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# A THEORY OF CAPACITIES FOR POTENTIALS OF FUNCTIONS IN LEBESGUE CLASSES

NORMAN G. MEYERS

## Introduction.

This paper develops a theory of capacities for potentials of functions in the Lebesgue classes $\mathscr{L}_{\mu;p}$, where $\mu$ is a fixed positive measure on Euclidean $n$-space and $1<p<\infty$. This contrasts with the classical or older theories which are concerned with the case $p=2$ and with potentials of measures or what we will call the space $\mathscr{L}_1$. The theory for $\mathscr{L}_{\mu;p}$ parallels the classical theory in many ways and I have taken some pains to draw the parallel as closely as possible. However the two are not completely parallel because the special advantages of $\mathscr{L}_{\mu;p}$ over $\mathscr{L}_1$ permit us to treat a much broader class of potentials and often in a simpler way.

Section 2 is concerned with potentials of functions in $\mathscr{L}_{\mu;p}$ with respect to a positive lower semi-continuous kernel, $k$. It introduces a capacity $C_{k;\mu;p}$ and studies such questions as convergence of potentials, convergence of sequences of potentials under the weak and strong topologies in $\mathscr{L}_{\mu;p}$, capacitary distributions, capacitary potentials and the capacitability of sets. The capacitability results become important later in proving certain theorems in proper generality.

Section 3 presents a parallel theory for the case $\mathscr{L}_1$ and its corresponding capacity $C_{k;1}$. Here it is apparently necessary to make further restrictions on the kernel and though I have not attempted to make the class of kernels as large as possible I have striven for a reasonable degree of generality.

In Section 4 I study the capacities as functions of $p$. The main results are presented in Theorems 10 and 11 and give continuity properties of the capacities; in particular they relate the classical and non-classical capacities by showing in considerable generality that

$$C_{k;m;p}(K) \to C_{k;1}(K) \qquad \text{as } p \to 1^+$$

for every compact $K$, $m$ being Lebesgue measure.

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A different capacity $c_{k;\mu;p}$ and its counterpart $c_{k;1}$ are introduced in Section 5. These capacities unlike the previous ones are based on potentials of measures in $\mathscr{L}_1$. The two capacities are however closely related as is shown in Section 6. In particular Theorem 14 gives the equality

$$c^*_{k;\mu;p}(A) = (C_{k;\mu;p}(A))^{1/p}$$

for all sets $A$, where $c^*_{k;\mu;p}$ is the outer capacity corresponding to $c_{k;\mu;p}$. This permits one to state results for $c_{k;\mu;p}$ which would be difficult to reach in any other way; for example the capacitability of analytic sets. Theorem 16 then shows that in the case of compact sets of finite capacity the $C_{k;\mu;p}$-capacitary distribution can be simply expressed by means of the $c_{k;\mu;p}$-capacitary potential.

This completes the general theory. The reader should note that while I have confined considerations to a single Euclidean space many of the results hold in greater generality; for example in the context of pairs of locally compact Hausdorff spaces or groups. The rest of the paper is concerned with the case of Bessel potentials (Section 7). The theory of the previous sections is sufficiently general so that all of it applies intact to this particular case. In addition I apply it to get more specific results, the most interesting of which is the relation between Hausdorff measure and the Bessel capacities (for example see Theorem 21).

Variants of the capacities $C_{k;\mu;p}$ appear in the literature in at least two different connections. First in the general theory of functional completion and the trace theory for strongly differentiable functions (see [1], [2]) and second in the theory of removable singularities for solutions of quasi-linear elliptic equations of second order (see [12]). Recently Littman (see [11]) has studied removable singularities for higher order equations but has found it necessary to introduce a different capacity. Though ours and his are very closely related they appear to be different; the exact relation between the two is, however, unknown.

## 1. Preliminaries.

The underlying space in this entire paper will be the Euclidean $n$-space $E^n$, $n \geqq 1$; all point sets are tacitly assumed to be subsets of $E^n$. By $x, y$ and $z$ we denote points of $E^n$. By $G$ and $K$ we denote open and compact subsets of $E^n$ respectively.

A function whose values are real numbers or $\pm\infty$ will be called *real valued* and if further, the function assumes only values greater or equal to zero it will be called *positive*.

Definition 1. For the purposes of this paper we define a *capacity* to be a positive set function $C$ given on a $\sigma$-additive class of sets $\mathscr{F}$, which contains the compact sets and has the properties:

(i) $C(\emptyset)=0$, $\emptyset=$ the empty set.
(ii) If $A_1$ and $A_2$ are in $\mathscr{F}$ and $A_1 \subset A_2$, then $C(A_1) \leqq C(A_2)$.
(iii) If $A_i$, $i=1,2,\ldots$, are in $\mathscr{F}$, then $C(\bigcup_{i=1}^{\infty} A_i) \leqq \sum_{i=1}^{\infty} C(A_i)$.

If in addition $C$ has the property

(iv) For every $A \in \mathscr{F}$, $C(A)=\sup_{K \subset A} C(K)$,

then $C$ is called an *inner capacity*. If $C$ has the property

(iv)′ For every $A \in \mathscr{F}$, $C(A)=\inf_{G \supset A} C(G)$,

then $C$ is called an *outer capacity*.

Note that if $\mathscr{F}' \subset \mathscr{F}$ is a $\sigma$-additive class and for all $A$ we define

$$C^{\mathscr{F}'}(A) = \inf_{A \subset A' \in \mathscr{F}'} C(A') .$$

then $C^{\mathscr{F}'}$ is a capacity defined on all subsets of $E^n$ and agrees with $C$ on $\mathscr{F}'$. In particular, if we take $\mathscr{F}'$ to be the class of all open sets then $C^{\mathscr{F}'}$ is an outer capacity and we denote it by $C^*$.

If a statement holds except on a set $A$ where $C^{\mathscr{F}}(A)=0$, then we say that the statement holds $C$-a.e. If $f$ and the elements of the sequence $\{f_i\}$ are real valued functions which are finite $C$-a.e. and for every $\varepsilon>0$

$$C^{\mathscr{F}}_{i\to\infty}(\{x \mid |f_i(x)-f(x)| \geqq \varepsilon\}) \to 0 ,$$

then we say $f_i \to f$ in $C$-cap. If for every $\varepsilon>0$ there is a set $A$ depending on $\varepsilon$, such that

$$C^{\mathscr{F}}(A) \leqq \varepsilon \quad \text{and} \quad f_i \to f \text{ uniformly on } E^n - A ,$$

we say that $f_i \to f$ $C$-a.u.

Definition 2. By a measure we mean the completion of a real valued, $\sigma$-additive set function $\mu$, defined on the Borel field and having the property

$$\mu(K) \text{ is finite for all } K .$$

'$m$' will stand for Lebesgue measure in $E^n$ and we will generally drop '$m$' from the notation when no confusion can result; thus $\int \ldots dx$ means integration with respect to '$m$'.

If $\mu$ is a measure and $A$ is a $\mu$-measurable set, we say that $\mu$ is *con-*

*centrated* on $A$ if $\mu(B)=0$ for all sets $B$ which are $\mu$-measurable and contained in $E^n - A$.

Let $\mu$ be a measure and $A$ a Borel set. By $\mu | A$ we shall mean the measure which for every Borel set, $B$, is given by

$$\mu | A(B) = \mu(A \cap B) .$$

DEFINITION 3. Let $\mathscr{M}$ be the vector space of all Radon measures. Though these elements are measures only locally we use the same notation as for measures. We define a topology on $\mathscr{M}$ by saying that a net $\{\mu_\alpha\}$, $\alpha \in \mathscr{A}$, converges to $\mu$ *weakly* in $\mathscr{M}$ if

$$\lim_{\mathscr{A}} \int \varphi(x)\, d\mu_\alpha(x) = \int \varphi(x)\, d\mu(x)$$

for every test function $\varphi$; that is, $\varphi$ is finite real valued, continuous and has a compact support. The cone of positive elements, which are necessarily measures, will be denoted by $\mathscr{M}^+$ and this illustrates a procedure we will follow generally without further comment; a superscript $+$ on a symbol for a set indicates the subset of positive elements.

If $\mu \in \mathscr{M}$ and $f$ is a $\mu$-measurable function on every bounded open set and

$$\int_K f(x)\, d\mu(x) \quad \text{exists and is finite for all } K ,$$

then we define the Radon measure $f\mu$ by

$$\langle f\mu, \varphi\rangle = \int \varphi(x) f(x)\, d\mu(x) ,$$

where $\varphi$ is any test function.

DEFINITION 4. If $\mu \in \mathscr{M}^+$ and $1 < p < \infty$, then $\mathscr{L}_{\mu;p}$ will be the Banach space of all $\mu$-measurable functions $f$ such that

$$\|f\|_{\mu;p} = \left(\int |f(x)|^p\, d\mu(x)\right)^{1/p} < \infty .$$

$\mathscr{L}_1$ will stand for the Banach space of all measures with norm

$$\|\mu\|_1 = \text{total variation of } \mu < \infty .$$

We will deal with weak topologies in these spaces. We take the usual weak topology in $\mathscr{L}_{\mu;p}$ and in $\mathscr{L}_1$ we take the topology induced by $\mathscr{M}$.

## 2. The capacities $C_{k;\mu;p}$.

DEFINITION 5. Let $k = k(x,y)$ be a positive, lower semi-continuous kernel defined on $E^n \times E^n$. Let $\nu$ and $\mu$ be in $\mathscr{M}$. If $\sigma = \nu_x \oplus \mu_y$ is the tensor product, then $\sigma \in \mathscr{M}$ with respect to the product space. $\sigma$ has a canonical Jordan decomposition $\sigma=\sigma^+-\sigma^-$, where $\sigma^+$ and $\sigma^-$ are in $\mathscr{M}^+$. We define

$$k(\nu,\mu) = \int k(x,y)\, d\sigma^+(x,y) - \int k(x,y)\, d\sigma^-(x,y)\,,$$

provided one of the terms on the right is finite. So that $k(\nu,\mu)$ can always have a value we put

$$k(\nu,\mu) = +\infty \quad \text{otherwise}.$$

Let $\delta_x$ denote the Dirac measure concentrated at $x$. Then $k(x,\mu)=k(\delta_x,\mu)$, and $k(\nu,y)$ is defined similarly.

LEMMA 1. *The functions $k(\nu,\mu)$, $k(x,\mu)$ and $k(\nu,y)$ are lower semi-continuous functions on $\mathscr{M}^+\times\mathscr{M}^+$, $E^n\times\mathscr{M}^+$ and $\mathscr{M}^+\times E^n$ respectively.*

The lower semi-continuity of the second and third are simple consequences of the lower semi-continuity of the first. For a proof see Lemma 2.2.1 of [7].

DEFINITION 6. Let $k$ be a kernel as given in Definition 5 and let $\mu\in\mathscr{M}^+$. If $A$ is an arbitrary point set, then

$$C_{k;\mu;p}(A) = C_{\mu;p}(A) = \inf_f \|f\|^p_{\mu;p}\,,$$

where the above infimum is taken over all $f$ in $\mathscr{L}^+_{\mu;p}$ such that

$$k(x,f\mu) \geqq 1 \quad \text{on } A\,.$$

We follow the convention that the infimum over an empty set equals $+\infty$; thus $C_{\mu;p}(A)=+\infty$ if and only if no such $f$ exists. We shall call a function $f$ in $\mathscr{L}^+_{\mu;p}$ such that $k(x,f\mu)\geqq 1$ on $A$, a *test function* for $C_{\mu;p}(A)$. One can easily show that

$$C_{\mu;p}(A) = \inf_f\{\inf_{x\in A} k(x,f\mu)\}^{-p} = \{\sup_f \inf_{x\in A} k(x,f\mu)\}^{-p},$$

where $f\in\mathscr{L}^+_{\mu;p}$ and $\|f\|_{\mu;p}\leqq 1$.

THEOREM 1. *$C_{\mu;p}$ is an outer capacity defined on all subsets of $E^n$.*

PROOF. Conditions (i) and (ii) of Definition 1 are trivial to verify. To verify (iii) we may as well assume that $\sum_{i=1}^\infty C_{\mu;p}(A_i)<+\infty$; then $C_{\mu;p}(A_i)<+\infty$ and for every $\varepsilon>0$ there exists a test function $f_i$ such that

$$\|f_i\|_{\mu;p}^p \leqq C_{\mu;p}(A_i) + 2^{-i}\varepsilon .$$

Put $f(x)=\sup_i f_i(x)$. Then $(f(x))^p \leqq \sum_{i=1}^{\infty}(f_i(x))^p$ which implies

$$\|f\|_{\mu;p}^p \leqq \sum_{i=1}^{\infty}\|f_i\|_{\mu;p}^p \leqq \sum_{i=1}^{\infty} C_{\mu;p}(A_i) + \varepsilon . \tag{1}$$

Since $k(x,f\mu) \geqq k(x,f_i\mu)$ it is clear that $f$ is a test function for $C_{\mu;p}(\bigcup_{i=1}^{\infty} A_i)$, and (iii) easily follows from (1).

