MATH. SCAND. 26 (1970), 293–302

UNIFORM MEASURES AND SPHERICAL HARMONICS

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This paper deals with miscellaneous results about uniform measures which are generalisations of a theorem due to Chr. Berg. Furthermore, we develop a theory which is a generalisation of the usual spherical harmonics. Finally we apply the results to two-point homogeneous spaces.

Let (M,d) be a locally compact metric space. Let u be a positive Radon measure defined on the Borel field of M. Suppose u is uniform with respect to d, which means that u has the property

$$\forall x,y \in M \ \ \forall r > 0: \ u(S(x,r)) = u(S(y,r)) ,$$

where $S(x,r)$ is the open ball with center x and radius r (see [1]). Let A be a closed subspace of M and $\hat{u}$ a positive uniform Radon measure on A. Of course, $\hat{u}$ may be considered as a measure on M. All the measures we speak about in the sequel are Radon measures defined on the Borel field of the space under consideration. Unless otherwise stated the measures are tacitly assumed to be positive.

Theorem 1. *Suppose that for every compact set $K \subseteqq M$ there are an $\varepsilon_K > 0$ and a $C_K > 0$ such that for $0 < \varepsilon < \varepsilon_K$ and all $x \in K$,*

$$\hat{u}(S(x,\varepsilon)) \leqq C_K u(S(x,\varepsilon)) .$$

Then there is a $\lambda > 0$ such that $\hat{u} = \lambda u|A$ and

$$\lambda = \lim_{\varepsilon \to 0} c_\varepsilon(\hat{u})/c_\varepsilon(u) ,$$

where $x \in A$, $c_\varepsilon(\hat{u}) = \hat{u}(S(x,\varepsilon))$, and $c_\varepsilon(u) = u(S(x,\varepsilon))$.

Proof. We define the kernel function

$$\begin{aligned} K_\varepsilon(x,y) &= c_\varepsilon(u)^{-1} \quad \text{for } d(x,y) < \varepsilon , \\ &= 0 \qquad\qquad \text{for } d(x,y) \geqq \varepsilon . \end{aligned}$$

Let φ be a continuous function on M of compact support. Define $K_\varepsilon\varphi$ by

Received May 5, 1969.

$$(K_\varepsilon\varphi)(x) = \int_M K_\varepsilon(x,y)\varphi(y)\,du(y)\,.$$

The following calculations and the functions involved make sense for ε sufficiently small. We easily obtain

$$|\varphi(x)-K_\varepsilon\varphi(x)| \leqq \omega_\varphi(\varepsilon) = \sup\{|\varphi(x)-\varphi(y)| \mid d(x,y)\leqq\varepsilon\}\,.$$

We choose ε_0 so that $K_\varepsilon\varphi$ is supported by a fixed compact set K for all $0<\varepsilon<\varepsilon_0$. Then we have

$$\lim_{\varepsilon\to 0}\int_M K_\varepsilon\varphi(x)\,d\hat{u}(x) = \int_M \varphi(x)\,d\hat{u}(x)\,.$$

Put $\lambda_\varepsilon = c_\varepsilon(\hat{u})/c_\varepsilon(u)$. By applying Fubini's theorem we obtain

$$\int_M K_\varepsilon\varphi(x)\,d\hat{u}(x) = \lambda_\varepsilon\int_A \varphi(y)\,du(y) + \int_{M\setminus A}\big(\hat{u}(S(y,\varepsilon))/c_\varepsilon(u)\big)\varphi(y)\,du(y)\,.$$

The theorem about dominated convergence shows that the last integral tends to zero as ε tends to zero. If $\int_M\varphi(x)d\hat{u}(x)\neq 0$, this implies that $\lim_{\varepsilon\to 0}\lambda_\varepsilon$ exists and the theorem is now obvious.

Let now the situation be that of theorem 1.

THEOREM 2. *Under the assumptions of theorem* 1, *the interior* $\mathring{A}$ *of* A *is closed. Therefore, if* $\mathring{A}\neq\emptyset$ *and* M *is connected, then* $A=M$. *If* $\mathring{A}\neq\emptyset$, *then* A *is open.*

PROOF. If $\mathring{A}\neq\emptyset$, then $\lambda_\varepsilon=\lambda$ for all $\varepsilon<\varepsilon_0$, where ε_0 is the radius of an open ball contained in $\mathring{A}$. Let x belong to A. We then have $\hat{u}(S(x,\varepsilon)) = \lambda u(S(x,\varepsilon))$ for all $\varepsilon<\varepsilon_0$. Further, because of $\hat{u}=\lambda u|A$ this shows $u((M\setminus A)\cap S(x,\varepsilon))=0$. This set, being open, must then be empty. Consequently, we have $S(x,\varepsilon)\subseteq A$ and theorem 2 is proved.

