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STABILITY AND CONVERGENCE RATES IN L_p FOR CERTAIN DIFFERENCE SCHEMES

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1. Introduction.

Let $1 \leqq p \leqq \infty$ and $L_p = L_p(\mathsf{R}^1)$ with

$$\|v\|_p = \left(\int_{-\infty}^{\infty} |v(x)|^p \, dx \right)^{1/p}, \quad 1 \leqq p < \infty ,$$

$$\|v\|_\infty = \sup_{\mathsf{R}^1} |v(x)| .$$

Consider a finite difference operator

$$(1.1) \quad E_k v(x) = \sum_{j=-\infty}^{\infty} a_j v(x-jh), \ \textstyle\sum_j |a_j| < +\infty, \quad kh^{-1} = \lambda = \text{constant} ,$$

consistent with the initial value problem

$$(1.2) \qquad \frac{\partial u}{\partial t} = \frac{\varrho}{2\pi} \frac{\partial u}{\partial x}, \quad \varrho \text{ real constant} ,$$

$$u(x,0) = v(x) .$$

We shall discuss the question of stability of such an operator in L_p, that is, the question of validity of an estimate of the form

$$\|E_k^n v\|_p \leqq C \|v\|_p, \quad n = 1,2,\ldots, \qquad v \in L_p ,$$

and the related question of estimating the error at $t = nk$ between the approximate solution $E_k^n v$ and the exact solution

$$E(t)v = v\left(\cdot + \frac{\varrho t}{2\pi}\right)$$

of (1.1).

The results will be expressed in terms of the characteristic function of E_k, namely

$$(1.3) \qquad a(y) = \textstyle\sum_j a_j e^{2\pi i j y}$$

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In the case of L_2 the necessary and sufficient condition for stability is

$$|a(y)| \leqq 1, \quad y \text{ real}.$$

We shall always assume that this condition is satisfied. For L_p, $p \neq 2$, this condition is still necessary but not sufficient. For simplicity, we shall consider the case when

(a) $|a(y)| \equiv 1$, or
(b) $|a(y)| < 1$ for $0 < |y| \leqq \frac{1}{2}$.

For small y, we can then write

$$a(y) = \exp(-i\lambda\varrho y + \psi(y))$$

where (unless $\psi \equiv 0$ and E_k is exact)

$$\psi(y) = \psi_0 y^r(1+o(1)), \quad \psi_0 \neq 0, \ r > 1 .$$

In case (a), ψ is purely imaginary and in case (b),

$$\operatorname{Re}\psi(y) = -\gamma y^s(1+o(1)), \quad \gamma > 0 .$$

Here $r-1$ and s can be interpreted as the orders of accuracy and dissipation of the operator, respectively. In the results below case (a) is included in the statements by setting $s = \infty$.

We shall now present the main results of our paper.

THEOREM 1.1. *There are constants c and C such that for any n and k,*

$$c n^{r|\frac{1}{2}-p^{-1}|(r^{-1}-s^{-1})} \leqq \|E_k^n\|_p \leqq C n^{r|\frac{1}{2}-p^{-1}|(r^{-1}-s^{-1})} .$$

In particular E_k is stable in L_p if and only if $r = s$ or $p = 2$.

Here and below c and C will denote small and large positive constants, respectively, not necessarily the same each time.

To formulate the result on the rate of convergence we need in addition to L_p the homogeneous Besov spaces $B_p^{\alpha, q*}$ which are defined as follows (cf. e.g. [10]). Set

$$\omega_{1,p}(t,u) = \sup_{|h| \leqq t} \|u(\cdot + h) - u\|_p ,$$
$$\omega_{2,p}(t,u) = \sup_{|h| \leqq t} \|u(\cdot + h) - 2u + u(\cdot - h)\|_p .$$

For $\alpha > 0$ let $\alpha = (\alpha) + \bar{\alpha}$, where (α) is the largest integer $< \alpha$ and $0 < \bar{\alpha} \leqq 1$. Then $B_p^{\alpha, q*}$ is defined as the completion of $\mathscr{S}$ in the (semi-) norm

$$(1.4)\qquad \|u\|_{B_p^{\alpha,q*}} = \begin{cases} \left(\int_0^\infty (t^{-\bar\alpha}\omega_{1,p}(t,D^{(\alpha)}u))^q \frac{dt}{t}\right)^{1/q}, & 0<\bar\alpha<1, \\ \left(\int_0^\infty (t^{-1}\omega_{2,p}(t,D^{(\alpha)}u))^q \frac{dt}{t}\right)^{1/q}, & \bar\alpha=1, \end{cases}$$

with the usual interpretations for $q=\infty$, and with $D=d/dx$.