We have shown that $C_{\mu;p}$ is a capacity and it remains only to verify condition (iv)′ of Definition 1. Again we may assume that $C_{\mu;p}(A) < \infty$. Then if $0 < \varepsilon < 1$ there must exist a test function for $C_{\mu;p}(A)$, call it $f$, such that

$$\|f\|_{\mu;p}^p \leqq C_{\mu;p}(A) + \varepsilon .$$

Put $f_\varepsilon = (1-\varepsilon)^{-1} f$ and let

$$G = \{x \mid k(x,f_\varepsilon\mu) > 1\} .$$

Since $k(x,f_\varepsilon\mu)$ is lower semi-continuous in $x$, $G$ is an open set and since $k(x,f_\varepsilon\mu) \geqq (1-\varepsilon)^{-1}$ on $A$, $G \supset A$. Therefore $f_\varepsilon$ is a test function for $C_{\mu;p}(G)$ and we have

$$C_{\mu;p}(G) \leqq \|f_\varepsilon\|_{\mu;p}^p \leqq C_{\mu;p}(A)(1-\varepsilon)^{-p} + \varepsilon(1-\varepsilon)^{-p} . \tag{2}$$

(iv)′ is now an easy consequence of (2).

**Theorem 2.** *For all $f \in \mathscr{L}_{\mu;p}$ and $0 < a < \infty$ we have*

$$C_{\mu;p}(\{x \mid |k(x,f\mu| \geqq a\}) \leqq a^{-p}\|f\|_{\mu;p}^p .$$

**Proof.** Note first that $|k(x,f\mu)| \leqq k(x,|f|\mu)$ and therefore it is sufficient to take $f \in \mathscr{L}_{\mu;p}^+$. But then $a^{-1}f$ is a test function for the capacity of the set under consideration and Theorem 2 follows directly from Definition 6.

**Theorem 3.** *If $f \in \mathscr{L}_{\mu;p}$ and $|k(x,f\mu)| = +\infty$ on $A$ then $C_{\mu;p}(A)=0$. If $C_{\mu;p}(A)=0$, then there exists $f \in \mathscr{L}_{\mu;p}^+$ such that $k(x,f\mu) = +\infty$ on $A$.*

**Proof.** The first statement follows by letting $a \to \infty$ in the statement of Theorem 2. To prove the second statement take a sequence of test functions for $C_{\mu;p}(A)$, $\{f_i\}$, such that

$$\|f_i\|_{\mu;p} \leqq 2^{-i} .$$

Set $f = \sum_{i=1}^{\infty} f_i$.

COROLLARY. *If $f=f_1+f_2$ $\mu$-a.e., where $f_1$ and $f_2$ are in $\mathscr{L}_{\mu;p}$, then*

$$k(x,f\mu) = k(x,f_1\mu)+k(x,f_2\mu) \quad C_{\mu;p}\text{-a.e.}$$

*and*

$$k(x,af\mu) = ak(x,f\mu) \quad C_{\mu;p}\text{-a.e.},$$

*where 'a' is any finite constant.*

PROOF. The only points $x$, where the above equalities can fail are points where one of the terms on the right is infinite. Since these are sets of $C_{\mu;p}$-capacity zero the equalities must hold $C_{\mu;p}$-a.e.

THEOREM 4. *Each of the following statements implies the succeeding one:*

(i) $f_i \to f$ *strongly in* $\mathscr{L}_{\mu;p}$.
(ii) $k(x,f_i\mu) \to k(x,f\mu)$ *in* $C_{\mu;p}$-cap.
(iii) *There exists a subsequence* $\{f_i'\}$ *such that*

$$k(x,f_i'\mu) \to k(x,f\mu) \quad C_{\mu;p}\text{-a.u.}$$

(iv) $k(x,f_i'\mu) \to k(x,f\mu)$ $C_{\mu;p}$-a.e.

PROOF. The potentials $k(x,f\mu)$ and $k(x,f_i\mu)$ are finite $C_{\mu;p}$-a.e. From the preceding Corollary and Theorem 2,

$$(3) \qquad C_{\mu;p}(\{x \mid |k(x,f_i\mu)-k(x,f\mu)| \geqq \varepsilon\}) = C_{\mu;p}(\{x \mid |k(x,(f_i-f)\mu)| \geqq \varepsilon\}) \leqq \varepsilon^{-p}\|f_i-f\|_{\mu;p}^p .$$

From (3) we can immediately infer that (i) $\Rightarrow$ (ii).

Now suppose that (ii) holds. Then given $\varepsilon>0$ there must exist a subsequence $\{f_{i_j}\}$ and sets $\{A_j\}$ for which

$$|k(x,f_{i_j}\mu)-k(x,f\mu)| \leqq j^{-1} \qquad \text{except on } A_j,$$

where

$$C_{\mu;p}(A_j) \leqq \varepsilon 2^{-j} .$$

Hence $k(x,f_{i_j}\mu) \to k(x,f\mu)$ as $j\to\infty$, uniformly on $E^n-\bigcup_j A_j$, where $C_{\mu;p}(\bigcup_j A_j) \leqq \varepsilon$. A simple diagonalization argument will now prove (iii). That (iii) $\Rightarrow$ (iv) is obvious.

COROLLARY. *Suppose $f_i$, $i=1,2,\ldots$, are in $\mathscr{L}_{\mu;p}$. If $\sum_{i=1}^\infty |f_i|$ is in $\mathscr{L}_{\mu;p}$, then*

$$\textstyle\sum_{i=1}^\infty k(x,f_i\mu) = k(x,(\sum_{i=1}^\infty f_i)\mu) \quad C_{\mu;p}\text{-a.e.}$$

PROOF. Set $f_i^+(x) = \max(f_i(x), 0)$ and $f_i^-(x) = \max(-f(x), 0)$ wherever $f_i(x)$ is defined. Then $\sum_{i=1}^\infty f_i^+$ and $\sum_{i=1}^\infty f_i^-$ are in $\mathscr{L}_{\mu;p}$. If the result holds for $f_i^+$ and $f_i^-$, then it will hold for the $f_i$. Hence we may assume that $f_i(x) \geqq 0$. By Theorem 4 a subsequence of the partial sums

$$\sum_{i=1}^n k(x, f_i\mu)$$

converges to $k(x, (\sum_{i=1}^\infty f_i)\mu)$ $C_{\mu;p}$-a.e. But since all terms of the series are positive the full series must converge wherever the subsequence of partial sums converges, thus proving the result. We also see that the partial sums converge $C_{\mu;p}$-a.u.

THEOREM 5. (i) *If $f_i \to f$ weakly in $\mathscr{L}_{\mu;p}$, then*

$$\liminf k(x, f_i\mu) \leqq k(x, f\mu) \leqq \limsup k(x, f_i\mu) \quad C_{\mu;p}\text{-a.e.}$$

(ii) *If $f_i \to f$ weakly in $\mathscr{L}^+_{\mu;p}$, then*

$$k(x, f\mu) \leqq \liminf k(x, f_i\mu) \quad \textit{everywhere}$$

*and*

$$k(x, f\mu) = \liminf k(x, f_i\mu) \quad C_{\mu;p}\text{-a.e.}$$

PROOF. Suppose $f_i \to f$ weakly in $\mathscr{L}_{\mu;p}$. Then by the Banach–Saks Theorem a subsequence $\{f_i'\}$ exists such that

$$g_j = j^{-1} \sum_{i=1}^j f_i'$$

converges strongly to $f$ in $\mathscr{L}_{\mu;p}$. Therefore, from Theorem 4, a subsequence $\{g_j'\}$ exists such that

$$k(x, f\mu) = \lim k(x, g_j'\mu) \quad C_{\mu;p}\text{-a.e.}$$

(i) now follows by observing that

$$\lim k(x, g_j'\mu) \geqq \liminf k(x, f_i'\mu) \geqq \liminf k(x, f_i\mu) \quad C_{\mu;p}\text{-a.e.}$$

This proves the first inequality in (i). The second follows by replacing $f_i$ and $f$ by $-f_i$ and $-f$ respectively.

We now turn to the proof of (ii). If $f_i \to f$ weakly in $\mathscr{L}^+_{\mu;p}$, then clearly

$$f_i\mu \to f\mu \qquad \text{weakly in } \mathscr{M}.$$

The first statement in (ii) is then a consequence of Lemma 1. Assertion (i) and the first part of (ii) clearly imply the second part of (ii).

THEOREM 6. *If $A_i \uparrow A$, then $C_{\mu;p}(A_i) \uparrow C_{\mu;p}(A)$.*

PROOF. Without loss of generality we assume that the sequence $C_{\mu;p}(A_i)$ converges to a finite limit, $l$. Let $f_i$ be a test function for $C_{\mu;p}(A_i)$ such that

$$\|f_i\|_{\mu;p}^p \leqq C_{\mu;p}(A_i) + 1/i \,. \tag{4}$$

Since the $f_i$ form a bounded sequence in $\mathscr{L}^+_{\mu;p}$, a subsequence $\{f_{i_j}\}$ converges weakly to a function $f \in \mathscr{L}^+_{\mu;p}$. From Theorem 5,

$$k(x,f\mu) \geqq 1 \quad \text{on } A_i \quad C_{\mu;p}\text{-a.e.}$$

Hence

$$k(x,f\mu) \geqq 1 \quad \text{on } A \quad C_{\mu;p}\text{-a.e.} \tag{5}$$

Let $B$ be the subset of $A$ where (5) holds. Then from (4) and the weak convergence

$$C_{\mu;p}(A) = C_{\mu;p}(B) \leqq \|f\|_{\mu;p} \leqq l \,. \tag{6}$$

The desired result is now a simple consequence of (6).

We can now prove a lower semi-continuity property of the capacities which is an analogue of Fatou's Lemma.

COROLLARY. *If $\{A_i\}$ is any sequence of sets, then*

$$C_{\mu;p}(\liminf_{i\to\infty} A_i) \leqq \liminf_{i\to\infty} C_{\mu;p}(A_i) \,.$$

PROOF. Let $B = \liminf_{i\to\infty} A_i = \bigcup_j \bigcap_{k\geqq j} A_k$ and $B_i = \bigcup_{j=1}^i \bigcap_{k\geqq j} A_k$. Then

$$B_i \uparrow B \,. \tag{7}$$

Therefore, from Theorem 6,

$$C_{\mu;p}(B) = \lim_{i\to\infty} C(B_i) \leqq \liminf_{i\to\infty} C(A_i) \,. \tag{8}$$

THEOREM 7. *If $K_i \downarrow K$, then $C_{\mu;p}(K_i) \downarrow C_{\mu;p}(K)$.*

PROOF. This proposition holds not only for $C_{\mu;p}$ but for any outer capacity. Let $G \supset K$; then for sufficiently large values of $i$ we must have $K_i \subset G$. Therefore

$$\lim C_{\mu;p}(K_i) \leqq C_{\mu;p}(G) \,, \tag{9}$$

and since $C_{\mu;p}$ is an outer capacity (9) must hold with $G$ replaced by $K$; we can then easily infer Theorem 7.

DEFINITION 7. If $C$ is a capacity, we say that a set $A$ is *$C$-capacitable* if

$$\sup_{K\subset A} C(K) = \inf_{G\supset A} C(G) = C^*(A) \,.$$

Capacitability has been studied in a very general context by Choquet [4]. Theorem 1 of [4] together with our Theorems 6 and 7 give

THEOREM 8. *All analytic sets, and hence all Borel sets, are $C_{\mu;p}$-capacitable.*

Theorem 8 is an interesting result in itself but its main importance to us is that it will allow us to extend results known for compact sets to all sets, thus giving a more general and aesthetically pleasing theory.

We now take up a brief discussion of capacitary distributions and potentials for the $C_{\mu;p}$. Let $A$ be any set such that $C_{\mu;p}(A) < \infty$. We consider the following variational problem:

$$\min_f \|f\|^p_{\mu;p}, \tag{10}$$

the minimum being taken over all $f$ in $\mathscr{L}_{\mu;p}$ such that

$$k(x,f\mu) \geqq 1 \quad \text{on } A \quad C_{\mu;p}\text{-a.e.}$$

We will call such a function $f$, *a test function for $A$ in* (10).

