{R}^n)$, see remark 1.2.) Now it suffices to prove

$$B^\alpha(\mathsf{R}^n) \subset \left(\mathscr{L}_0^{p,n}(\mathsf{R}^n), \mathscr{L}_k^{p,n+kp}(\mathsf{R}^n)\right)_{\alpha/k,\infty}$$

with $\alpha = (\lambda - n)/p$ and $0 < \alpha/k < 1$. Let I be the identity mapping. We will show that

(3.1) $$I\colon\ C^0(\mathsf{R}^n) \to \mathscr{L}_0^{p,n}(\mathsf{R}^n)$$

(3.2) $$I\colon\ C^k(\mathsf{R}^n) \to \mathscr{L}_k^{p,n+kp}(\mathsf{R}^n)$$

To prove (3.1) let us take $f \in C^0(\mathsf{R}^n)$. Then

$$\left(\int_{|x-x_0|\leq r} |f(x)|^p\,dx\right)^{1/p} \leqq |f|_{C^0(\mathsf{R}^n)} \left(\int_{|x-x_0|\leq r} 1\,dx\right)^{1/p} = |f|_{C^0(\mathsf{R}^n)} r^{n/p} C ,$$

which means $f \in \mathscr{L}_0^{p,n}(\mathsf{R}^n)$. Next we prove (3.2). Let us take $f \in C^k(\mathsf{R}^n)$. Then from Taylor's formula we get

$$f(x) - (\text{polynomial of degree} < k) = (k!)^{-1} \textstyle\sum_{|l|=k} (D^l f)(x_0 + \theta(x-x_0))(x-x_0)^l,$$

where

$$(x-x_0)^l = (x_1 - x_{01})^{l_1} \ldots (x_n - x_{0n})^{l_n} .$$

It follows immediately that

$$|f(x) - q_k(x)| \leqq C \sup\nolimits_{|l|=k} |(x-x_0)^l (D^l f)(x_0 + \theta(x-x_0))| .$$

From this we get

$$\left(\int_{|x-x_0|\leq r} |f(x) - q_k(x)|^p\,dx\right)^{1/p} \leqq C r^k \sup\nolimits_{|l|=k} |D^l f| \left(\int_{|x-x_0|\leq r} 1\,dx\right)^{1/p}$$
$$\leqq C r^{(n+pk)/p}\, |f|_{C^k(\mathsf{R}^n)} .$$

The desired norm inequalities also follow from the above.

By means of (3.1) and (3.2) we conclude, using also theorem 1.1., that

$$I\colon\ B^\alpha(\mathsf{R}^n) = \left(C^0(\mathsf{R}^n), C^k(\mathsf{R}^n)\right)_{\alpha/k,\infty} \to \left(\mathscr{L}_0^{p,n}(\mathsf{R}^n), \mathscr{L}_k^{p,n+kp}(\mathsf{R}^n)\right)_{\alpha/k,\infty} .$$

Now we show that $\mathscr{L}_k^{p,\lambda}(\mathsf{R}^n) \subset B^\alpha(\mathsf{R}^n)$, $\alpha = (\lambda - n)/p$, by proving that $f \in \mathscr{L}_k^{p,\lambda}(\mathsf{R}^n)$ implies (see (2.2))

$$|\varphi_\nu * f|_{L^\infty} \leqq C 2^{\nu\alpha} .$$

We take a function $X \in C_0^\infty(\mathsf{R}^n)$ with support in a neighbourhood of the origin and such that

$$\int q_k(x)\, X(x)\, dx = 0$$

for any polynomial q_k of degree $< k$ and

$$\hat{X}(\xi) \neq 0 \quad \text{for } \tfrac{1}{2} < |\xi| < 2 .$$

Let $X_\nu(x) = 2^{-\nu n} X(2^{-\nu} x)$. Such a function X exists, see lemma 3.1 below. Using Hölder's inequality we get