THEOREM 1.2. *For* $0\leqq\alpha<r$ *and* $\alpha\neq r|\frac{1}{2}-p^{-1}|$ *we have*

$$\|(E_k^n-E(nk))v\|_p \leqq Ch^{\beta(\alpha)}\|v\|_{B_p^{\alpha,\infty*}},$$

where

$$\beta(\alpha) = \alpha(1-r^{-1}) + \min(0,(\alpha-r|\tfrac{1}{2}-p^{-1}|)(r^{-1}-s^{-1})).$$

In the stable cases, that is, if $r=s$ or $p=2$, the order of convergence is $\beta(\alpha)=\alpha(1-r^{-1})$ when $0\leqq\alpha<r$. In the opposite case the error is larger; for small α, $\beta(\alpha)$ is then negative and for $\alpha=0$ we recognize the exponent in Theorem 1.1. We will also prove a corresponding lower estimate for small h in Theorem 5.2 below.

One may ask if by some smoothing device it is possible to curb the effect of non-stability. We shall indeed construct mean value type operators G_h depending on parameters μ and ν such that the following result holds.

THEOREM 1.3. *For* $0\leqq\alpha<\min(\mu,r)$, $\alpha\neq r|\frac{1}{2}-p^{-1}|-\nu$, *we have*

$$\|(E_k^nG_h-E(nk))v\|_p \leqq Ch^{\tilde\beta(\alpha)}\|v\|_{B_p^{\alpha,\infty*}},$$

where

$$(1.5)\qquad \tilde\beta(\alpha) = \alpha(1-r^{-1}) + \min(0,(\alpha+\nu-r|\tfrac{1}{2}-p^{-1})(r^{-1}-s^{-1})).$$

In particular, if $\nu>r|\frac{1}{2}-p^{-1}|$, we have the full rate of convergence $\alpha(1-r^{-1})$ for $0\leqq\alpha<\min(\mu,r)$.

The standard examples of difference operators which are stable in L_2 but not in L_p, $p\neq 2$, are the operators corresponding to

$$(1.6)\qquad a_1(y) = \varrho^2\lambda^2\cos(2\pi y)-i\varrho\lambda\sin(2\pi y)+1-\varrho^2\lambda^2,$$

$$(1.7)\qquad a_2(y) = \frac{1-\varrho\lambda+(1+\varrho\lambda)\exp(2\pi iy)}{1+\varrho\lambda+(1-\varrho\lambda)\exp(2\pi iy)}.$$

The first operator, the Lax–Wendroff [8] operator, satisfies condition (b) and $r=3$, $s=4$, and the second operator, proposed by Wendroff [17] (cf. also [15]) satisfies (a) and $r=3$, $s=\infty$. In these cases we have by Theorem 1.2 convergence for $v\in B_p^\alpha$ with $\alpha>\frac{1}{3}|\frac{1}{2}-p^{-1}|$ and $\alpha>|\frac{1}{2}-p^{-1}|$, respectively.

Using a smoothing operator with $\nu=2$ and $\mu=3$, we get convergence for $0<\alpha\leqq 3$ with order $\frac{2}{3}\alpha$. Already for $\nu=1$ we get convergence for $\alpha>\frac{1}{3}|\frac{1}{2}-p^{-1}|-\frac{1}{6}$ and for $\alpha>|\frac{1}{2}-p^{-1}|-\frac{1}{3}$, respectively. Examples of such smoothing operators will be discussed in Section 6 below.

In the stable cases the results in Theorems 1.1 and 1.2 were contained in [15] and [10]. Our main interest here is in the case of non-stability. For $p=\infty$ one can easily see that

$$\|E_k^n\|_\infty = \textstyle\sum_j |a_{nj}| ,$$

where a_{nj} are the Fourier coefficients of $a(y)^n$. Using the saddle point method for estimating a_{nj}, Serdjukova [11], [12], and Hedstrom [2], [3], [4] were able to prove the results in Theorem 1.1 for this case, and in [4] Hedstrom succeded in obtaining the result corresponding to Theorem 1.2.