DEFINITION 8. We call a solution, $f$, of problem (10) a *$C_{\mu;p}$-capacitary distribution of $A$* and we call $k(x,f\mu)$ a *$C_{\mu;p}$-capacitary potential of $A$.*

THEOREM 9. *If $C_{\mu;p}(A) < \infty$, then $A$ has a unique $C_{\mu;p}$-capacitary distribution $f$; $f \in \mathscr{L}^+_{\mu;p}$, $\|f\|^p_{\mu;p} = C_{\mu;p}(A)$ and*

$$\int (f(x))^{p-1}\, g(x)\, d\mu(x) \geqq 0$$

*for all $g \in \mathscr{L}_{\mu;p}$ such that*

$$k(x,g\mu) \geqq 0 \quad \text{on } A \quad C_{\mu;p}\text{-a.e.}$$

PROOF. The set of test functions in (10) is obviously convex and by Theorem 4 is strongly closed and hence weakly closed in $\mathscr{L}_{\mu;p}$. The existence of a unique minimizing function, $f$, now follows by well-known techniques of the calculus of variations. Now $f^+(x) = \max(f(x), 0)$ is also a test function and $\|f^+\|^p_{\mu;p} \leqq \|f\|^p_{\mu;p}$; therefore $f = f^+$ so that $f \in \mathscr{L}^+_{\mu;p}$. If $B$ is the subset of $A$ where $k(x,f\mu) < 1$, then

$$C_{\mu;p}(A) = C_{\mu;p}(A-B) \leqq \|f\|^p_{\mu;p} \leqq C_{\mu;p}(A),$$

so that $\|f\|^p_{\mu;p} = C_{\mu;p}(A)$. Finally, if $g \in \mathscr{L}_{\mu;p}$ and

$$k(x,g\mu) \geqq 0 \quad \text{on } A \quad C_{\mu;p}\text{-a.e.},$$

then $f + tg$ is a test function in (10) when $t \geqq 0$; therefore

$$\frac{d}{dt^+}\|f+tg\|^p_{\mu;p}\Big]_{t=0} = p\int (f(x))^{p-1}\, g(x)\, d\mu(x) \geqq 0 ,$$

thus completing the proof.

We now present another corollary which is closely associated with some later work; see Theorem 16.

COROLLARY. *Let $0 < C_{\mu;p}(A) < \infty$ and let $f$ be the $C_{\mu;p}$-capacitary distribution of $A$. Then for every $g \in \mathscr{L}_{\mu;p}$ we have*

$$\left|\int f^{p-1}\, g\, d\mu(x)\right| \leqq C_{\mu;p}(A)\, \sup_A |k(x,g\mu)| .$$

PROOF. The inequality is clearly true if $\sup_A |k(x,g\mu)| = \infty$. On the other hand if $\sup_A |k(x,g\mu)| = 0$, then from Theorem 9 the left side is also zero; so let us assume that

$$0 < a = \sup_A |k(x,g\mu)| < \infty .$$

Since $k(x,f\mu)$ is the $C_{\mu;p}$-capacitary potential of $A$,

$$k(x, (f-a^{-1}g)\mu) \geqq 0 \quad \text{on } A \quad C_{\mu;p}\text{-a.e.}$$

From this and Theorem 9 we then have

$$\int f^{p-1}\, (f-a^{-1}g)\, d\mu(x) \geqq 0$$

or

$$\int f^{p-1}\, g\, d\mu(x) \leqq a \int f^p\, d\mu(x) .$$

Since the same inequality holds with $-g$ in place of $g$ we have proved the inequality.

COROLLARY. *Let $A_i$, $i = 1,2,\ldots$, and $A$ be sets with capacitary distributions $f_i$ and $f$ respectively. If $A \subset \liminf A_i$ and $\lim C_{\mu;p}(A_i) = C_{\mu;p}(A)$, then $f_i \to f$ strongly in $\mathscr{L}_{\mu;p}$. In particular this conclusion holds if $A_i \uparrow A$ or the $A_i$ are compact and $A_i \downarrow A$.*

PROOF. The $f_i$ form a bounded set in $\mathscr{L}_{\mu;p}$ so that a subsequence $\{f_{i_j}\}$ must converge weakly in $\mathscr{L}^+_{\mu;p}$ to some function $g$. From Theorem 5 $k(x,g\mu) \geqq 1$ on $A$ $C_{\mu;p}$-a.e. Therefore $g$ is a test function for $A$ in the variational problem (10); further from the weak convergence

$$\|g\|^p_{\mu;p} \leqq C_{\mu;p}(A) . \tag{11}$$

But then

(12) $$g = f \quad \text{and} \quad \|f_{i_j}\|_{\mu;p} \to \|f\|_{\mu;p} \quad \text{as } j \to \infty .$$

From (12) it follows that $f_{i_j} \to f$ strongly as $j \to \infty$. By a simple argument one concludes this is true for the full sequence.

## 3. The classical capacity $C_{k;1}$.

For the sake of completeness and comparison with the capacities $C_{k;\mu;p}$ we now summarize some known results concerning the classical capacity $C_{k;1}$. The conditions we give on the kernel are not the most general known but still contain most cases of interest.

Definition 9. If $A$ is an arbitrary point set, then

$$C_{k;1}(A) = C_1(A) = \inf \|\mu\|_1,$$

where the above infimum is taken over all $\mu \in \mathscr{L}_1^+$ such that

$$k(x,\mu) \geqq 1 \quad \text{on } A .$$

We call such a measure $\mu$ *a test measure for* $C_1(A)$.

*The natural analogues of Theorems* 1–4 *and their Corollaries are true for* $C_1$.

The proofs proceed as before except with minor changes; for example in Theorem 1 the $f_i$ are replaced by positive measures $\mu_i$ and $f$ is replaced by $\mu = \sum_{i=1}^{\infty} \mu_i$.

To carry the theory beyond this point it is apparently necessary to place additional restrictions on the kernel $k(x,y)$.

Definition 10. We say that a kernel $k(x,y) \in \mathscr{K}$ if

(i) $$k(x,y) = k_1(|x-y|) ,$$

where $k_1(r)$ is a positive, decreasing, continuous function of $r$ for $0 \leqq r < \infty$ which is finite for $0 < r < \infty$;

(ii) $$\lim_{r \to \infty} k_1(r) = 0 .$$

In place of Theorem 5 one has:

*If* $k \in \mathscr{K}$ *and* $\mu_i \to \mu$ *weakly in* $\mathscr{L}_1^+$ *with* $\|\mu_i\| \leqq M < \infty$, *then*

$$k(x,\mu) = \liminf k(x,\mu_i) \quad C_1\text{-a.e.}$$

The above result follows from the work in [9] and the fact that $k$ is *regular.*

Also:

*If $k \in \mathscr{K}$ then the natural analogues of Theorems* 6–8 *and their Corollaries are true for $C_1$.*

Definition 11. Let $A$ be an arbitrary set such that $C_1(A) < \infty$. Consider the variational problem

$$\min_\mu \|\mu\|_1 ,$$

where the above minimum is taken over all $\mu \in \mathscr{L}_1$ such that

$$k(x,\mu) \geqq 1 \quad \text{on } A \quad C_1\text{-a.e.}$$

We call a solution, $\mu$, of this problem a *$C_1$-capacitary distribution of $A$* and we call $k(x,\mu)$ a *$C_1$-capacitary potential of $A$.*

One can now easily show

*If $C_1(A) < \infty$, then $A$ has a $C_1$-capacitary distribution and every $C_1$-capacitary distribution of $A$ is in $\mathscr{L}_1{}^+$.*

## 4. The capacities $C_{\mu;p}$ as functions of $p$.

Here we propose to study the behavior of $C_{\mu;p}(A)$ as a function of $p$ for a fixed measure $\mu$, a fixed kernel $k$ and a fixed set $A$.

Definition 12. Let $\mathscr{L}_\mu$ be the vector space of all $\mu$-simple functions; by $\mu$-simple we mean that the function is $\mu$-measurable, vanishes except on a set of finite $\mu$-measure and assumes only finitely many values each of which is a finite real number. If $A$ is an arbitrary set we define

$$S_{\mu;p}(A) = \inf_s \|s\|_{\mu;p}^p ,$$

where the above infimum is taken over all $s$ in $\mathscr{L}_\mu{}^+$ such that

$$k(x,s\mu) \geqq 1 \quad \text{on } A .$$

Lemma 2. *$S_{\mu;p}(K) = C_{\mu;p}(K)$ for all $K$.*

Proof. It is obvious that $S_{\mu;p}(K) \geqq C_{\mu;p}(K)$, so we need only prove the opposite inequality. For this purpose we may assume that $C_{\mu;p}(K) < \infty$. Let $f$ be a test function for $C_{\mu;p}(K)$, let $0 < \varepsilon < 1$, and put $f_\varepsilon = (1-\varepsilon)^{-1}f$. There exists a sequence of functions in $\mathscr{L}_\mu{}^+$, $\{s_i\}$, such that

$$s_i \to f_\varepsilon \quad \text{strongly in } \mathscr{L}_{\mu;p}^+ . \tag{13}$$

I claim that

(14) $k(x, s_i\mu) \geqq 1$ on $K$ for $i$ sufficiently large .

For if (14) is false, then for arbitrarily large values of $i$ there exist points $x_i \in K$ where $k(x_i, s_i\mu) < 1$. The points $x_i$ have a limit point $x$ in $K$. From Lemma 1 we must then have $k(x, f_\varepsilon\mu) \leqq 1$ which is not possible. Therefore from (13) and (14)

$$S_{\mu;p} \leqq (1-\varepsilon)^{-p} C_{\mu;p}(K) ,$$

and the desired inequality follows immediately.

LEMMA 3.

(i) *$C_{\mu;p}(K)$ is an upper semi-continuous function of $p$.*

*If $0 < \|\mu\|_1 < \infty$, then*

(ii) *$(C_{\mu;p}(A)/\|\mu\|_1)^{1/p}$ is an increasing function of $p$*

*and*

(iii) *$C_{\mu;p}(K)$ is continuous from the right.*

PROOF. Consider the set of all functions $s \in \mathscr{L}_\mu{}^+$ such that

$$k(x, s\mu) \geqq 1 \quad \text{on } K .$$

If this set is empty, then by Lemma 2, $C_{\mu;p}(K) = \infty$ for all $p$ and there is nothing more to prove. If the set is not empty, then by Lemma 2, $C_{\mu;p}(K)$ is the infimum of continuous functions of $p$ and hence upper semi-continuous thus proving (i). The assertion (ii) is a consequence of the Hölder inequality applied to test functions for $C_{\mu;p}(A)$, and (iii) follows from (i) and (ii).

THEOREM 10. *Let $k(x,y)$ satisfy the condition that for every $K$ there exists $R$ such that*

$$\int_{\{|y| \geqq R\}} \sup_{x \in K} k(x,y)\, d\mu(y) < \infty \quad \textit{and} \quad \sup_{|y| \geqq R,\, x \in K} k(x,y) < \infty .$$

*Then $C_{\mu;p}(K)$ is upper semi-continuous and continuous from the right in $p$, $1 < p$.*

PROOF. In view of Lemma 2 we may assume that $C_{\mu;p}(K) < \infty$ for all $p$; for if even one were infinite then all would be and the result would hold. Now choose $1 < q < 2$ and restrict considerations to those values of $p$ such that

$$q \leqq p \leqq q' \quad \text{where } q^{-1} + q'^{-1} = 1 .$$

Since, according to Lemma 2, $C_{\mu;p}(K)$ is an upper semi-continuous function of $p$ it must attain a maximum value on the interval $q \leqq p \leqq q'$ and must thus be bounded. That is,

$$C_{\mu;p}(K) \leqq M < \infty \qquad \text{for } q \leqq p \leqq q' .$$

For each value of $p$ choose a test function, $f$, for $C_{\mu;p}(K)$ such that

$$\|f\|_{\mu;p}^p \leqq C_{\mu;p}(K) + \varepsilon ,$$

where $0 < \varepsilon < 1$. Now

$$(15) \quad \int_{\{|y| \geqq R\}} k(x,y)\, f(y)\, d\mu(y) \leqq \left( \int_{\{|y| \geqq R\}} |k(x,y)|^{p'} d\mu(y) \right)^{1/p'} \|f\|_{\mu;p}$$
$$\leqq \left( \int_{\{|y| \geqq R\}} k(x,y)\, d\mu(y) \right)^{1/p'} (\sup_{\{|y| \geqq R\}} k(x,y))^{1/p} \|f\|_{\mu;p} .$$

From our assumption it is then clear that for $R$ sufficiently large and independent of $p$,

$$\sup_{x \in K} \int_{\{|y| \geqq R\}} k(x,y)\, f(y)\, d\mu(y) < \varepsilon ,$$

and hence if we set $\nu = \mu \,|\, \{|x| < R\}$, then $(1-\varepsilon)^{-1} f$ is a test function for $C_{\nu;p}(K)$. Now

$$C_{\nu;p}(K) \geqq C_{\mu;p}(K) ,$$

since if $g$ is a test function for $C_{\nu;p}(K)$ and we extend it by setting $g(x) = 0$ for $|x| \geqq R$ it will be a test function for $C_{\mu;p}(K)$, while

$$\|g\|_{\mu;p} = \|g\|_{\nu;p} .$$

On the other hand we have

$$C_{\nu;p}(K) \leqq (1-\varepsilon)^{-p} \|f\|_{\mu;p}^p \leqq (1-\varepsilon)^{-q'} \left( C_{\mu;p}(K) + \varepsilon \right) .$$

We conclude that $C_{\nu;p}(K) \to C_{\mu;p}(K)$ uniformly on $q \leqq p \leqq q'$ as $R \to \infty$. Now according to Lemma 3, $C_{\nu;p}(K)$ is continuous from the right; therefore $C_{\mu;p}(K)$ is continuous from the right for $q \leqq p < q'$ and finally for $1 < p$.