In the special case $M=\mathsf{R}^n$ with the usual euclidean metric a much stronger result can be obtained. We first prove an inequality in the general case. Let the situation be as described before theorem 1.

Let $K\subseteq M$ be a compact set and put

$$S(K,\varepsilon) = \{y\in M \mid \exists k\in K: d(k,y)<\varepsilon\}\,.$$

For $\varepsilon>0$ sufficiently small (such that $c_{2\varepsilon}(\hat{u})<\infty$ and $c_\varepsilon(u)<\infty$) we have the inequality

$$\hat{u}(K) \leqq \big(c_{2\varepsilon}(\hat{u})/c_\varepsilon(u)\big)\,u(S(K,\varepsilon))\,. \tag{1}$$

To prove this we put

$$1_K(x) = 1 \quad \text{for } x \in K\,, \qquad = 0 \quad \text{otherwise}\,.$$

Then we have

$$\hat{u}(K) = \int_M 1_K(y)\,d\hat{u}(y) = \int_M \left(\int_M K_\varepsilon(x,y)\,1_K(y)\,du(x)\right) d\hat{u}(y)$$
$$= \int_M \left(\int_M K_\varepsilon(x,y)\,1_K(y)\,d\hat{u}(y)\right) du(x) \leqq \left(c_{2\varepsilon}(\hat{u})/c_\varepsilon(u)\right) u\big(S(K,\varepsilon)\big)\,.$$

Roughly speaking, we can say that $\hat{u}$ does not increase much faster than u.

THEOREM 3. *Let $A \subseteqq \mathsf{R}^n$ be a closed subset with the uniform measure $\hat{u}$ (with respect to euclidean metric). Let Q be a connected k-dimensional analytic manifold and $P \subseteqq Q$ a non-empty open set. Let $\varphi\colon Q \to \mathsf{R}^n$ be an analytic mapping with $\varphi(P) \subseteqq A$. Then $\varphi(Q) \subseteqq A$.*

PROOF. We define for $\lambda > 0$

$$\tilde{u}_\lambda(x) = \int_{\mathsf{R}^n} \exp\left(-\lambda(x-y)^2\right) d\hat{u}(y)\,.$$

From (1) it follows that $\tilde{u}_\lambda$ is a well-defined real analytic function of $x \in \mathsf{R}^n$. It is constant on A, hence $\tilde{u}_\lambda(\varphi(y)) = \tilde{u}_\lambda(x)$ for $y \in Q$ and $x \in A$. The uniqueness theorem for Laplace transforms of measures now shows that $\varphi(y) \in A$, and theorem 3 is proved.

Christian Berg has proved that if A is a compact subset of a sphere with a uniform probability $\hat{u}$ and A has non-empty interior, then A must be the whole sphere and the measure must be the natural one.

Berg's theorem together with its analogue for R^n can easily be derived from theorem 3 and the uniqueness theorem for uniform measures (see [1]). Thus the Lebesgue measure on R^n is the only measure which is uniform on a closed subset with non-empty interior.

Let (M,d) be a locally compact metric space with the uniform measure u. Then for every continuous function f,

$$\int_M f\big(d(x,y)\big)\,du(y) \quad \text{is independent of } x \in M$$

(if it is well defined for some $x_0 \in M$). M is compact if and only if u is finite (supposing M complete, but finiteness of u always implies that

(M,d) is a precompact metric space; in this case we always assume that u is a probability). Let A be a compact subset of R^n with the uniform probability $\hat{u}$ and suppose $\int_A \boldsymbol{y}\, d\hat{u}(\boldsymbol{y}) = \boldsymbol{0}$. Then we obtain

$$\int_A (\boldsymbol{x}-\boldsymbol{y})^2 \, d\hat{u}(\boldsymbol{y}) = \boldsymbol{x}^2 + \int_A \boldsymbol{y}^2 \, d\hat{u}(\boldsymbol{y}) .$$

Since the left hand side is independent of $\boldsymbol{x} \in A$, we have

THEOREM 4. *A compact subset A of R^n with uniform probability $\hat{u}$ is contained in a sphere with center in the center of mass.*

Let (M,d) be a compact metric space. We call M a *spherical space* if there is a probability measure u on M which satisfies the following condition: For any continuous functions f and g there is a continuous function h such that

$$\forall x,z \in M: \quad \int_M f(d(x,y))g(d(y,z)) \, du(z) = h(d(x,z)) .$$

The measure u is called the *spherical measure* of M. It is unique because it is easily seen to be uniform (let g be the constant 1).