In our approach we notice that

$$\|E_k^n\|_p = M_p(a^n) ,$$

where $M_p(\cdot)$ denotes the Fourier multiplier norm

$$\begin{aligned} M_p(\varphi) &= \sup\{\|\hat{\varphi}*v\|_p ;\ \|v\|_p \leqq 1\} \\ &= \sup\{\|\varphi\hat{v}\|_p ;\ \|\hat{v}\|_p \leqq 1\} , \end{aligned}$$

where $\hat{\varphi}$ is the Fourier transform of φ. We then apply a number of known properties of these norms, as described in e.g. Hörmander [5]. The central tool in the estimate will be the Carlson–Beurling inequality

$$\|\hat{\varphi}\|_1 \leqq (2\|\varphi\|_2\|D\varphi\|_2)^{\frac{1}{2}} .$$

This inequality was used for similar purposes by J.-P. Kahane (see [6, p. 103]. In the technical parts of the paper we shall not work with the norm (1.4) but rather with a Bessel potential type norm, which is computationally more convenient. By a simple interpolation argument we can then conclude that the same results hold in the norm (1.4).

The qualitative question of stability in L_p was discussed by multiplier methods in a paper presented by one of the authors [15] at the XIV Congress of Scandinavian mathematicians, Copenhagen 1964. Since it now appears that the proceedings of that conference will not be published we include in Section 2 below the main result of that paper (cf. also [16]).

The construction of the mean-value operators G_h in Theorem 3 is adapted from [7], where similar operators were used to increase the rate of convergence for parabolic initial-value problems with non-smooth initial data.

Throughout the paper we assume analyticity of the characteristic function a. Except in Theorem 3.1 this is not essential. We will actually only use what amounts to a C^1-condition in proving the upper estimates, and a C^2-condition in the proofs of the estimates from below. We do not pursue these questions further here.

2. L_p-spaces and Fourier multipliers.

In this section we shall describe some results on Fourier multipliers. Most of these are found in Hörmander [5]; the reader is referred to that paper for details.

Let $\mathscr{S}$ denote the topological space of C^∞ functions on R^1 defined by the seminorms

$$p_{mn}(u) = \sup_x |x^m (d/dx)^n u(x)|, \quad m,n = 0,1,\ldots ,$$

and let $\mathscr{S}'$ be its dual, the space of tempered distributions. For $u \in \mathscr{S}$, the Fourier transform $\mathscr{F}u = \hat{u} \in \mathscr{S}$ is defined by

$$(\mathscr{F}u)(y) = \int_{-\infty}^{\infty} e^{2\pi ixy}\, u(x)\, dx ,$$

and for $u \in \mathscr{S}'$, $\mathscr{F}u = \hat{u}$ is defined by $\hat{u}(v) = u(\hat{v})$ when $v \in \mathscr{S}$.

Let A be a bounded linear translation invariant operator on L_p. Then there exists $a \in \mathscr{S}'$, the symbol of A, such that

$$Au = \mathscr{F}^{-1}a * u = \mathscr{F}^{-1}a\mathscr{F}u = \mathscr{F}^{-1}(a\mathscr{F}u) , \quad u \in \mathscr{S} . \tag{2.1}$$

Conversely, any $a \in \mathscr{S}'$ such that

$$\begin{aligned} M_p(a) &= \sup\{\|\mathscr{F}^{-1}a*u\|_p ;\, u \in \mathscr{S}, \|u\|_p = 1\} \\ &= \sup\{\|\mathscr{F}(av)\|_p ;\, v \in \mathscr{S}, \|\hat{v}\|_p = 1\} < \infty \end{aligned} \tag{2.2}$$

defines by (2.1) a bounded linear translation invariant operator A on L_p and we have $\|A\|_p = M_p(a)$. The set of $a \in \mathscr{S}'$ for which (2.2) holds is denoted by M_p. Let us collect some of the fundamental facts about M_p in the following two lemmas.