REMARK. If we replace the assumption

$$\int_{\{|y| \geqq R\}} \sup_{x \in K} k(x,y)\, d\mu(y) < \infty$$

by

$$\int_{\{|y| \geqq R\}} (\sup_{x \in K} k(x,y))^q\, d\mu(y) < \infty ,$$

where $q$ is a fixed number, $1 \leqq q < \infty$, then we can prove by the same method that $C_{\mu;p}(K)$ is continuous from the right for $1<p<q'$. Examples, which we will not discuss here, show that $C_{\mu;p}(K)$ is not generally continuous from the left.

We now turn to the case $p=1$.

LEMMA 4. *$C_1(K)=S_1(K)=\inf_s\|s\cdot m\|_1$, where the infimum is taken over all $s$ in $\mathscr{L}_m{}^+$ such that $k(x,s\cdot m)\geqq 1$ on $K$.*

PROOF. The proof is similar to that of Lemma 2. Again we need only prove $S_1(K)\leqq C_1(K)$ in the case when the latter is finite. Then given $0<\varepsilon<1$ there must exist a test measure for $C_1(K),\mu$, such that

$$\|\mu\|_1 \leqq C_1(K) + \varepsilon .$$

Put $\mu_\varepsilon=(1-\varepsilon)^{-1}\mu$. There exists a sequence of continuous positive functions $\{f_i\}$, such that

$$f_i \to \mu_\varepsilon \text{ weakly in } \mathscr{L}_1 \quad \text{and} \quad \|f_i\|_1 \leqq \|\mu_\varepsilon\|_1 .$$

This can be proved by mollifying $\mu_\varepsilon$ with a positive continuous kernel $J=J(x)$ having compact support and $\int J(x)dx=1$. Then, for example,

$$f_i(x) = (i)^{-n}\int J(i(x-y))\, d\mu_\varepsilon(y) .$$

Since each $f_i$ can be approached strongly in $\mathscr{L}_1$ and from below by a sequence of positive functions in $\mathscr{L}_m{}^+$ we conclude that there exists a sequence of positive functions $\{s_i\}$ in $\mathscr{L}_m{}^+$ such that

$$s_i \to \mu_\varepsilon \text{ weakly in } \mathscr{L}_1 \quad \text{and} \quad \|s_i\|_1 \leqq \|\mu_\varepsilon\|_1 .$$

The remainder of the proof is the same as that of Lemma 2.

We now show that under quite general circumstances the classical capacity is the limit of our capacities. We put $C_{m;p}=C_p$.

THEOREM 11. *Let $k(x,y)$ satisfy the condition that for every $K$ there exists $R$ such that*

$$\int_{\{|y|\geqq R\}} (\sup_{x\in K} k(x,y))^q\, dy < \infty ,$$

*where $q$ is a number, $1\leqq q<\infty$. Further let $k(x,y)$ satisfy the condition*

$$\lim_{|y|\to\infty} \sup_{x\in K} k(x,y) = 0 .$$

*Then $C_p(K)$ is continuous from the right at $p=1$.*

PROOF. We first need a replacement for (i) in Lemma 3. Just as before we can show that $C_p(K)$ *is an upper semi-continuous function of* $p$ *for* $1 \leqq p$. Now proceed as in the proof of Theorem 10 but restrict $p$ to $1 \leqq p < q'$ and in place of the test functions $f$ use test functions $s$ in $\mathscr{L}_m{}^+$. Replace (15) by

$$\int_{\{|y| \geqq R\}} k(x,y)s(y)\,dy \leqq \left(\int_{\{|y| \geqq R\}} (k(x,y))^q\,dy\right)^{1/p'} (\sup_{\{|y| \geqq R\}} k(x,y))^{1-q/p'} \|s\|_p$$

for $1 < p$ and

$$\int_{\{|y| \geqq R\}} k(x,y)s(y)\,dy \leqq \|s\|_1 \sup_{\{|y| \geqq R\}} k(x,y)$$

for $1 = p$. Put $\nu = m \,|\, \{|x| < R\}$. Again we consider the function $C_{\nu;p}(K)$ and for $p = 1$ we take

$$\inf_s \|s\nu\|_1 ,$$

the infimum being over all $s$ in $\mathscr{L}_m{}^+$ such that

$$k(x, s\nu) \geqq 1 \quad \text{on } K .$$

The resultant function of $p$ is continuous from the right for $1 \leqq p$. The proof is completed by showing that these functions tend uniformly to $C_p(K)$ in a right neighborhood of 1.

REMARKS. If $k \in \mathscr{K}$ then open sets are $C_1$-capacitable and Theorem 11 yields the following characterization of $C_1(A)$:

$$C_1(A) = \inf_{G \supset A} \sup_{K \subset G} \lim_{p \to 1+} C_p(K) .$$

Simple examples show that the kernel $k$ must decrease to zero at infinity with sufficient rapidity in order for Theorem 11 to hold.

## 5. The capacities $c_{\mu;p}$.

In this section we introduce new capacities which are based on measures as test elements and we develop some of their simpler properties.

DEFINITION 13. Let $\mu \in \mathscr{M}^+$, $1 < p < \infty$ and $A \in \mathscr{F}_1$. Here $\mathscr{F}_1$ denotes the $\sigma$-field of sets which are $\nu$-measurable for *all* measures $\nu$ in $\mathscr{L}_1{}^+$. We then define

$$c_{\mu;p}(A) = \sup \|\nu\|_1 ,$$

where the above supremum is taken over all measures $\nu$ in $\mathscr{L}_1{}^+$ such that $\nu$ is concentrated on $A$ and

$$\|k(\nu, y)\|_{\mu;p'} \leqq 1 .$$

We will call such a $\nu$ a *test measure for* $c_{\mu;p}(A)$.

From Definition 13 one can see that for $A \in \mathscr{F}_1$

$$c_{\mu;p}(A) = (\inf_\nu \|k(\nu,y)\|_{\mu;p'})^{-1} = (\inf_\nu \sup_f k(\nu,f\mu))^{-1},$$

where $\nu \in \mathscr{L}_1^+$, is concentrated on $A$ and $\|\nu\|_1 \geqq 1$ while $f \in \mathscr{L}^+_{\mu;p}$ and $\|f\|_{\mu;p} \leqq 1$. Without harm we may restrict either $\|\nu\|_1$ or $\|f\|_{\mu;p}$ to equal one.

THEOREM 12. *$c_{\mu;p}$ is an inner capacity on $\mathscr{F}_1$.*

PROOF. Clearly $c_{\mu;p}$ has properties (i) and (ii) of Definition 1. To prove that it has property (iii) let $A = \bigcup_{i=1}^\infty A_i$ and without loss of generality assume that the sets $A_i$ are disjoint. Let $\nu$ be a test measure for $c_{\mu;p}(A)$ and set $\nu_i = \nu|A_i$; then $\nu = \sum_{i=1}^\infty \nu_i$ and $\nu_i$ is clearly a test measure for $c_{\mu;p}(A_i)$. Therefore

$$\|\nu\|_1 = \sum_{i=1}^\infty \|\nu_i\|_1 \leqq \sum_{i=1}^\infty c_{\mu;p}(A_i)$$

and (iii) follows immediately. We now show that $c_{\mu;p}$ is an inner capacity. To this purpose let $A \in \mathscr{F}_1$ and let $\nu$ be a test measure for $c_{\mu;p}(A)$. If $K \subset A$, then $\nu|K$ is a test measure for $c_{\mu;p}(K)$ and

$$\sup_{K \subset A} \|\nu|K\|_1 = \|\nu\|_1 .$$

Therefore we must have

$$\sup_{K \subset A} c_{\mu;p}(K) \geqq c_{\mu;p}(A)$$

which implies that $c_{\mu;p}$ is an inner capacity.

We can form the corresponding outer capacity $c^*_{\mu;p}$.

DEFINITION 14. Let $A \in \mathscr{F}_1$; then $\nu$ is called a $c_{\mu;p}$-capacitary distribution, and $k(\nu,y)$ is called a $c_{\mu;p}$-capacitary potential for $A$ if $\nu$ is a test measure for $c_{\mu;p}(A)$ and

$$\|\nu\|_1 = c_{\mu;p}(A) .$$

THEOREM 13. *If $A$ is a closed set such that $c_{\mu;p}(A) < \infty$ and*

$$\lim_{r\to\infty} c_{\mu;p}(A_r) = 0 \quad \textit{where} \quad A_r = A \cap \{x \mid |x| > r\},$$

*then $A$ has a $c_{\mu;p}$-capacitary distribution. In particular, if $c_{\mu;p}(K) < \infty$ then $K$ has a $c_{\mu;p}$-capacitary distribution.*

PROOF. Let $K_i \uparrow A$ as $i \to \infty$ and let $\nu_i$ be test measures for $c_{\mu;p}(K_i)$ such that

$$\|\nu_i\|_1 \uparrow c_{\mu;p}(A) \quad \text{as } i \to \infty .$$

The existence of such $\nu_i$ follows from Theorem 12. A subsequence $\{\nu_{i_j}\}$ then converges weakly in $\mathscr{L}_1{}^+$ to a test measure for $c_{\mu;p}(A), \nu$. Now $\nu_{i_j}|A_r$ is obviously a test measure for $A_r$; therefore

$$\|\nu_{i_j}|A_r\|_1 \leqq c_{\mu;p}(A_R) \leqq \varepsilon \qquad \text{for } r \geqq R\,.$$

Thus

$$\|\nu_{i_j}|A-A_r\|_1 \geqq \|\nu_{i_j}\|_1 - \varepsilon \qquad \text{for } r \geqq R\,,$$

and we then infer

$$\|\nu\|_1 \geqq \|\nu|A - A_r\|_1 \geqq c_{\mu;p}(A) - \varepsilon$$

from which the desired result follows.

Definition 15. Let $A \in \mathscr{F}_1$. We define

$$c_1(A) = \sup \|\nu\|_1\,,$$

where the above supremum is taken over all measures $\nu$ in $\mathscr{L}_1{}^+$ for which

$$\sup_{y \in E^n} k(\nu,y) \leqq 1\,.$$

We call such a $\nu$ a test measure for $c_1(A)$. Clearly $c_1(A)$ may also be defined by

$$c_1(A) = \left(\inf_\nu \sup_y k(\nu,y)\right)^{-1},$$

where the infimum is taken over all measures $\nu \in \mathscr{L}_1{}^+$ such that $\nu$ is concentrated on $A$ and $\|\nu\|_1 \geqq 1$.

Our theorems concerning $c_{\mu;p}$ will also hold for $c_1$; the proofs are the same.

## 6. The relation between $C_{\mu;p}$ and $c_{\mu;p}$ .

We will show in Theorem 14 that the capacities $C_{\mu;p}$ and $c_{\mu;p}$ are very closely related. The method of proof is due to Fuglede in [8], where he has shown, among other things, that $C_1(K)=c_1(K)$, and in [10] where he has extended his previous results to capacities of the type considered here and proved a result overlapping Theorem 14.

Theorem 14.

(i) $c^*_{\mu;p}(A)=(C_{\mu;p}(A))^{1/p}$ *for all sets* $A$.

(ii) $c_{\mu;p}(A)=(C_{\mu;p}(A))^{1/p}$ *for all analytic sets* $A$.

Proof. Following Definition 6 we have seen that

$$(C_{\mu;p}(K))^{-1/p} = \sup_{\|f\|_{\mu;p}\leqq 1} \inf_{x\in K} k(x,f\mu), \quad f \in \mathscr{L}^+_{\mu;p}\,.$$

Hence

$$(C_{\mu;p}(K))^{-1/p} = \sup_{\|f\|_{\mu;p}\leq 1} \inf_{\|\nu\|_1=1} k(\nu, f\mu)$$

where $f \in \mathscr{L}^+_{\mu;p}$ and $\nu \in \mathscr{L}_1{}^+$ and is concentrated on $K$. Following Definition 13 we have seen that

$$(c_{\mu;p}(K))^{-1} = \inf_{\|\nu\|_1=1} \sup_{\|f\|_{\mu;p}\leq 1} k(\nu, f\mu) ,$$

where again $f \in \mathscr{L}^+_{\mu;p}$ and $\nu \in \mathscr{L}_1{}^+$ and is concentrated on $K$. In the weak topologies of $\mathscr{L}_1$ and $\mathscr{L}_{\mu;p}$ the sets over which $\nu$ and $f$ respectively vary are compact and convex. The following extension of von Neumann's Minimax Theorem is known; see [6] where an even more general result is derived.