$$\begin{aligned} |X_\nu * f(x)|_{L^\infty} &= \left| \int X_\nu(x-y)\, f(y)\, dy \right|_{L^\infty} \\ &= \left| \int_{|x-y| \leqq C2^\nu} X_\nu(x-y)\big(f(y) - q_k(y)\big)\, dy \right|_{L^\infty} \\ &\leqq |X_\nu|_{L^{p'}}\, |f - q_k|_{L^p(I_{x,C2^\nu})} \leqq C\, |X_\nu|_{L^{p'}} (2^\nu)^{\lambda/p} . \end{aligned}$$

But

$$\begin{aligned} \left(\int |X_\nu(x)|^{p'}\, dx \right)^{1/p'} &= \left(\int |X(x2^{-\nu})|^{p'}\, dx \right)^{1/p'} 2^{-\nu n} \\ &= \left(\int |X(x)|^{p'}\, dx \right)^{1/p'} 2^{\nu n/p'} 2^{-\nu n} = 2^{-\nu n/p} |X|_{L^{p'}} . \end{aligned}$$

We get

$$|X_\nu * f|_{L^\infty} \leqq C\, |X|_{L^{p'}} (2^\nu)^{(\lambda - n)/p} .$$

Now we take $\psi \in S$ such that $\hat{\psi}(\xi) \neq 0$ for $\tfrac{1}{2} < |\xi| < 2$ and

$$\operatorname{supp} \hat{\psi} = \{\xi \mid \tfrac{1}{2} \leqq |\xi| \leqq 2\}$$

and X as above. Then we get

$$|\psi_\nu * X_\nu * f|_{L^\infty} \leqq |\psi_\nu|_{L^1}\, |X_\nu * f|_{L^\infty} \leqq |\psi|_{L^1} C 2^{\nu\alpha} \leqq \text{const.}\, 2^{\nu\alpha} ,$$

where $\alpha = (\lambda - n)/p$. But $\varphi = \psi * X \in S$ is a function such as φ in (2.2). So we have proved that f in the norm (2.2) is bounded.

Lemma 3.1. *There exists a function $X \in C_0^\infty(\mathsf{R}^n)$ with support in a neighbourhood of the origin, $\int q_k(x) X(x)\, dx = 0$ for all polynomials q_k of degree $< k$ and $\hat{X}(\xi) \neq 0$ for $\tfrac{1}{2} < |\xi| < 2$.*

PROOF. We take a function $\theta(x) \in C_0^\infty(\mathsf{R})$ with support in $|x| \leqq C$. Let $g(x) = D^k\theta(x)$. Then $g(x) \in C_0^\infty(\mathsf{R})$ with support in $|x| \leqq C$. Of course we have

$$\int g(x)dx = 0, \quad \int xg(x)dx = 0, \quad \ldots, \quad \int x^{k-1}g(x)dx = 0 .$$

Further let $\psi(x) = g(x_1)g(x_2)\ldots g(x_n)$. Obviously $\psi(x) \in C_0^\infty(\mathsf{R}^n)$ and $\int q_k(x)\psi(x)dx = 0$ for all polynomials q_k of degree $< k$. We have $\hat{\psi}(\xi) \in S$ and $\hat{\psi}(\xi)$ must be $\neq 0$ in some point ξ_0. Let us suppose that $|\xi_0| = 1$ (otherwise we can make a homothetic transformation that does not change the properties above of ψ). We may suppose that $\hat{\psi}(\xi) \geqq 0$. Then $\hat{\psi}(\xi) > 0$ in a neighbourhood of ξ_0.