LEMMA 2.1. (i) *M_p is a Banach algebra under pointwise addition and multiplication, with norm $M_p(\cdot)$.*

(ii) *$M_2 = L_\infty$ and $M_2(a) = \|a\|_\infty$.*

(iii) *M_1 is the set of Fourier–Stieltjes transforms on R^1 and*

$$M_1(a) = \int |d\mu| \quad \text{if} \quad a(y) = \int e^{2\pi ixy}\, d\mu .$$

In particular, $a \in M_1$ if $\hat{a} \in L_1$ and $M_1(a) = \|\hat{a}\|_1$.

(iv) *For* $p^{-1}+p'^{-1}=1$ *we have* $M_p=M_{p'}$ *with equality of norms, and* $M_1 \subseteqq M_p \subseteqq M_2$. $M_p(a)$ *is logarithmically convex in* p^{-1}. *In particular, for* $p \geqq 2$,

$$M_p(a) \leqq M_1(a)^{1-2p^{-1}} M_2(a)^{2p^{-1}}.$$

(v) *If* α, β, γ, δ *are real numbers and* $\tilde{a}(y)=e^{i(\alpha+\beta y)}a(\gamma+\delta y)$, *then* $\tilde{a} \in M_p$ *if and only if* $a \in M_p$ *and* $M_p(\tilde{a})=M_p(a)$.

LEMMA 2.2. (i) *Assume that* $\{a_n\}_{n=1}^{\infty} \subset M_p$ *is a sequence such that* $\sup_n M_p(a_n) < \infty$ *and* $a_n \to a$ *in* $\mathscr{S}'$ *as* $n \to \infty$. *Then* $a \in M_p$.

(ii) *Let* $p \neq 2$ *and assume that the function* $f \in C^2$ *is real and such that* $\sup_n M_p(\exp(inf)) < \infty$. *Then* f *is linear.*

We need the following simple consequence.

LEMMA 2.3. *If* α *is real,* $\nu > 1$, *then* $\exp(i\alpha y^\nu) \notin M_p$ *for* $p \neq 2$.

PROOF. Assuming $\exp(i\alpha y^\nu) \in M_p$ we obtain by Lemma 2.1 (v) that

$$M_p(\exp(in\alpha y^\nu)) = M_p(\exp(i\alpha y^\nu)) = \text{constant},$$

and so by Lemma 2.2 (ii) that αy^ν is linear contrary to our assumptions.

The following lemma is closely related.

LEMMA 2.4. *Let* $u \in C_0^1$ *and let* $\psi \in C^2$ *be real and* $|\psi''| \geqq \delta > 0$ *in an interval containing the support of* u. *Then*

$$\|\mathscr{F}(\exp(i\psi)u)\|_\infty \leqq 8\delta^{-\frac{1}{2}}\|u'\|_1.$$

PROOF. Let y_0 be in the support of u. We have

$$\int \exp(i\psi(y)+2\pi ixy)\, u(y)\, dy = \int u'(y)\left(\int_{y_0}^{y} \exp(i\psi(y')+2\pi ixy')\, dy'\right) dy.$$

By van der Corput's lemma (cf. [18, p. 197]) we have

$$\left|\int_{y_0}^{y} \exp(i\psi(y')+2\pi ixy')\, dy'\right| \leqq 8\delta^{-\frac{1}{2}},$$

which proves the result.

The main technical tools below will be the following estimates.

LEMMA 2.5. *Assume that* $a \in L_2$, $a' \in L_2$. *Then*

(i) $\hat{a} \in L_1$ *and*

$$\|\hat{a}\|_1 \leqq (2\|a\|_2\|a'\|_2)^{\frac{1}{2}},$$

(ii) $a \in M_p$ *and for* $p \geqq 2$,

$$M_p(a) \leqq 2^{\frac{1}{2}-p^{-1}} \|a\|_\infty^{2p^{-1}} \|a\|_2^{\frac{1}{2}-p^{-1}} \|a'\|_2^{\frac{1}{2}-p^{-1}} .$$

PROOF. The first inequality is the Carlson–Beurling inequality (cf. e.g. [1]). The second inequality follows from the first and Lemma 2.1 (iv).

We shall also use the following result.

LEMMA 2.6. *Let* $a \in M_p$. *Then*

$$\lim_{n\to\infty} M_p(a^n)^{1/n} = \|a\|_\infty .$$

PROOF. For $p=2$ this follows at once from Lemma 2.1 (ii) and for $p=\infty$ it is a result by Beurling [1]. In the general case it follows then from Lemma 2.1 (iv).