Put

$$\mathscr{A} = \{k: M^2 \to \mathsf{R} \mid k(x,y) = f(d(x,y)) \text{ with } f \text{ real and continuous}\} .$$

$\mathscr{A}$ is a real commutative Banach algebra with the norm

$$\|k\|_\infty = \sup\{|k(x,y)| \mid (x,y) \in M^2\} ,$$

and the multiplication

$$(k \circ h)(x,y) = \int_M k(x,z)h(z,y) \, du(z) .$$

We do not always distinguish between $k \in \mathscr{A}$ considered as an integral operator (integration with respect to u) and k considered as a function on M^2.

We call a complex continuous function φ a *spherical function* if φ is an eigenfunction for each $k \in \mathscr{A}$. Spherical functions φ and ψ are said to be associated (φ ass. ψ) if they belong to the same eigenvalue with respect to each particular $k \in \mathscr{A}$. This is an equivalence relation, and for a spherical function φ its equivalence class together with the constant 0 is a linear space V of spherical functions. We define the kernel

$$\overline{K}_\varepsilon(x,y) = k(\varepsilon)(\varepsilon - d(x,y))^+ ,$$

where

$$k(\varepsilon)^{-1} = \int_M (\varepsilon - d(x,y))^+ \, du(y), \quad x \in M \text{ arbitrary}.$$

The arguments in the proof of theorem 1 hold also for this kernel and yield

$$\lim_{\varepsilon \to 0} \|\bar{K}_\varepsilon \varphi - \varphi\|_\infty = 0 .$$

In particular $\bar{K}_\varepsilon \varphi \neq 0$ for ε sufficiently small, so the eigenvalue for φ corresponding to $\bar{K}_\varepsilon$ is different from zero. $\bar{K}_\varepsilon$ being a compact operator, this shows that the space V is finite-dimensional. Because every $k \in \mathscr{A}$ is self-adjoint, any two non associated spherical functions φ and ψ are orthogonal. We choose a numbering V_i, $i \in \mathsf{N}$, of the spaces of spherical functions described above. Let k_i, $i \in \mathsf{N}$, be the kernel of the orthogonal projection on V_i (there is a denumerable infinity of V_i's if and only if the space M is infinite). A simple and standard application of the theory of eigenfunction expansions yields the following facts (the commutativity is essential for this):

For every $k \in \mathscr{A}$,

$$k(x,y) \,(=)\, \textstyle\sum_i \lambda_i k_i(x,y) ,$$

where the series on the right hand side converges in $\mathscr{L}_2$ norm to the left hand side. Each k_i, being a limit of kernels in $\mathscr{A}$, belongs to $\mathscr{A}$. Let f be an arbitrary continuous function on M. Then

$$\bar{K}_\varepsilon f(x) = \textstyle\sum_i \lambda_i^\varepsilon f_i(x) ,$$

where

$$\bar{K}_\varepsilon(x,y) \,(=)\, \textstyle\sum_i \lambda_i^\varepsilon k_i(x,y)$$

and

$$f_i(x) = \int_M k_i(x,y) \, f(y) \, du(y) .$$

The series $\sum_i \lambda_i^\varepsilon f_i(x)$ converges uniformly as we know from the theory of eigenfunction expansions. Furthermore $\bar{K}_\varepsilon f \to f$ uniformly as $\varepsilon \to 0$. This gives

THEOREM 5. *Every continuous function f on M can be uniformly approximated by finite sums of spherical functions.*

For a continuous function f on M we form the (formal) series

$$f(x) \sim \textstyle\sum_i f_i(x)$$

with f_i as above. For a finite signed measure v we form the formal series

$$v \sim \textstyle\sum_i v_i(x) ,$$

where

$$v_i(x) = \int_M k_i(x,y)\, dv(y) .$$

The main theorems on spherical harmonics (Parseval's equation, uniqueness theorems, for these formal series) are easily derived from theorem 5.