Consider $\{\hat{\psi}_B(\xi)\}$, where $\hat{\psi}_B(\xi) = \hat{\psi}(B^{-1}\xi)$ and B an orthogonal matrix. $D^l\hat{\psi}_B(0) = 0$ because $D^l\hat{\psi}(0) = 0$. Further $\hat{\psi}_B(\xi) \neq 0$ in $\xi_B = B\xi_0$. We have a set of functions $\{\hat{\psi}_B(\xi)\}$ such that $\hat{\psi}_B(\xi) > 0$ in a neighbourhood of a point on the unit sphere. Now we cover the unit sphere by a finite subset of such neighbourhoods corresponding to $\hat{\psi}_{B_1}, \hat{\psi}_{B_2}, \ldots, \hat{\psi}_{B_N}$. We get

$$\hat{X}(\xi) = \sum_{\nu=1}^{N} \hat{\psi}_{B_\nu}(\xi) \neq 0 \quad \text{on } |\xi| = 1$$

and in a neighbourhood of this set. We may suppose that this neighbourhood is $\frac{1}{2} < |\xi| < 2$ (otherwise we can repeat the covering argument above, now with $\hat{X}_t(\xi) = \hat{X}(t\xi)$, t constant). The function $X(x)$ has the desired properties.

4. The case bounded $\Omega \subset \mathsf{R}^n$.

We shall say that the open, bounded set $\Omega \subset \mathsf{R}^n$ satisfies assumption (H), if it has

1) the lifting property,
2) the cone property,

which properties we now define.

DEFINITION 3.1. The set $\Omega \subset \mathsf{R}^n$ has the lifting property if there is a linear continuous mapping L such that

$$L\colon\ C^j(\Omega) \to C^j(\mathsf{R}^n) \quad \text{for } j = 0, k$$

and $R \circ L$ is the identity mapping on $C^j(\Omega)$ if R is the restriction to $\bar{\Omega}$ of a function defined in R^n.

DEFINITION 3.2. The set $\Omega \subset \mathsf{R}^n$ has the cone property if to every point x in $\bar{\Omega}$ there exists a neighbourhood O_x of x and a corresponding bounded cone C_x with vertex at the origin and the property $y+C_x \subset \Omega$ for $y \in \Omega \cap O_x$.

REMARK 4.1. Ω has the lifting property if the boundary of Ω is of class C^k. See S. Agmon [1, p. 128] and J. Peetre [8].

If Ω is of class C^k it has the cone property. See Agmon [1, p. 129].

A convex set has the cone property.

THEOREM 4.1. *If Ω satisfies the above assumption* (H), *then* $\mathscr{L}_k^{p,\lambda}(\Omega) = B^\alpha(\Omega)$ *for* $0 < \alpha = (\lambda - n)/p < k$.

PROOF. We carry out the proof by showing that

$$\mathscr{L}_k^{p,\lambda}(\Omega) \longrightarrow (C^0(\Omega), C^k(\Omega))_{\alpha/k,\infty} \xrightarrow{L} (C^0(\mathsf{R}^n), C^k(\mathsf{R}^n))_{\alpha/k,\infty} \longrightarrow$$
$$\longrightarrow \mathscr{L}_k^{p,\lambda}(\mathsf{R}^n) \xrightarrow{R} \mathscr{L}_k^{p,\lambda}(\Omega) .$$

Here steps two and four are immediate. Step three follows from section 1. Therefore only step one remains to be proved.

Choose a finite, open covering $\{O_i\}_{i=1}^{\nu}$ of $\bar{\Omega}$, such that to each O_i we can find a bounded cone C_i and $x + C_i \subset \Omega$ for $x \in \Omega \cap O_i$. This is possible, because Ω is bounded and has the cone property (use the Heine–Borel theorem). Now take $f \in \mathscr{L}_k^{p,\lambda}(\Omega)$. We shall consider $\Omega \cap O_i$ and prove that $f \in B^\alpha(\Omega \cap O_i)$. Let C_i be the cone corresponding to O_i.