Consider now in particular the translation invariant operator E_k in (1.1). This operator has the symbol $a(hy)$ where a is defined by

$$a(y) = \textstyle\sum_j a_j \exp(2\pi ijy) .$$

We assume that a is analytic; in applications to difference schemes a is always a rational trigonometric function. We have here by Lemma 2.1 (iii),

$$\|E_k\|_\infty = M_\infty(a) = \textstyle\sum_j |a_j| < +\infty ,$$

so that $a \in M_p$ for all p. We shall sometimes use the equivalent norm to $M_p(a)$ described in the following lemma.

LEMMA 2.7. *Let* $\eta \in C_0^\infty$ *and* $\eta=1$ *in an interval of length* 1. *Then for any* $a \in M_p$ *which is periodic of period* 1,

$$c M_p(a) \leqq M_p(\eta a) \leqq C M_p(a) ,$$

where c, C *are independent of* a.

PROOF. Trivial consequence of the closed graph theorem.

For real $\alpha>0$ let $\alpha=[\alpha]+\underline{\alpha}$ where $[\alpha]$ is the integral part of α and $0 \leqq \underline{\alpha} < 1$ and set $\omega_\alpha(y)=y^{[\alpha]}|y|^{\underline{\alpha}}$. For $u \in \mathscr{S}$ we define

$$\|u\|_{p,\alpha}^* = \|\mathscr{F}^{-1}\omega_\alpha \mathscr{F}u\|_p . \tag{2.3}$$

This is well defined since $|y|^{\underline{\alpha}}$ is locally in $FL_1 \subseteqq M_p$ (cf. [18, p. 241]). The closure of $\mathscr{S}$ in this norm is denoted $L_{p,\alpha}^*$. For α integer we have

$$\|u\|_{p,\alpha}^* = \|D^\alpha u\|_p .$$

For $1<p<\infty$ we could have used $|y|^\alpha$ instead of $\omega_\alpha(y)$ in (2.3) since $\operatorname{sign}(y) \in M_p$ for such p.

In order to describe these spaces we shall compare them with the spaces $B_p^{\alpha,q*}$. We shall need the following well-known partition of unity (cf. [5, p. 121]).

Lemma 2.8. *There is a function $\varphi \in C_0^\infty$ with support in $\{y; \frac{1}{2} < |y| < 2\}$ such that*

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}y) = 1, \quad y \neq 0 .$$

We can now prove the following embedding result (cf. e.g. [14]).

Lemma 2.9. *For any $\alpha > 0$, $B_p^{\alpha,1*} \subseteq L_{p,\alpha}^* \subseteq B_p^{\alpha,\infty*}$.*

Proof. Let $\psi_j(y) = \varphi(2^{-j}y)$ where φ is the function in Lemma 2.8. By [9] it is known that an equivalent norm for $B_p^{\alpha,q*}$ is

$$|||u|||_{B_p^{\alpha,q*}} = \left\{\sum_{-\infty}^{\infty} (\|\mathscr{F}^{-1}\psi_j \mathscr{F}u\|_p 2^{\alpha j})^q\right\}^{1/q},$$

again with the usual interpretation for $q = \infty$. Simple calculations give

$$\|\psi_j \omega_\alpha\|_\infty \leqq C 2^{\alpha j},$$
$$\|\psi_j \omega_\alpha\|_2 \leqq C 2^{(\alpha+\frac{1}{2})j},$$
$$\|D(\psi_j \omega_\alpha)\|_2 \leqq C 2^{(\alpha-\frac{1}{2})j},$$

and hence by Lemma 2.5 (ii),

$$M_p(\psi_j \omega_\alpha) \leqq C 2^{\alpha j}.$$

In the same way we have

$$M_p(\psi_j \omega_\alpha^{-1}) \leqq C 2^{-\alpha j}.$$

We therefore have

$$\|\mathscr{F}^{-1}\psi_j \mathscr{F}u\|_p$$
$$= \|\mathscr{F}^{-1}\psi_j \omega_\alpha^{-1} \omega_\alpha \mathscr{F}u\|_p \leqq M_p(\psi_j \omega_\alpha^{-1}) \|\mathscr{F}^{-1}\omega_\alpha \mathscr{F}u\|_p \leqq C 2^{-\alpha j} \|u\|_{p,\alpha}^*,$$