*Let $V_1$ be a real linear space and $V_2$ a Hausdorff real linear topological space. Let $H \subset V_1$, $K \subset V_2$ where $H$ is non-empty, convex and $K$ is non-empty, convex and compact. Let $\Phi = \Phi(x_1, x_2)$ be a real valued function on $H \times K$, $\Phi(x_1, x_2) > -\infty$, which is concave in $x_1$ for each fixed $x_2$ and convex and lower semi-continuous in $x_2$ for each fixed $x_1$. Then*

$$\sup_{x_1} \inf_{x_2} \Phi(x_1, x_2) = \inf_{x_2} \sup_{x_1} \Phi(x_1, x_2) .$$

We conclude that

$$c_{\mu;p}(K) = (C_{\mu;p}(K))^{1/p} .$$

From Theorem 12 and Theorem 8 we further conclude

$$c^*_{\mu;p}(G) = (C_{\mu;p}(G))^{1/p},$$

and since $c^*_{\mu;p}$ and $C_{\mu;p}$ are both outer capacities, (i) follows.

From (i) and Theorem 8 we infer that analytic sets are $c^*_{\mu;p}$-capacitable. Since analytic sets are in $\mathscr{F}_1$ this observation and Theorem 12 imply that

$$c_{\mu;p}(A) = c^*_{\mu;p}(A), \quad A \text{ analytic.}$$

(ii) now follows.

REMARK. In the case $p=1$ it is true that $c_1(K) = C_1(K)$. However the equality $c_1{}^*(A) = C_1(A)$ apparently requires additional restrictions on the kernel $k(x,y)$; for example $k \in \mathscr{K}$ is sufficient. For further information see [9] where this problem is discussed in great generality and detail.

In particular Theorem 14 implies that $c^*_{\mu;p}$ and $C_{\mu;p}$ have the same null sets. These sets have already been characterized in several ways. It also implies that sets of infinite capacity are the same in both cases.

We now give a simple characterization of compact sets of infinite capacity.

THEOREM 15. *The following statements are equivalent.*

(i) $C_{\mu;p}(K) = +\infty$.

(ii) $c_{\mu;p}(K) = +\infty$.

(iii) *There exists a point* $x' \in K$ *such that* $k(x',y)=0$ $\mu$-a.e.

PROOF. From Theorem 14 we know that (i) and (ii) are equivalent. Hence it suffices to prove the equivalence of (i) and (iii). Suppose that (i) holds. If we take a covering of $K$ by a finite number of cubes then the intersection of $K$ with at least one of the cubes must have infinite $C_{\mu;p}$-capacity. By a familiar nesting procedure we can then find a point $x' \in K$ and a sequence of closed cubes $\{\Gamma_i\}$ where

$$\Gamma_i \downarrow \{x'\} \quad \text{as } i \to \infty .$$

But then it follows from Theorem 7 that $C_{\mu;p}(x')=\infty$. Since there can be no test functions for $C_{\mu;p}(x')$ we must have

$$k(x', \mu|A) = 0$$

for every $\mu$-measurable set $A$ and we get (iii). On the other hand if (iii) holds, it is obvious that no test function can exist for $C_{\mu;p}(x')$ so that $C_{\mu;p}(x')=\infty$, and hence $C_{\mu;p}(K)=\infty$.

LEMMA 5. *If* $\nu \in \mathscr{M}^+$ *and* $k(\nu,y) \in \mathscr{L}_{\mu;p'}$, *then for all sets* $A$ *we have*

$$\nu^*(A) \leqq \|k(\nu,y)\|_{\mu;p'}\, c^*_{\mu;p}(A) .$$

PROOF. Let $C_{\mu;p}(G) < \infty$ and let $f$ be a test function for $C_{\mu;p}(G)$. Then

$$\int k(\nu|G,y)f(y)\,d\mu(y) = \int k(x,f\mu)\,d(\nu|G)(x) .$$

Since $k(x,f\mu) \geqq 1$ on $G$ we have from Hölder's Inequality

$$\nu(G) \leqq \|k(\nu,y)\|_{\mu;p'}\,\|f\|_{\mu;p} ,$$

from which we can easily infer the desired result.

The following proposition establishes the connection between the two capacitary distributions, the most important of the results being the potential representation of the $C_{\mu;p}$-capacitary distribution.

THEOREM 16. *Let $A$ be an analytic set. Then*

(i) $(c_{\mu;p}(A))^{-1} = \inf_\gamma \sup_g k(\gamma, g\mu) = \sup_g \inf_\gamma k(\gamma, g\mu)$, *where* $g \in \mathscr{L}^+_{\mu;p}$ *with* $\|g\|_{\mu;p} = 1$ *and* $\gamma \in \mathscr{L}_1{}^+$, *is concentrated on* $A$, $\|\gamma\|_1 = 1$ *and* $k(\gamma, y) \in \mathscr{L}_{\mu;p'}$.

(ii) *$A$ has a $c_{\mu;p}$-capacitary distribution different from zero if and only if the functional $k(\gamma, g\mu)$, where $\gamma$ and $g$ vary over the sets defined in* (i), *has a 'saddle point' $(\gamma', g')$ (that is, $k(\gamma', g'\mu) \leqq k(\gamma, g'\mu)$ and $k(\gamma', g'\mu) \geqq k(\gamma', g\mu)$), where $k(\gamma', g'\mu) > 0$. In this case*

$$(c_{\mu;p}(A))^{-1} = k(\gamma', g'\mu), \qquad \nu = c_{\mu;p}(A)\,\gamma', \quad f = c_{\mu;p}(A)\,g',$$

*where $\nu$ and $f$ are the respective capacitary distributions.*

(iii) *If $A$ has capacitary distributions $\nu$ and $f$, then*

$$(f(y))^{p-1} = (c_{\mu;p}(A))^{p-1}\,k(\nu, y) \qquad \mu\text{-a.e.}$$

*Furthermore $\nu$ is concentrated on the set $B = \{x \mid k(x, f\mu) = 1\} \cap A$ and $c_{\mu;p}(B) = c_{\mu;p}(A)$.*

PROOF. By Theorem 15, $c_{\mu;p}(A) = (C_{\mu;p}(A))^{1/p}$. Hence

$$(c_{\mu;p}(A))^{-1} = \inf_\gamma \sup_g k(\gamma, g\mu) = \sup_g \inf_\gamma k(\gamma, g\mu)\,,$$

where $g \in \mathscr{L}^+_{\mu;p}$, $\|g\|_{\mu;p} = 1$, $\gamma \in \mathscr{L}_1{}^+$, is concentrated on $A$ and $\|\gamma\|_1 = 1$. If $c_{\mu;p}(A) > 0$, then $\inf_\gamma \sup_g k(\gamma, g\mu) < \infty$, and it thus does no harm to demand that $k(\gamma, y) \in \mathscr{L}_{\mu;p'}$. If $c_{\mu;p}(A) = 0$, then $k(\gamma, y) \notin \mathscr{L}_{\mu;p'}$ if $\gamma \neq 0$ and again (i) must hold.

We now turn to the proof of (ii). Suppose $A$ has non-zero capacitary distributions $\nu$ and $f$. Put $\gamma' = (c_{\mu;p}(A))^{-1}\nu$ and $g' = (c_{\mu;p}(A))^{-1}f$. From Lemma 5 it follows that $k(\gamma, g'\mu) \geqq (c_{\mu;p}(A))^{-1}$. On the other hand

$$k(\gamma', g\mu) \leqq \|k(\gamma', y)\|_{\mu;p'} = (c_{\mu;p}(A))^{-1}\,.$$

Hence $(\gamma', g')$ is a saddle point and $k(\gamma', g'\mu) = (c_{\mu;p}(A))^{-1}$. We now prove the converse. Since $(\gamma', g')$ is a saddle point it is easily shown from (i) that $k(\gamma', g'\mu) = (c_{\mu;p}(A))^{-1}$. Hence $k(\gamma, g'\mu) \geqq (c_{\mu;p}(A))^{-1}$. From this it follows that

$$k(x, g'\mu) \geqq (c_{\mu;p}(A))^{-1} \text{ on } A \qquad C_{\mu;p}\text{-a.e.};$$

for if this is not true then $k(x, g'\mu) < (c_{\mu;p}(A))^{-1}$ on some compact subset $K$, of non-zero capacity. We may then take $\gamma$ equal to $(c_{\mu;p}(K))^{-1} \times$ (a $c_{\mu;p}$-capacitary distribution of $K$) and derive a contradiction. It now follows easily that $f = c_{\mu;p}(A)\,g'$. On the other hand

$$k(\gamma', g\mu) \leqq k(\gamma', g'\mu) = (c_{\mu;p}(A))^{-1}\,.$$

Hence $\|k(\gamma',y)\|_{\mu;p'} \leqq (c_{\mu;p}(A))^{-1}$ from which we have $\nu = \gamma' c_{\mu;p}(A)$, completing the proof of (ii).

If $c_{\mu;p}(A) = 0$, then (iii) obviously holds so we consider the case $c_{\mu;p}(A) > 0$. From (ii)

$$k(\nu, f\mu) = c_{\mu;p}(A) = \|f\|_{\mu;p} . \tag{17}$$

Since $\|k(\nu,y)\|_{\mu;p'} = 1$ we must have

$$k(\nu,y) = (f(y)/\|f\|_{\mu;p})^{p-1} \quad \mu\text{-a.e.}$$

which gives the potential representation. Further, since $\nu$ is concentrated on the set where $k(x,f\mu) \geqq 1$ and has total variation $c_{\mu;p}(A)$, it follows from (17) that $\nu$ is concentrated on $B$. Since $B \subset A$ we then have $c_{\mu;p}(B) = c_{\mu;p}(A)$.

Our next task is to extend the potential representation to all capacitary distributions, $f$.

Corollary. *Let $\mathscr{D}$ be a dense subset of $\mathscr{L}^+_{\mu;p}$ and let $k(x,y)$ be such that $\varphi \in \mathscr{D}$ implies $k(x,\varphi\mu)$ is continuous and tends to zero at infinity. Under this condition, if $0 < C_{\mu;p}(A) < \infty$ and $A$ has the $C_{\mu;p}$-capacity distribution $f$, then there exists $\nu \in \mathscr{L}_1{}^+$ such that $\|\nu\|_1 = c^*_{\mu;p}(A)$,* supp $\nu \subset \bar{A}$ *and*

$$(f(y))^{p-1} = (c^*_{\mu;p}(A))^{p-1}\, k(\nu,y) \quad \mu\text{-a.e.}$$

Proof. There exists a $G_\delta$-set $H$ where $A \subset H \subset \bar{A}$ and $C_{\mu;p}(H) = C_{\mu;p}(A)$. In turn there exists a $K_\sigma$-set, $S$, where $S \subset H$ and $C_{\mu;p}(S) = C_{\mu;p}(H)$. It is easy to see that $f$ is the capacitary distribution of $S$. Let $K_i \uparrow S$ and let $\{\nu_i\}$ be a sequence of $c_{\mu;p}$-capacitary distributions of the $K_i$. Further let $f_i$ be the $C_{\mu;p}$-capacitary distribution of $K_i$. Then

$$(f_i(y))^{p-1} = (c_{\mu;p}(K_i))^{p-1}\, k(\nu_i,y) \quad \mu\text{-a.e.}$$

Since the $\nu_i$ form a bounded sequence in $\mathscr{L}_1$, a subsequence converges weakly in $\mathscr{L}_1$ to a measure $\nu$. Without loss of generality we assume it is the full sequence. Then, if $\varphi \in \mathscr{D}$,

$$\int k(x,\varphi\mu)\, d\nu_i(x) \to \int k(x,\varphi\mu)\, d\nu(x).$$

Thus

$$\int k(\nu_i,y)\, \varphi(y)\, d\mu(y) \to \int k(\nu,y)\, \varphi(y)\, d\mu(y).$$

On the other hand $f_i \to f$ strongly in $\mathscr{L}_{\mu;p}$ and hence $f_i^{p-1} \to f^{p-1}$ weakly in $\mathscr{L}_{\mu;p'}$. Also $c_{\mu;p}(K_i) \uparrow c^*_{\mu;p}(A)$. It then follows that

$$\int (f(y))^{p-1} \varphi(y)\, d\mu(y) = (c^*_{\mu;p}(A))^{p-1} \int k(\nu,y)\, \varphi(y)\, d\mu(y)$$

which implies the representation. That $\operatorname{supp}\nu \subset \bar{A}$ is obvious, so it remains only to evaluate $\|\nu\|_1$. To this end let $\bar{x} \in \operatorname{supp}\nu$. Since $\nu_i$ converges to $\nu$ weakly, there must exist a sequence of points $x_i$, where $x_i \in \operatorname{supp}\nu_i$ and $x_i \to \bar{x}$ as $i \to \infty$. From Theorem 16, $k(x_i, f_i\mu) \leqq 1$, and hence Lemma 1 implies $k(\bar{x}, f\mu) \leqq 1$ or

$$k(x,f\mu) \leqq 1 \quad \text{everywhere on } \operatorname{supp}\nu .$$

From the representation formula we conclude $\|\nu\|_1 \geqq c^*_{\mu;p}(A)$; but from the weak convergence the opposite inequality also holds and hence equality.