We now suppose that the spherical space M is infinite. We define $H_n = \sum_{i=1}^n V_i$ and $h_n = \sum_{i=1}^n k_i$. The mapping $\theta_n \colon M \to H_n$ is defined by

$$\theta_n(a)(x) = h_n(a,x) .$$

The mapping θ_n has the fundamental property that $\|\theta_n(a) - \theta_n(b)\|_2$ is a function of $d(a,b)$. Furthermore θ_n is continuous. Hence the image of u under θ_n is a uniform probability on $\theta_n(M)$. We put

$$\varepsilon_n = \sup \{d(x,y) \mid \theta_n(x) = \theta_n(y)\} .$$

Then $\varepsilon_n \to 0$ as $n \to \infty$ (if this were not the case we would obtain a contradiction with the uniqueness theorem for the formal series associated with a measure, in this case a point measure). Suppose now $\varepsilon_n > 0$ for all n. For a point $a \in M$ we put

$$K_n(a) = \{x \in M \mid \theta_n(x) = \theta_n(a)\} .$$

The function

$$d(x, K_n(a)) = \inf \{d(x,y) \mid y \in K_n(a)\}$$

cannot assume the value ε_n. Hence the set

$$B_n(a) = \{x \in M \mid d(x, K_n(a)) < \varepsilon_n\}$$

is both open and closed. For every $x \in B_n(a)$, we have $d(x,a) < 2\varepsilon_n$. Hence the point $a \in M$ has a neighbourhood base consisting of sets which are both open and closed. If M is not totally disconnected, this shows $\varepsilon_{n_0} = 0$ for a suitable n_0, hence $\varepsilon_n = 0$ for $n \geqq n_0$. This gives

THEOREM 6. *Let (M,d) be a spherical space which is not totally disconnected. Then there is a positive continuous function φ such that $D = \varphi \circ d$ is a new metric on M, and M with this metric is isometric with a subset of a sphere in a euclidean space.*

However, M with the new metric D is not necessarily a spherical space but only a *prespherical space* (the smallest algebra containing $\mathscr{A}$ and closed under pointwise multiplication is commutative).

Let now (M,d) be an infinite totally disconnected spherical space. Let $A \subseteq M$ be a set which is both open and closed and $A \neq \emptyset, M$. Put

$$\delta_A = d(A, \complement A) = \inf \{d(x,y) \mid x \in A \wedge y \notin A\} .$$

The function 1_A is continuous and is an eigenfunction for $\bar{K}_\varepsilon$ for every $\varepsilon < \delta_A$, the corresponding eigenvalue being 1. Then 1_A is a finite sum of spherical functions and the corresponding kernels k_i satisfy

$$(2) \qquad k_i \circ \bar{K}_\varepsilon = k_i \quad \text{for } \varepsilon \text{ sufficiently small} .$$

The space spanned by the functions 1_A, A open and closed, is dense in the space of continuous functions (it is an algebra and separates points). Let k_i be one of the fundamental kernels and φ a spherical function corresponding to k_i. Let S be a finite linear kombination of functions 1_A with

$$\|\varphi - S\|_\infty < \delta .$$

S is a finite sum $S = \sum_v \varphi_v$ of non associated spherical functions. The inequality above implies

$$\|\varphi - S\|_2^2 = \|\varphi - \varphi_\tau\|_2^2 + \sum_{v \neq \tau} \|\varphi_v\|_2^2 < \delta^2 ,$$

where φ_τ is the member of the sum which is associated with φ. We choose $\delta < \|\varphi\|_2$ and the inequality above then implies $\varphi_\tau \neq 0$. This shows that (2) is satisfied by k_i. But for $i \in \mathsf{N}$, $x,y \in M$ we have

$$k_i(x,y) \leqq k_i(x,x) = k_i(y,y) > 0$$

because k_i is positive semidefinite and only depends on $d(x,y)$. Let now ε be sufficiently small for the kernel k_i, namely such that $k_i \circ \bar{K}_\varepsilon = k_i$. Then

$$k_i(x,x) = \int_M \bar{K}_\varepsilon(x,z)\, k_i(z,x)\, du(z) \leqq k_i(x,x) \int_M \bar{K}_\varepsilon(z,x)\, du(z) = k_i(x,x) .$$

This shows

$$k_i(z,x) = k_i(x,x) \quad \text{for } d(z,x) \text{ sufficiently small} .$$

Suppose now $D = \varphi \circ d$ is a new metric on M such that M with this metric is isometric to a subset A of R^n. Owing to theorem 4, A must be a subset of a sphere with center in the center of mass. We suppose A is contained in the unit sphere with $\mathbf{0}$ as center of mass. Let $\boldsymbol{\theta}\colon M \to \mathsf{R}^n$ be the D-isometric imbedding. The usual scalar product $k(x,y) = \boldsymbol{\theta}(x) \cdot \boldsymbol{\theta}(y)$ is a kernel which is in $\mathscr{A}$. The kernel k must be a finite sum of kernels k_i. Hence, for all $x,y \in M$ with $d(x,y) < \varepsilon_0$ (ε_0 sufficiently small) we must have $k(x,x)=k(x,y)$. This contradicts, however, that θ is an isometry. Thus we have