so that

$$\|u\|_{B_p^{\alpha,\infty*}} \leqq C \|u\|_{p,\alpha}^* .$$

On the other hand

$$\|\mathscr{F}^{-1}\omega_\alpha \mathscr{F}u\|_p = \textstyle\sum_{|j-1|\leqq 1} \|\mathscr{F}^{-1}(\psi_1 \psi_j \omega_\alpha)\mathscr{F}u\|_p$$
$$\leqq \textstyle\sum_{|j-1|\leqq 1} M_p(\psi_1 \omega_\alpha) \|\mathscr{F}^{-1}\psi_j \mathscr{F}u\|_p \leqq C \sum_j 2^{\alpha j} \|\mathscr{F}^{-1}\psi_j \mathscr{F}u\|_p ,$$

or

$$\|u\|_{p,\alpha}^* \leqq C \|u\|_{B_p^{\alpha,1*}} ,$$

which completes the proof.

The spaces $B_p^{\alpha, q*}$ have the following interpolation property.

LEMMA 2.10. *Let* $0<\alpha_0<\alpha_1$, $1 \leqq q_j \leqq \infty$, $j=0,1$, *and let* $0<\theta<1$. *Then there is a constant* C *such that if the operator* A *has the property*

$$\|Au\|_p \leqq C_j \|u\|_{B_p^{\alpha_j, q_j*}}, \quad j=0,1, \tag{2.4}$$

then

$$\|Au\|_p \leqq C C_0^\theta C_1^{1-\theta} \|u\|_{B_p^{\alpha,\infty*}}, \quad \alpha=(1-\theta)\alpha_0+\theta\alpha_1 .$$

The same conclusion holds if in (2.4) *we replace the norm in* B_p^{α, q_j*} *by the* $L^*_{p,\alpha}$*-norm.*

PROOF. For the first part see [10]. The second part then follows by Lemma 2.9.

We shall need the following lemma, which is a trivial consequence of our definitions.

LEMMA 2.11. *Let* $a \in M_p$. *Then*

$$\sup\{\|\mathscr{F}^{-1}a\mathscr{F}u\|_p;\ \|u\|^*_{p,\alpha} \leqq 1\} = M_p(\omega_\alpha^{-1}a) .$$

If the sup *is infinite we interpret this to mean* $\omega_\alpha^{-1}a \notin M_p$.

We shall make frequent use of the following trivial consequence of Lemma 2.1 (v); in fact it is this observation together with Lemma 2.11 which makes the $L^*_{p,\alpha}$-spaces more convenient for our purposes than the $B_p^{\alpha, q*}$-spaces.

LEMMA 2.12. *For* $\lambda > 0$ *we have*

$$M_p(\omega_\alpha^{-1}a(\lambda\cdot)) = \lambda^\alpha M_p(\omega_\alpha^{-1}a) .$$

3. Stability.

In this section we shall study the question of stability in L_p of operators of the form (1.1). By the above, stability in L_p is equivalent to

$$\sup_n M_p(a^n) < +\infty , \tag{3.1}$$

where a is the characteristic function (1.3) of E_k. By Lemma 2.1 (ii), we find that E_k is stable in L_2 if and only if

$$|a(y)| \leqq 1, \quad y \in R . \tag{3.2}$$

For $p \neq 2$ the situation is more complicated. We then have the following theorem.

Theorem 3.1. *Let $p \neq 2$. Then E_k is stable in L_p if and only if one of the following two conditions is satisfied, namely*

(i) $a(y) = c \exp(2\pi i j y)$, $|c| = 1$, *some* j,

(ii) $|a(y)| < 1$ *except for at most a finite number of points* y_q, $q = 1, \ldots, Q$, *in* $[0,1)$ *where* $|a(y)| = 1$. *For* $q = 1, \ldots, Q$ *there are constants* α_q, β_q, ν_q *where* α_q *is real,* $\operatorname{Re}\beta_q > 0$, *and* ν_q *is an even natural number such that*

$$a(y_q + y) = a(y_q) \exp\left(i\alpha_q y - \beta_q y^{\nu_q}(1 + o(1))\right) \quad \text{as } y \to 0 .$$

Proof. The sufficiency of these conditions for stability was established for $p = \infty$ by Strang [13] (cf. also [16]). By Lemma 2.1, stability in this case implies stability in L_p for $1 \leqq p \leqq \infty$.