REMARKS. The kernel $k$ will satisfy the conditions of the above Corollary if for each $K$ and $x$,

$$\int_K k(x,y)\, d\mu(y) < \infty \text{ and tends to 0 as } |x| \to \infty$$

and

$$\lim_{z\to x} \int_K |k(z,y) - k(x,y)|\, d\mu(y) = 0 .$$

If $A$ is a bounded set, then we can dispense with the condition

$$\lim_{|x|\to\infty} \int_K k(x,y)\, d\mu(y) = 0 ,$$

and in the general statement, with the condition $\lim_{|x|\to\infty} k(x,\varphi\mu) = 0$.

For kernels which satisfy the conditions of the above Corollary and the additional condition

$$(18) \qquad k(\nu,y) = 0 \quad \mu\text{-a.e.}, \ \nu \in \mathscr{L}_1, \quad \text{implies} \quad \nu = 0$$

this gives a simple and natural extension of the concept of $c_{\mu;p}$-capacitary distribution. Given any set $A$ with $0 \leqq c^*_{\mu;p}(A) < \infty$ we can define its capacitary distribution to be the *unique* measure $\nu$ such that

$$(f(y))^{p-1} = (c^*_{\mu;p}(A))^{p-1}\, k(\nu,y) \qquad \mu\text{-a.e.}$$

In the case of analytic sets this agrees with the $c_{\mu;p}$-capacitary distribution if the latter exists. (18) is satisfied in many cases of interest. If $k$ satisfies (18) and the conditions of the Corollary, we get the following convergence result for the capacitary distributions $\nu$.

*Let $A_i$, $i = 1,2,\ldots$, and $A$ be sets with capacitary distributions $\nu_i$ and $\nu$ respectively. If $A \subset \liminf A_i$ and $\lim c^*_{\mu;p}(A_i) = c^*_{\mu;p}(A)$, then $\nu_i \to \nu$ weakly* *in* $\mathscr{L}_1$. *In particular this conclusion holds if* $A_i \uparrow A$ *or the* $A_i$ *are compact and* $A_i \downarrow A$.

## 7. Bessel potentials.

We now turn to the potentials which are of principal interest in this paper. As references for the following formulas see [2] and [13].

The kernels $k(x,y) = g_\alpha(x-y)$ ($g_\alpha$ also depends on $n$) where

$$(19)\qquad g_\alpha(x) = 2^{-\frac{1}{2}(n+\alpha-2)}\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}\alpha)^{-1}|x|^{\frac{1}{2}(\alpha-n)}K_{\frac{1}{2}(n-\alpha)}(|x|), \qquad \alpha > 0 .$$

$\Gamma$ stands for the Gamma function and $K_\nu$ stands for the modified Bessel function of the third kind of order $\nu$, where $g_\alpha(x)$ is an analytic function of $x$ except at $x=0$ and $g_\alpha(x) > 0$. Obviously $g_\alpha$ is in $\mathscr{L}_1$ and has a Fourier transform given by

$$(20)\qquad \hat{g}_\alpha(\xi) = (2\pi)^{-\frac{1}{2}n}(1+|\xi|^2)^{-\frac{1}{2}\alpha},$$

where

$$\hat{\varphi}(\xi) = (2\pi)^{-\frac{1}{2}n}\int \varphi(x)\, e^{(-i\xi x)}\, dx \quad \text{for } \varphi \in \mathscr{L}_1 .$$

From (20) it is obvious that

$$(21)\qquad g_\alpha * g_\beta = g_{\alpha+\beta} .$$

The behaviors of $g_\alpha$ at $x=0$ and $x=\infty$ are of particular interest. We have as $x \to 0$

$$(22)\qquad g_\alpha(x) = 2^{-\alpha}\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}(n-\alpha))\Gamma(\tfrac{1}{2}\alpha)^{-1}|x|^{\alpha-n} + o(|x|^{\alpha-n}), \qquad 0<\alpha<n;$$

$$(23)\qquad g_n(x) = 2^{1-n}\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)^{-1}\log|x|^{-1} + O(1);$$

(24) $g_\alpha(x)$ is finite and continuous at $x=0$, $\alpha > n$.

As $x \to \infty$ we have

$$(25)\qquad g_\alpha(x) \sim 2^{-\frac{1}{2}(n+\alpha-1)}\pi^{-\frac{1}{2}(n-1)}\Gamma(\tfrac{1}{2}\alpha)^{-1}|x|^{\frac{1}{2}(\alpha-n-1)}e^{-|x|} .$$

Letting $r = |x|$, we have

$$\frac{\partial g_\alpha}{\partial r}(x) = -(2)^{-\frac{1}{2}(n+\alpha-2)}\pi^{-\frac{1}{2}n}\Gamma(\tfrac{1}{2}\alpha)^{-1}|x|^{\frac{1}{2}(\alpha-n)}K_{\frac{1}{2}(n-\alpha+2)}(|x|) < 0$$

for $r > 0$.

In general the asymptotic behavior of the derivatives of $g_\alpha$ is given by the crude but adequate estimate

$$\left|\left(\frac{\partial}{\partial r}\right)^q g_\alpha(x)\right| \leqq \text{const}\, e^{-\frac{1}{2}|x|} \qquad \text{for } |x| > 1 , \tag{26}$$

where the *constant* depends on $q$ and $\alpha$.

Let $\mathscr{S}$ be the space of rapidly decreasing infinitely continuously differentiable functions and $\mathscr{S}'$ the space of tempered distributions. If $T \in \mathscr{S}'$, then

$$\widehat{g_\alpha * T} = (2\pi)^{\frac{1}{2}n}\, \hat{g}_\alpha \cdot \hat{T}$$

from which it follows that $g_\alpha$, under the convolution, maps $\mathscr{S}$ and $\mathscr{S}'$ onto themselves in a one-to-one manner.

DEFINITION 16. We now introduce the capacities corresponding to the kernels $g_\alpha$. We set

$$B_{\alpha;p} = C_{g_\alpha;m;p} \quad \text{and} \quad b_{\alpha;p} = c_{g_\alpha;m;p}, \qquad 1 < p < \infty;$$

we set

$$B_{\alpha;1} = C_{g_\alpha;1} \quad \text{and} \quad b_{\alpha;1} = c_{g_\alpha;1} .$$

THEOREM 17. *The kernels $g_\alpha$ satisfy all the assumptions concerning kernels made in Sections* 1–6 *and therefore all the previous results apply to their potentials and their corresponding capacities $B_{\alpha;p}$ and $b_{\alpha;p}$, $1 \leqq p < \infty$.*

THEOREM 18. *If $f \in \mathscr{L}_p$, $1 < p$, then given $\varepsilon > 0$ there exists $G$ such tha $B_{\alpha;p}(G) < \varepsilon$ and the restriction of $g_\alpha * f$ to $E^n - G$ is continuous.*

PROOF. There exists a sequence $\{\psi_i\}$ such that $\psi_i \in \mathscr{S}$ and $\psi_i \to f$ strongly in $\mathscr{L}_p$. Since $g_\alpha * \psi_i$ is a continuous function the rest follows from (iii) of Theorem 4.

We now prove a regularity theorem for capacitary distributions and capacitary potentials.

THEOREM 19. *If $1 < p$ and $0 < B_{\alpha;p}(A)$, then*

(i) *the capacitary distribution $f$ is a strictly positive, lower semi-continuous function,*

(ii) *$f$ is finite $B_{\alpha;1}$*-a.e.,

(iii) *$f$ is a $C^\infty$-function in $E^n - \bar{A}$ and has an exponential order of decrease at infinity if $A$ is bounded,*

(iv) *if $\alpha \leqq n$, then $f \in \mathscr{L}_q$ for all $q$,*

$$\max(1, p-1) \leqq q < \max(p, (p-1)n(n-\alpha)^{-1})$$

*and we can replace* $\max(1, p-1)$ *by* 1 *if $A$ is bounded.*

*If $\alpha > n$, then $f \in \mathscr{L}_q$, where $q = \max(1, p-1)$ and we may take $q = 1$ if $A$ is bounded; furthermore $f$ is continuous and tends to zero at infinity.*

Proof. The basis of the proof is the Corollary to Theorem 16 from which

$$(f(x))^{p-1} = g_\alpha * \nu(x) , \tag{27}$$

where $\nu \in \mathscr{L}_1{}^+$ with

$$\operatorname{supp}\nu \subset \bar{A} \quad \text{and} \quad \|\nu\|_1 = B_{\alpha;p}(A) .$$

Since $g_\alpha$ is strictly positive $f$ is strictly positive and since $g_\alpha * \nu$ is lower semi-continuous so is $f$. From Section 3 we have $g_\alpha * \nu$ finite $B_{\alpha;1}$-a.e. and hence the same is true for $f$. This completes (i) and (ii). If $x \in E^n - \bar{A}$ or more generally $x \in E^n - \operatorname{supp}\nu$, it follows from the continuity of $(\partial/\partial x)^\beta g_\alpha(x)$ for $x \neq 0$ and the estimate (26) that

$$(\partial/\partial x)^\beta (g_\alpha * \nu)(x) = \big((\partial/\partial x)^\beta g_\alpha\big) * \nu(x) .$$

Continuity of the right side will now follow from the same facts. Thus $g_\alpha * \nu$ is a $C^\infty$-function on $E^n - \bar{A}$ and $f$ must also be since it is strictly positive. To complete the proof of (iii) we note that if $A$ is a bounded set, then (25) implies

$$f(x) \leqq \text{const}\, e^{-\frac{1}{2}(p-1)^{-1}|x|} \qquad \text{for } |x| \text{ sufficiently large.}$$

If $\alpha \leqq n$, then Young's Inequality shows

$$g_\alpha * \nu \in \mathscr{L}_r, \qquad 1 \leqq r < n(n-\alpha)^{-1} . \tag{28}$$

From (28), $f \in \mathscr{L}_{p-1}$ if $p \geqq 2$; on the other hand if $1 < p < 2$, since $f \in \mathscr{L}_p$ and from (28), $f^{p-1} \in \mathscr{L}_1$, Hölder's Inequality gives $f \in \mathscr{L}_1$. The upper bound on $q$ follows immediately from $f \in \mathscr{L}_p$ and (28). *Note that the upper bound gives us new information only in the case $\alpha p > n$.* Finally if $A$ is bounded, we easily see from (iii) and $f \in \mathscr{L}_p$ that $f \in \mathscr{L}_1$.

If $\alpha > n$, then $g_\alpha * \nu = g_{\alpha-\varepsilon} * (g_\varepsilon * \nu)$ for $0 < \varepsilon < \alpha - n$. $g_\varepsilon * \nu \in \mathscr{L}_1$ while $g_{\alpha-\varepsilon}$ is a bounded (continuous) function. It is well known that in such a case the convolution is continuous and tends to zero at infinity. Therefore $f$ has the same properties. That $f \in \mathscr{L}_q$ follows just as before, which completes the proof.

REMARKS. Theorem 19 can be strengthened in several respects. Firstly $f$ is actually an analytic function on $E^n - \bar{A}$. Secondly in the case $\alpha > n$, $f$ is Hölder continuous and if $\alpha > n+k$, $k = 0, 1, 2, \ldots$, $f$ will have Hölder continuous derivatives up to the order $k$. All the results in Theorem 16 can be used to improve the regularity of the capacitary potential $g_\alpha * f$. Thus from (iii), $g_\alpha * f$ is a $C^\infty$-function (in fact analytic) on $E^n - \bar{A}$ and has exponential order of decrease at infinity if $A$ is bounded. For example the latter statement is a consequence of

$$g_\alpha * f(x) \leqq \sup_{\{|y| < \frac{1}{2}|x|\}} g_\alpha(x-y) \int f(y)\,dy + \sup_{\{|y| > \frac{1}{2}|x|\}} f(y) \int g_\alpha(y)\,dy\,.$$

The increased smoothness in $f$ which we have from (iv) will be reflected in increased smoothness of $g_\alpha * f$ but we will not go into details.

It is worth noting that when $\alpha p \leqq n$ it is not generally possible to increase the Lebesgue exponent of $f$ beyond $p$. For if $f \in \mathscr{L}_q$, $q > p$, then $g_\alpha * \nu \in \mathscr{L}_{q/p-1}$. But then for $A$ closed this implies $B_{\alpha;(q/p-1)'}(A) > 0$. Since $(q/p-1)' < p$, later considerations will show this is not always possible.

## 8. The relation between measure and capacity.

The rest of the paper is devoted to working out the relation between Hausdorff measure and capacity. We begin with the case of $n$-dimensional or Lebesgue measure. This case is only of passing interest but the relations take a particularly simple form.

LEMMA 6. *$B_{\alpha;p}(A)$ and $b_{\alpha;p}(A)$ are invariant under orthogonal transformations and translations of the set $A$.*

This is obvious.