THEOREM 7. *Let* (M,d) *be an infinite totally disconnected spherical space. Then the conclusion in theorem* 6 *fails to hold.*

Let (M,d) be a compact metric space and G its group of isometries. M is a two-point homogeneous space if for all $x_1,y_1,x_2,y_2 \in M$,

$$d(x_1,y_1) = d(x_2,y_2) \Rightarrow \exists \varphi \in G: \varphi(x_1)=x_2 \wedge \varphi(y_1)=y_2 .$$

In particular G is transitive on M. Hence there is a unique G invariant probability, which is easily seen to be a spherical measure. Since a kernel k is G invariant if and only if $k(x,y)$ is a function of $d(x,y)$, it is easily seen that the spaces V_i are precisely the finite dimensional spaces of continuous functions which are irreducible under the action of the group G; for a projection kernel corresponding to such a space is G invariant.

THEOREM 8. *Let* (M,d) *be a compact two-point homogeneous space which is not totally disconnected. Then the group* G *of isometries is a Lie group in its natural compact topology. Hence the space* M *has a unique analytic structure such that* G *is a Lie transformation group on* M.

PROOF. The mapping θ_n from the proof of theorem 6 is injective for a suitable n. Denoting by $O(H_n)$ the Lie group of orthogonal transformations of H_n, we consider

$$\hat{\theta}_n : G \to O(H_n)$$

defined by

$$(\hat{\theta}_n(I)f)(x) = f(I^{-1}(x)), \quad I \in G .$$

$\hat{\theta}_n$ is a continuous group homomorphism. Let I be different from the identical mapping of M onto M. Then there is $x_0 \in M$ with $x_0 \neq I(x_0)$, which implies $\theta_n(x_0) \neq \theta_n(I(x_0))$ or

$$h_n(x_0,x) \neq h_n(I(x_0),x) = h_n(x_0,I^{-1}(x))$$

or $\hat{\theta}_n$ is injective. $\hat{\theta}_n$ is then an isomorphism between G and $\hat{\theta}_n(G)$. Being a closed subgroup of a Lie group, $\hat{\theta}_n(G)$ is a Lie group. Now theorem 8 follows from a well-known theorem about Lie groups.

The conclusions of theorem 8 are known under the condition of connectedness of M (see [2]).

Consider the group $\{0,1\}$ and put $G=\{0,1\}^{\mathsf{N}}$. With the usual product topology, G is a compact abelian group. For $x,y \in G$, where $x = (x_1,x_2,\ldots)$ and $y=(y_1,y_2,\ldots)$, we put

$$d(x,y) = \sum_{i=1}^{\infty} |x_i - y_i| \, 3^{-i} .$$

This metric determines the product topology on G, and with it G is a two-point homogeneous space. Moreover G is infinite and totally disconnected.

The following problems in connection with the preceding results remain open.

1) If $A \subseteq \mathsf{R}^n$ is a closed subset with a uniform measure u, it seems probable that A is an analytic submanifold. Perhaps more dubious is the conjecture that A has a transitive group of isometries.

2) How many of the preceding results have analogues in the non compact case? The definition of a spherical space carries over with a minor modification. To avoid irregularities we assume connectedness. It is easily seen that the set

$$\{y \in M \mid d(x,y) = r\}$$

has a uniform probability if it is non-empty and

$$\bar{S}(x,r) = \{y \in M \mid d(x,y) \leqq r\}$$

is compact. This suggests a natural definition of harmonic functions. Are there "many" non trivial harmonic functions?

3) Is it possible to choose the function φ in theorem 6 such that (M,d) is not only a prespherical space but a spherical space? If (M,d) is a compact metric space with a group G of isometries satisfying

$$\forall x,y \in M \;\; \exists \varphi \in G: \quad \varphi(x) = y \wedge \varphi(y) = x ,$$

then it is easily seen that (M,d) is a prespherical space. How many results carry over to this case?

Acknowledgements. One of the preceding results is, as stated, due to Chr. Berg with whom I also had stimulating discussions. The conclusion of theorem 2 is made stronger by a suggestion from Bent Fuglede. Moreover I had helpful discussions with Flemming Topsøe.

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