Choose a function $X \in C_0^\infty(\mathsf{R}^n)$ with

a) the support in $-C_i = \{y;\ -y \in C_i\}$ and
b) $\int X(x)\,dx = 1$ and
c) $\int q(x) X(x)\,dx = 0$ for all polynomials q of degree less than k and with no constant term.

LEMMA 4.1. *There exists a function $X \in C_0^\infty(\mathsf{R}^n)$ with the properties* a), b) *and* c) *above.*

PROOF. Let us take $\theta(x) \in C_0^\infty(\mathsf{R})$ with the support in $[a,b]$ such that

$$\int \frac{\theta(x)}{x^k}\,dx = \frac{1}{(k-1)!} .$$

Then we have

$$\int \frac{D^{k-1}\theta(x)}{x}\,dx = \quad \ldots \quad = (k-1)! \int \frac{\theta(x)}{x^k}\,dx = 1$$

and

$$\int x \frac{D^{k-1}\theta(x)}{x}\, dx = 0, \quad \ldots, \quad \int x^{k-1} \frac{D^{k-1}\theta(x)}{x}\, dx = 0 .$$

Now set $\Phi(x) = x^{-1} D^{k-1}\theta(x)$.

We want the support of X to be in $-C_i$. Choose a "cube" in $-C_i$ and construct X as a product of n functions $\Phi(x_\nu)$ with their supports in the desired intervals. This completes the proof of the lemma.

Now let $\psi_\nu(x) = X(2^{\nu+1}x)\, 2^{(\nu+1)n} - X(2^\nu x)\, 2^{\nu n}$. For $\nu \geqq 0$, $\psi_\nu(x)$ has support in $-C_i$. Then we get

$$\sum_{\nu=0}^{\infty} \psi_\nu(x) + X(x) = \delta_0 ,$$

for if $g \in C_0^\infty(\mathsf{R}^n)$ we have, if $N > N_0$,

$$\left| \int \left(\sum_{\nu=0}^{N} \psi_\nu(x) + X(x) \right) g(x)\, dx - g(0) \right| = \left| \int X(2^{N+1}x)\, 2^{(N+1)n} g(x)\, dx - g(0) \right|$$

$$= \left| \int X(y) \left(g\left(\frac{y}{2^{(N+1)n}} \right) - g(0) \right) dy \right|$$

$$\leqq \int |X(y)| \left| g\left(\frac{y}{2^{(N+1)n}} \right) - g(0) \right| dy < \varepsilon .$$

Thus

$$f(x) = \sum_{\nu=0}^{\infty} \psi_\nu * f(x) + X * f(x) \quad \text{for } x \in \overline{\Omega \cap O_i} .$$

Note that the terms on the right side are well defined functions and they are continuously differentiable up to the order we want.

Let $f_\nu(x) = \psi_\nu * f(x)$ for $\nu \geqq 0$ and $f_{-1}(x) = X * f(x)$. We have $f(x) = \sum_{-1}^{\infty} f_\nu(x)$ for $x \in \overline{\Omega \cap O_i}$. We shall prove that

$$2^{\nu\alpha} J(2^{-\nu}, f_\nu) \leqq C |f|_{\mathscr{L}_k^{p,\lambda}(\Omega)} \quad \text{for all } \nu .$$

For $\nu \geqq 0$ we have, taking supremum over all $x \in \overline{\Omega \cap O_i}$,

$$|f_\nu|_{C^0(\overline{\Omega \cap O_i})} = \sup \left| \int \psi_\nu(x-y) f(y)\, dy \right|$$

$$= \sup \left| \int \psi_\nu(x-y)(f(y) - q_k(y))\, dy \right|$$

$$= \sup \left| 2^{\nu n} \int \psi_0((x-y)2^\nu)(f(y) - q_k(y))\, dy \right|$$

$$\leqq \sup 2^{\nu n} \left(\int |\psi_0((x-y)2^\nu)|^{p'} dy \right)^{1/p'} \left(\int_{(x-y)2^\nu \in \operatorname{supp} \psi_0} |f(y) - q_k(y)|^p\, dy \right)^{1/p}$$