THEOREM 20. *Let $1 < p$. The following relations hold:*

(i) *If $\alpha p < n$, then $B_{\alpha;p}(A) \geqq \varkappa (m^*(A))^{(n-\alpha p)n^{-1}}$.*

(ii) *If $\alpha p = n$, then $B_{\alpha;p}(A) \geqq \varkappa (m^*(A))^{\varepsilon}$ for $0 < \varepsilon \leqq 1$.*

(iii) *If $\alpha p > n$, then $B_{\alpha;p}(A) \geqq \varkappa > 0$ for $A \neq \emptyset$.*

*In each case $\varkappa$ is a strictly positive constant independent of the set $A$ but dependent on the numerical parameters present.*

PROOF. It is easy to show that we need prove (i) only in case $A$ is a bounded, open, non-empty set. In this case it is obvious that $B_{\alpha;p}(A) < \infty$. Let $f$ be a test function for $B_{\alpha;p}(A)$; then

$$m(A) \leqq \int_A g_\alpha * f\, dx .$$

Setting $p^* = np(n-\alpha p)^{-1}$ we have

$$\int_A g_\alpha * f\, dx \leqq (m(A))^{1/p^{*\prime}} \|g_\alpha * f\|_{p^*} ,$$

and from the Sobolev Inequality

$$\|g_\alpha * f\|_{p^*} \leqq \varkappa \|f\|_p .$$

Therefore

$$m(A) \leqq \varkappa (m(A))^{1/p^{*\prime}} \|f\|_p ,$$

and (i) now follows easily. Assertion (ii) follows in a similar way from the inequality

$$\|g_\alpha * f\|_q \leqq \varkappa \|f\|_p , \qquad p \leqq q < \infty .$$

As for (iii), we note that $g_\alpha \in \mathscr{L}_{p'}$ in this case; therefore

$$g_\alpha * f(0) \leqq \|g_\alpha\|_{p'} \|f\|_p .$$

Hence

$$B_{\alpha;p}(\{0\}) \geqq \|g_\alpha\|_{p'}^{-p} ,$$

and the rest follows easily from Lemma 6.

DEFINITION 17. For $\varrho > 0$, $\sigma_\varrho(x_0) = \sigma_\varrho$ will denote the open solid sphere with center $x_0$ and radius $\varrho$.

LEMMA 7. *If* $1 < p$ *and* $\alpha p < n$, *then there exists a finite positive constant* $\varkappa$, *independent of* $\varrho$, *such that*

$$\varkappa^{-1} \varrho^{n-\alpha p} \leqq B_{\alpha;p}(\sigma_\varrho) \leqq \varkappa\, \varrho^{n-\alpha p} \quad \textit{for}\ 0 < \varrho \leqq 1 .$$

PROOF. The first inequality follows immediately from Theorem 20(i). To prove the second inequality let $f$ be a test function for $B_{\alpha;p}(\sigma_4)$; then

$$\int g_\alpha(x-y) f(y)\, dy \geqq 1 \quad \text{on } \sigma_4 .$$

This means that

$$\int g_\alpha\left(\frac{x-z}{\varrho}\right) \varrho^{-n} f\left(\frac{z}{\varrho}\right) dz \geqq 1 \quad \text{on } \sigma_{4\varrho} . \tag{29}$$

From (22) and (25) there must exist a finite positive constant $\varkappa_1$ such that

(30) $\varkappa_1^{-1} r^{\alpha-n} e^{-2r} \leqq g_\alpha(r) \leqq \varkappa_1 r^{\alpha-n} e^{-\frac{1}{2}r}, \quad g_\alpha(r) = g_\alpha(r, 0, \ldots, 0)$.

Therefore

(31)
$$g_\alpha(r/\varrho) \leqq \varkappa_1 \varrho^{n-\alpha} r^{\alpha-n} e^{-\frac{1}{2}r\varrho^{-1}} \leqq \varkappa_1 \varrho^{n-\alpha} r^{\alpha-n} e^{-2r} \leqq \varkappa_1^2 \varrho^{n-\alpha} g_\alpha(r) \quad \text{for } 0<\varrho\leqq \tfrac{1}{4}.$$

(29) and (31) yield

$$\int g_\alpha(x-z)\, \varkappa_1^2 \varrho^{-\alpha} f\left(\frac{z}{\varrho}\right) dz \geqq 1 \quad \text{on } \sigma_{4\varrho}, \quad 0<\varrho\leqq \tfrac{1}{4}.$$

Since

$$\int \left(\varkappa_1^2 \varrho^{-\alpha} f\left(\frac{z}{\varrho}\right)\right)^p dz = \varkappa_1^{2p} \varrho^{n-\alpha p} \int (f(y))^p \, dy,$$

it easily follows that

$$B_{\alpha;p}(\sigma_{4\varrho}) \leqq \varkappa_1^{2p} \varrho^{n-\alpha p} B_{\alpha;p}(\sigma_4), \quad 0<\varrho\leqq \tfrac{1}{4};$$

and if we replace $\varrho$ by $\frac{1}{4}\varrho$, we get the desired inequality.

We now consider the case $\alpha p = n$.

LEMMA 8. *If $1<p$, $\alpha p = n$ and $0<\bar{\varrho}<1$, then there exists a finite positive constant $\varkappa$, independent of $\varrho$, such that*

(32)
$$\varkappa^{-1}(\log \varrho^{-1})^{1-p} \leqq B_{\alpha;p}(\sigma_\varrho) \leqq \varkappa(\log \varrho^{-1})^{1-p}$$

*for* $0<\varrho\leqq\bar{\varrho}<1$.

PROOF. Let $\nu_\varrho$ be the $b_{\alpha;p}$-capacitary distribution of $\bar{\sigma}_\varrho$. Then $\|g_\alpha * \nu_\varrho\|_{\mu;p'} = 1$, and (22) then implies the existence of a constant $\varkappa$, independent of $\varrho$, such that

$$\int_{\{|x|\leqq 1\}} \left(\int |x-y|^{\alpha-n}\, d\nu_\varrho(y)\right)^{p'} dx \leqq \varkappa.$$

($\varkappa$ simply denotes a constant, not necessarily the same constant in two different formulas. We shall continue this practice later without further mention.) For $|x|\geqq\varrho$ we have $|x-y|\leqq 2|x|$, so that

$$\int_{\{\varrho\leqq|x|\leqq 1\}} |x|^{-n} dx \cdot \|\nu_\varrho\|_1^{p'} \leqq \varkappa,$$

from which we easily conclude

$$B_{\alpha;p}(\sigma_\varrho) \leqq \varkappa(\log \varrho^{-1})^{1-p}.$$

We now derive the lower bound on $B_{\alpha;p}(\sigma_\varrho)$. To this end let $m_\varrho = m\,|\,\sigma_\varrho$. Now, again from (22),

$$g_\alpha * m_\varrho(x) \leqq \varkappa \int_{\{|y|\leqq\varrho\}} |x-y|^{\alpha-n}\,dy \quad \text{for } |x| \leqq 1 .$$

It is easy to prove

$$\int_{\{|y|\leqq\varrho\}} |x-y|^{\alpha-n}\,dy \leqq \varkappa\varrho^\alpha \quad \text{for } |x| \leqq \varrho/\bar{\varrho}$$

and

$$\int_{\{|y|\leqq\varrho\}} |x-y|^{\alpha-n}\,dy \leqq \varkappa\,\varrho^n\,|x|^{\alpha-n} \quad \text{for } \varrho/\bar{\varrho} < |x| \leqq 1 .$$

This gives us a bound on $g_\alpha * m_\varrho(x)$ for $|x| \leqq 1$. From (25) we then bound $g_\alpha * m_\varrho(x)$ for the remaining $x$ by

$$g_\alpha * m_\varrho(x) \leqq \varkappa\,\varrho^n e^{-|x|} \quad \text{for } |x| > 1 .$$

Integration of these upper bounds yields

$$\|g_\alpha * m_\varrho\|_{p'} \leqq \varkappa\varrho^n(\log\varrho^{-1})^{1/p'} ,$$

from which the lower bound for $B_{\alpha;p}(\sigma_\varrho)$ follows.

LEMMA 9. *Let $k(r)$, $0 \leqq r < \infty$, be strictly positive, decreasing and continuous from the right. Let $\mu \in \mathscr{M}^+$ and*

$$\int k(|x-y|)\,d\mu(y) \in \mathscr{L}_p, \qquad 1 < p .$$

*Then there exists a function $\bar{k}(r)$, $0 \leqq r < \infty$, where*

(i) *$\bar{k}(r)$ is strictly positive, decreasing and continuous from the right,*

(ii) $\int \bar{k}(|x-y|)\,d\mu(y) \in \mathscr{L}_p$,

(iii) *$\bar{k}(r) \geqq k(r)$ and* $\lim_{r\to 0} \bar{k}(r)\,k(r)^{-1} = +\infty$.

PROOF. Define

$$v_0(x) = \int_{\{x=y\}} k(|x-y|)\,d\mu(y)$$

and

$$v_i(x) = \int_{\{2^{-i}\leq|x-y|<2^{-i+1}\}} k(|x-y|)\,d\mu(y), \qquad i = 1, 2, \dots .$$

Note that $v_0(x) = 0$ as a function in $\mathscr{L}_p$ because the set of points on which $\mu$ concentrates non-zero mass is at most countable.

We will now prove the existence of a sequence $\{a_i\}$ of finite real numbers such that

(33) $$a_i \geqq 1, \qquad a_i \uparrow \infty,$$

and

(34) $$\sum_{i=1}^{\infty} a_i v_i(x) \in \mathscr{L}_p .$$

First notice that

$$\sum_{i=l}^{\infty} v_i \to 0 \quad \text{strongly in } \mathscr{L}_p \text{ as } l \to \infty .$$

Therefore there exists a subsequence of the positive integers, $\{l_j\}$, $l_1 = 1$, where

$$\sum_{j=1}^{\infty} \|\sum_{i=l_j}^{\infty} v_i\|_p < \infty .$$

But then there must exist a sequence of finite real numbers $\{b_j\}$ such that

$$b_j \geqq 1, \qquad b_j \uparrow \infty,$$

and

$$\sum_{j=1}^{\infty} b_j \|\sum_{i=l_j}^{\infty} v_i\|_p < \infty .$$

Define $a_i = b_j$ for $l_j \leqq i < l_{j+1}$. Then

$$\|\sum_{i=1}^{\infty} a_i v_i\|_p \leqq \sum_{j=1}^{\infty} \|\sum_{i=l_j}^{l_{j+1}-1} a_i v_i\|_p \leqq \sum_{j=1}^{\infty} b_j \|\sum_{i=l_j}^{\infty} v_i\|_p < \infty .$$

This completes the proof of (33) and (34). Now define $\bar{k}(r)$ by

$$\begin{aligned} \bar{k}(0) &= +\infty , \\ \bar{k}(r) &= a_i k(r) \quad \text{for } 2^{-i} \leqq r < 2^{-i+1};\ i = 1, 2, \ldots , \\ \bar{k}(r) &= k(r) \quad \text{for } 1 \leqq r . \end{aligned}$$

It is easy to verify that $\bar{k}(r)$ has properties (i)–(iii).

LEMMA 10. *Let $1 < p$ and $\alpha p \leqq n$. Suppose that $\bar{k}(r)$, $0 \leqq r < +\infty$, is a positive, decreasing function, continuous from the right, such that*

$$\bar{k}(r) \geqq g_\alpha(r) \quad \text{and} \quad \lim_{r\to 0} \bar{k}(r) g_\alpha(r)^{-1} = +\infty .$$

*Defining $\bar{B} = C_{\bar{k};m;p}$ we then have*

$$\lim_{\varrho\to 0} \bar{B}(\sigma_\varrho)\, B_{\alpha;p}(\sigma_\varrho)^{-1} = 0 .$$

PROOF. Note that $\bar{B}$ is invariant under translation so that the center of $\sigma_\varrho$ is of no consequence and we may take it to be 0.