To prove the necessity of the conditions, we first notice that stability in L_2, and thus (3.2) is a necessary condition. It follows, since a is analytic, that one of the following two conditions is satisfied, namely

(i)′ $|a(y)| \equiv 1$, $y \in \mathsf{R}$,

(ii)′ $|a(y)| < 1$ for all but a finite number of points y_q, $q = 1, \ldots, Q$ in $[0,1)$.

We shall prove that if (3.1) holds, then (i)′ implies (i) and (ii)′ implies (ii). Assume that this were not so. In both cases it would then have been possible to find $y_0 \in \mathsf{R}$ with $|a(y_0)| = 1$ and α, β, ν with α, β real, $\beta \neq 0$, $\nu > 1$, such that

$$a(y_0 + y) = a(y_0) \exp\left(i\alpha y + i\beta y^{\nu}(1 + o(1))\right) \quad \text{as } y \to 0 .$$

By Lemma 2.1 (v) we then conclude that (3.1) holds with a replaced by

$$b_n(y) = a^{-1}(y_0)\, a(y_0 + y n^{-\nu^{-1}}) \exp(-i\alpha y n^{-\nu^{-1}}) .$$

Therefore, since $|b_n(y)| \leqq 1$ and

$$\lim_{n\to\infty} b_n(y)^n = \exp(i\beta y^{\nu}) ,$$

uniformly on compact sets, it would follow by Lemma 2.2 (i) that $\exp(i\beta y^{\nu}) \in M_p$ which is in contradiction to Lemma 2.3.

4. The rate of growth.

In this section we shall study the growth rate of $\|E_k^n\|_p$ in the case that E_k is stable in L_2 but unstable in L_p for $p \neq 2$. We have then as above that (3.2) holds and hence that again (i)′ or (ii)′ is satisfied. Consider first the case (i)′. We then have the following result.

Theorem 4.1. *Assume that* (i)′, *but not* (i) *is satisfied. Then*

$$c n^{|\frac{1}{2} - p^{-1}|} \leqq \|E_k^n\|_p \leqq C n^{|\frac{1}{2} - p^{-1}|} .$$

PROOF. By Lemma 2.1 (iv) we may restrict ourselves to the case $p \geqq 2$. Let η be as in Lemma 2.7. We then have by Lemmas 2.7 and 2.5

$$M_p(a^n) \leqq CM_p(\eta a^n) \leqq C(\|\eta a^n\|_2 \|D(\eta a^n)\|_2)^{\frac{1}{2}-p^{-1}} \leqq Cn^{\frac{1}{2}-p^{-1}},$$

which proves the estimate from above.

To prove the estimate from below we write $a(y) = \exp(i\psi(y))$, where by assumption $\psi'' \not\equiv 0$. Let $\eta \in C_0^\infty$, $\eta \not\equiv 0$, have support in an interval where $\psi'' \neq 0$. We obtain by Parseval's relation and Hölder's inequality

$$(4.1)\quad 0 < \|\eta\|_2^{2(1-p^{-1})} = \|\eta a^n\|_2^{2(1-p^{-1})} = \|\mathscr{F}(\eta a^n)\|_2^{2(1-p^{-1})} \leqq C\|\mathscr{F}(\eta a^n)\|_{p'} \|\mathscr{F}(\eta a^n)\|_\infty^{1-2p^{-1}},$$

where $p'^{-1} + p^{-1} = 1$. On the other hand,

$$(4.2)\qquad \|\mathscr{F}(a^n\eta)\|_{p'} \leqq M_p(a^n)\, \|\mathscr{F}\eta\|_{p'},$$

and by Lemma 2.4,

$$(4.3)\qquad \|\mathscr{F}(\eta a^n)\|_\infty \leqq Cn^{-\frac{1}{2}}.$$

Together, (4.1), (4.2), and (4.3) prove the estimate from below.

We now turn to the case (ii)′. We shall first prove that the growth rate of $M_p(a^n)$ depends only upon the behavior of a in a neighborhood of the points y_q, $q = 1, \ldots, Q$. Let δ be a positive number, smaller than the distance modulo 1 between the y_q. Let η be a C^∞ periodic function with $|\eta| \leqq 1$ and

$$\eta(y) = 1 \quad \text{for } |y| \leqq \tfrac{1}{4}\delta, \qquad = 0 \quad \text{for } \tfrac{1}{2}\delta \leqq |y| \leqq \tfrac{1}{2},$$

and set $\eta_q(y) = \eta(y - y_q)$, $a_q = \eta_q a$. We then have the following result.