$$\leqq \sup 2^{\nu n} 2^{-\nu n/p'} C \left(\int_{\Omega \cap I_{x,c2^{-\nu}}} |f(y) - q_k(y)|^p \, dy \right)^{1/p}$$

$$\leqq (2^\nu)^{n/p} C |f|_{\mathscr{L}_k^{p,\lambda}(\Omega)} C 2^{-\nu\lambda/p} = C (2^{-\nu})^{(\lambda-n)/p} |f|_{\mathscr{L}_k^{p,\lambda}(\Omega)} .$$

Further we get

$$|f_\nu|_{C^k(\Omega \cap O_i)} = \sup_{|\alpha|=k} |D^\alpha f_\nu|_{C^0(\Omega \cap O_i)} = \sup |D^\alpha \psi_\nu * f(x)|$$

$$= \sup \left| \int D^\alpha \psi_0((x-y) 2^\nu) \, 2^{\nu n} f(y) \, dy \right|$$

$$= \sup \left| 2^{\nu k} \int \varphi_\alpha((x-y) 2^\nu) \, 2^{\nu n} f(y) \, dy \right| ,$$

where the last three suprema are taken for $x \in \overline{\Omega \cap O_i}$, $|\alpha| = k$. But $\varphi_\alpha = D^\alpha \psi_0$ is a function with essentially the same properties as ψ_0. Thus we can use the same estimate as above. We get

$$|f_\nu|_{C^k(\Omega \cap O_i)} \leqq 2^{\nu k} C (2^{-\nu})^{(\lambda-n)/p} |f|_{\mathscr{L}_k^{p,\lambda}(\Omega)} .$$

With similar methods we can treat $f_{-1}(x) = X * f(x)$ and get analogous estimates.

Then we have for all ν

$$(2^{\nu k})^{\alpha/k} \max (|f_\nu|_{C^0(\Omega \cap O_i)}, 2^{-\nu k} |f_\nu|_{C^k(\Omega \cap O_i)}) \leqq C |f|_{\mathscr{L}_k^{p,\lambda}(\Omega)} ,$$

where $\alpha = (\lambda - n)/p$, $0 < \alpha < k$.

We have proved that $f \in B^\alpha(\Omega \cap O_i)$ for $\alpha = (\lambda - n)/p$ and for an arbitrary set O_i in the construction above.

Lemma 4.2. *If $f(x) \in B^\alpha(\Omega \cap O)$ and $\eta(x) \in C_0^\infty(\mathsf{R}^n)$ and* $\operatorname{supp} \eta \subset O$, *we have $\eta(x) f(x) \in B^\alpha(\Omega)$.*

Proof. It suffices to notice that the mapping

$$F\colon f(x) \to \eta(x) f(x)$$

is such that

$$F\colon C^0(\Omega \cap O) \to C^0(\Omega), \quad F\colon C^k(\Omega \cap O) \to C^k(\Omega) .$$

The statement then follows by the interpolation theorem.

Now we can conclude the proof of theorem 4.1. In fact we choose to the finite, open covering $\{O_i\}_{i=1}^\nu$ of $\overline{\Omega}$ a partition of unity, that is, functions $(\eta_i)_{i=1}^\nu$ such that

$$\eta_i \text{ has support in } O_i \quad \text{and} \quad \sum_{i=1}^{\nu} \eta_i(x) = 1, \quad \text{when } x \in \bar{\Omega}.$$

We have shown that $\eta_i f \in B^\alpha(\Omega)$ (lemma 4.2). Thus we have also $\sum_{i=1}^{\nu} \eta_i f \in B^\alpha(\Omega)$. But $\sum_{i=1}^{\nu} \eta_i(x) f(x) = f(x)$ when $x \in \bar{\Omega}$. Thereby theorem 4.1 is proved.

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