Let $f$ be a test function for $B_{\alpha;p}(\sigma_\varrho)$ such that

(35) $$\|f\|_p^p \leqq 2 B_{\alpha;p}(\sigma_\varrho) .$$

For the moment let $\theta$ be a finite constant greater than one. Then we have

$$\int_{\{|y|<\varrho\theta\}} g_\alpha(x-y)\, f(y)\, dy + \int_{\{|y|\geqq\varrho\theta\}} g_\alpha(x-y)\, f(y)\, dy \geqq 1, \quad x \in \sigma_\varrho .$$

We now consider the second integral and try to make it small on $\sigma_\varrho$.

$$(36) \quad \int_{\{|y|\geqq\varrho\theta\}} g_\alpha(x-y)\, f(y)\, dy \leqq \left(\int_{\{|y|\geqq\varrho\theta\}} (g_\alpha(x-y))^{p'}\, dy\right)^{1/p'} \|f\|_p$$

and

$$(37) \quad \left(\int_{\{|y|\geqq\varrho\theta\}} (g_\alpha(x-y))^{p'}\, dy\right)^{1/p'} \leqq \left(\int_{\{|y|\geqq\varrho(\theta-1)\}} (g_\alpha(y))^{p'}\, dy\right)^{1/p'},$$

$$x \in \sigma_\varrho .$$

We now handle the cases $\alpha p < n$ and $\alpha p = n$ separately. In case $\alpha p < n$, (22) implies

$$\left(\int_{\{|y|\geqq\varrho(\theta-1)\}} (g_\alpha(y))^{p'}\, dy\right)^{1/p'} \leqq \varkappa(\varrho(\theta-1))^{\alpha-np^{-1}} .$$

Thus from (35)–(37) and Lemma 8

$$\int_{\{|y|\geqq\varrho\theta\}} g_\alpha(x-y)\, f(y)\, dy \leqq \varkappa(\theta-1)^{\alpha-np^{-1}}, \quad x \in \sigma_\varrho .$$

So for the proper choice of $\theta$

$$(38) \quad \int_{\{|y|\geqq\varrho\theta\}} g_\alpha(x-y)\, f(y)\, dy \leqq \tfrac{1}{2}, \quad x \in \sigma_\varrho,\ \alpha p < n .$$

In case $\alpha p = n$, (22) implies

$$\left(\int_{\{|y|\geqq\varrho(\theta-1)\}} (g_\alpha(y))^{p'}\, dy\right)^{1/p'} \leqq \varkappa(\log\varrho^{-1}(\theta-1)^{-1})^{1/p'}, \quad 0 < \varrho(\theta-1) \leqq \tfrac{1}{2} .$$

Thus from (35)–(37)

$$\int_{\{|y|\geqq\varrho\theta\}} g_\alpha(x-y)\, f(y)\, dy \leqq \varkappa(\log\varrho^{-1}(\theta-1)^{-1})^{1/p'}\, (B_{\alpha;p}(\sigma_\varrho))^{1/p'},$$

$$0 < \varrho(\theta-1) \leqq \tfrac{1}{2} .$$

If we now define $\theta$ as a function of $\varrho$ by the equation

$$\varrho(\theta-1) = \exp\left(-(B_{\alpha;p}(\sigma_\varrho))^{-\gamma}\right), \quad 0 < \gamma < p'-1 ,$$

it follows that $\lim_{\varrho\to 0}\varrho\theta = 0$, and the inequality (38) holds for sufficiently small values of $\varrho$. Therefore in both cases one can determine $\theta$ as a function of $\varrho$ so that

$$\lim_{\varrho\to 0}\varrho\theta = 0$$

and

$$\int_{\{|y|\geq \varrho\theta\}} g_\alpha(x-y)\,f(y)\,dy \;\leqq\; \tfrac{1}{2}\,, \qquad x \in \sigma_\varrho\,,\; 0<\varrho\leqq\varrho_1\,.$$

In both cases we have

$$\int_{\{|y|<\varrho\theta\}} g_\alpha(x-y)\,f(y)\,dy \;\geqq\; \tfrac{1}{2}\,, \qquad x \in \varrho_\varrho\,,\; 0<\varrho\leqq\varrho_1\,.$$

Now define

$$(\varepsilon(\varrho))^{-1} \;=\; \inf_{0<r<\varrho(\theta+1)} \bar{k}(r)\, g_\alpha(r)^{-1}\,.$$

Hence

$$\int \bar{k}(|x-y|)\; 2\varepsilon(\varrho)\, f(y)\,dy \;\geqq\; 1\,, \qquad x \in \sigma_\varrho\,,\; 0<\varrho\leqq\varrho_1\,,$$

so that from (35)

$$\bar{B}(\sigma_\varrho) \;\leqq\; \varkappa(\varepsilon(\varrho))^p\, B_{\alpha;p}(\sigma_\varrho), \qquad 0<\varrho\leqq\varrho_1\,,$$

and the proof is finished.

Definition 18. Let $\varphi(\varrho)$ be a positive, increasing function in some interval $0<\varrho<\varrho_1$ and let $\lim_{\varrho\to 0}\varphi(\varrho)=0$. If $A$ is an arbitrary set, then the Hausdorff $\varphi$-measure of $A$ is given by

$$H_{\varphi(\varrho)}(A) \;=\; \lim_{r\to 0}\{\inf \textstyle\sum_{i=1}^{\infty}\varphi(\varrho_i)\}\,,$$

where the above infimum is taken over all countable coverings of $A$ by spheres $\sigma_{\varrho_i}(x_i)$ such that $\varrho_i \leqq r$.

Note that $H_{\varphi(\varrho)}$ is a capacity and while it is clearly not in general an outer capacity it has the property

$$H_{\varphi(\varrho)}(A) \;=\; H_{\varphi(\varrho)}(D)\,, \tag{39}$$

where $D$ is a $G_\delta$-set containing $A$.

We now give the principal result of this section relating Hausdorff measure and capacity. We state the result in the case $p>1$ for two reasons: first it is already known for $p=1$ and second the case $p=1$ is contained in the case $p=2$ since $B_{2\alpha;1}(A)=0$, if and only if $B_{\alpha;2}(A)=0$.

Theorem 21. *If $1<p$ and $\alpha p\leqq n$, then $H_{\varphi(\varrho)}(A)<\infty$, $\varphi(\varrho)=B_{\alpha;p}(\sigma_\varrho)$ implies $B_{\alpha;p}(A)=0$.*

*If $\alpha p<n$, then $\varphi(\varrho)$ can be replaced by $\varrho^{n-\alpha p}$; and if $\alpha p=n$, then $\varphi(\varrho)$ can be replaced by $(\log \varrho^{-1})^{1-p}$.*

Proof. The second statement is an obvious consequence of the first and Lemmas (7), (8).

Turning to the proof of the first statement we see that in view of (39) and Theorems 8 and 17 it is sufficient to consider the case $A=K$. To derive a contradiction assume that $B_{\alpha;p}(K)>0$. Then from Theorems 14 and 17 there exists $\mu \in \mathscr{M}^+$ such that

$$\mu \neq 0, \qquad \operatorname{supp}\mu \subset K \qquad \text{and} \qquad g_\alpha * \mu \in \mathscr{L}_{p'} .$$

According to Lemma 9, with $p'$ and $g_\alpha$ replacing $p$ and $k$ respectively, a kernel $\bar{k}$ exists with properties (i)–(iii). If we set $\bar{B}=C_{\bar{k};m;p}$, we must have

$$\bar{B}(K) > 0 \tag{40}$$

by Theorems 14 and 17. Lemmas 9 and 10 imply that

$$\lim_{\varrho\to 0} \bar{B}(\sigma_\varrho)/B_{\alpha;p}(\sigma_\varrho) = 0 .$$

Now let $\{\sigma_{\varrho_i}(x_i)\}$ be a countable covering of $K$ by solid spheres; then

$$\bar{B}(K) \leqq \textstyle\sum_{i=1}^\infty \bar{B}(\sigma_{\varrho_i}(x_i)) = \sum_{i=1}^\infty \dfrac{\bar{B}(\sigma_{\varrho_i}(x_i))}{B_{\alpha;p}(\sigma_{\varrho_i}(x_i))} B_{\alpha;p}(\sigma_{\varrho_i}(x_i)) .$$

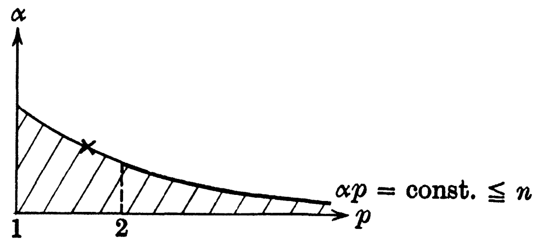
Since the ratios $\bar{B}(\sigma_{\varrho_i}(x_i))/B_{\alpha;p}(\sigma_{\varrho_i}(x_i))$ can be made as small as we wish while $\sum_{i=1}^\infty B_{\alpha;p}(\sigma_{\varrho_i}(x_i))$ remains bounded, we must have $\bar{B}(K)=0$ which contradicts (40).

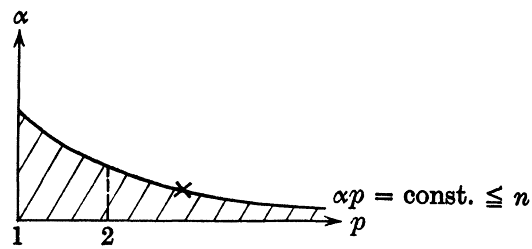
Definition 20. Given a capacity $B=B_{\alpha;p}$ we call $\alpha$, $p$ and $\alpha p$ the *order*, the *degree*, and the *weight* of $B$ respectively and denote them by $\operatorname{ord} B$, $\deg B$ and $\operatorname{wei} B$.

Fuglede and du-Plessis and others have investigated the sets on which the Riesz potential of an element in $\mathscr{L}_p$ can be infinite (see [5] and [8]). Whether one deals with a Riesz potential or a Bessel potential is irrelevent to the validity of their results. The following is a compact statement of their results in our language and shows the importance of the weight.

Theorem (du-Plessis, Fuglede). *Let $B$ and $B'$ be Bessel capacities. If $B(A)=0$, then $B'(A)=0$, provided* $\operatorname{wei} B' < \operatorname{wei} B$; *if in addition* $\deg B \leqq 2$, *then $B'(A)=0$, provided* $\operatorname{wei} B' = \operatorname{wei} B$ *and* $\deg B' \geqq 2$.

The above proposition can be easily visualized by means of the following figures where the cross indicates $B$ and the shaded regions indicate the possible $B'$. Of course, $\operatorname{wei} B \leqq n$ except in the trivial case of $A=\emptyset$.




THEOREM 22. *If* $H_{\varphi(\varrho)}(A) > 0$, $\varphi(\varrho) = \varrho^{n-\alpha p+\varepsilon}$, *where* $\varepsilon > 0$, *then* $B_{\alpha;p}(A) > 0$.

PROOF. It is sufficient to assume that $A = K$. The result now follows from known relations between Hausdorff measure and the classical capacity (see [3, Theorem 1, p. 28]) and the foregoing theorem.

We end the paper with an interesting consequence of the work of du-Plessis and Fuglede which elaborates Theorems 17 and 5.

THEOREM 23. *Suppose that* $\{f_i\}$ *is a bounded sequence of elements in* $\mathscr{L}_p$ *for some* $p \geqq 1$. *Then for each* $\alpha > 0$ *there exists a subsequence* $\{f_i'\}$ *such that*

$$f_i' \to f \quad \textit{weakly in } \mathscr{L}_p$$

*and*

$$g_\alpha * f_i'(x) \to g_\alpha * f(x) \qquad B\text{-a.e.}$$

*for all* $B$ *with* $\text{wei}\, B < \alpha p$.

PROOF. There is clearly no loss of generality in assuming that $f_i \in \mathscr{L}_p^+$. A weakly convergent subsequence exists; call this subsequence $\{f_{i_j}\}$ and the limit $f$. Now let $l$ be a fixed integer greater than $\alpha^{-1}$; then

$$g_{l-1} * f_{i_j}(x) \to g_{l-1} * f(x) \quad \text{strongly in } \mathscr{L}_p, \textit{ locally}.$$

This is a known result for Bessel potentials the proof of which is based on the Riesz compactness criterion. Now let $\chi_k$ be the characteristic function of the ball $\{|x| \leq k\}$; $k = 1, 2, \ldots$. Since $\chi_k \cdot (g_{l^{-1}} * f_{ij}) \to \chi_k \cdot (g_{l^{-1}} * f)$ strongly in $\mathscr{L}_p$ as $j \to \infty$, a subsequence of $j$ must exist such that the $(\alpha - l^{-1})$-potentials converge $B_{\alpha - l^{-1}; p}$-a.e. By diagonalization we may choose the subsequence independent of $k$. So as not to complicate matters we use the same notation for the subsequence as for the full sequence. Let $\bar{x}$ be any point such that

$$g_{\alpha - l^{-1}} * (\chi_k \cdot (g_{l^{-1}} * f_{i_j}))(\bar{x}) \to g_{\alpha - l^{-1}} * (\chi_k \cdot (g_{l^{-1}} * f))(\bar{x}) \quad \text{as } j \to \infty, \text{ for } all \ k .$$

Since $\|g_{l^{-1}} * f_{i_j}\|_p$, $\|g_{l^{-1}} * f\|_p \leq M$ independent of $j$ and since

$$\int_{\{|\bar{x} - y| \geq 1\}} |g_{\alpha - l^{-1}}(\bar{x} - y)|^{p'} \, dy < \infty \quad \text{for } 1 < p$$

while

$$\lim_{|y| \to \infty} g_{\alpha - l^{-1}}(\bar{x} - y) = 0 ,$$

we see that

$$g_{\alpha - l^{-1}} * [(1 - \chi_k) \cdot (g_{l^{-1}} * (f_{i_j} + f))](\bar{x}) \to 0 \quad \text{as } k \to \infty, \text{ uniformly in } j .$$

Hence

$$g_\alpha * f_{i_j}(\bar{x}) \to g_\alpha * f(\bar{x})$$

and

$$g_\alpha * f_{i_j}(x) \to g_\alpha * f(x) \quad B_{\alpha - l^{-1}; p}\text{-a.e.} \tag{41}$$

Again by diagonalization we choose a further subsequence so that (41) will hold independent of $l$. The required result can now be inferred from the theorem of du-Plessis and Fuglede.

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UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA, U.S.A.