LEMMA 4.1. *With the above notation there is a positive c such that*

$$c \max_{q=1,\ldots,Q} M_p(a_q^n) + o(1) \leqq M_p(a^n) \leqq \sum_{q=1}^{Q} M_p(a_q^n) + o(1) \quad \textit{as } n \to \infty.$$

PROOF. We first prove the estimate from above. We have

$$a(y)^n - \sum_{q=1}^{Q} a_q^n(y)^n = a(y)^n \left(1 - \sum_{q=1}^{Q} \eta(y-y_q)^n\right) = a(y)^n \chi_n(y).$$

Since $\chi_n(y)$ vanish in a constant neighborhood of the y_q we may change a in this neighborhood without changing the value of the product. Therefore, if $\varkappa$ satisfies

$$\sup\{|a(y)|;\ \chi_n(y) \neq 0\} < \varkappa < 1,$$

we obtain by Lemma 2.6 for large n,

$$M_p(a^n)-\sum_{q=1}^{Q} M_p(a_q{}^n) \leqq M_p(a^n\chi_n) \leqq \varkappa^n M_p(\chi_n) \leqq \varkappa^n(1+QM_p(\eta^n)) ,$$

and since $|\eta| \leqq 1$, one more application of Lemma 2.6 proves that the last expression is $o(1)$ as $n \to \infty$.

Consider now the estimate from below. Let $\zeta_q \in C^\infty$ be periodic and equal to 1 near y_q and have support where $\eta_q=1$. Then

$$a_q{}^n = \zeta_q a^n+(1-\zeta_q)(\eta_q a)^n .$$

By the same reasoning as above,

$$\lim_{n\to\infty} M_p((1-\zeta_q)(\eta_q a)^n) = 0 ,$$

and hence

$$M_p(a_q{}^n) \leqq M_p(\zeta_q)M_p(a^n) + o(1) \quad \text{as } n \to \infty .$$

Since q is arbitrary, this proves the estimate from below.

Consider the behavior of a in a neighborhood of y_q. Since (ii)$'$ holds, we may write

$$a(y_q+y) = a(y_q)\exp(i\alpha_q y+\psi_q(y)) ,$$

where α_q is real and $\operatorname{Re}\psi_q(y)<0$ for $0<|y|<\frac{1}{2}\delta$. By the analyticity we have as $y \to 0$,

$$\begin{aligned} \psi_q(y) &= \beta_q y^{r_q}(1+o(1)), & \beta_q \neq 0,\ r_q>1 ,\\ \operatorname{Re}\psi_q(y) &= -\gamma_q y^{s_q}(1+o(1)), & \gamma_q>0,\ s_q \geqq r_q . \end{aligned}$$

Setting

$$\mu = \max_{q=1,\ldots,Q}\left(1-\frac{r_q}{s_q}\right), \tag{4.4}$$

we have the following result.

THEOREM 4.2. *Assume that* (ii)$'$ *holds. Then*

$$cn^{|\frac{1}{2}-p^{-1}|\mu} \leqq \|E_k^n\|_p \leqq Cn^{|\frac{1}{2}-p^{-1}|\mu} ,$$

where μ *is defined by* (4.4).

PROOF. Again we can assume $p \geqq 2$. By Lemma 4.1 it is sufficient to consider the case $Q=1$, and by Lemma 2.1 (v) we can restrict ourselves to $y_1=0$, $a(y_1)=1$, $\alpha_1=0$. Further, for the case $r_1=s_1$, the result is contained in Theorem 3.1 so that we may here assume $r_1<s_1$. In that case β_1 is purely imaginary. Thus, dropping subscripts, let

$$\begin{aligned} |a(y)| &< 1, \quad 0<|y|\leqq \tfrac{1}{2} ,\\ a(y) &= \exp(\psi(y)) ,\\ \psi(y) &= i\beta y^r(1+o(1)), \quad \beta \neq 0 \text{ real},\ r>1 ,\\ \operatorname{Re}\psi(y) &= -\gamma y^s(1+o(1)), \quad \gamma>0,\ s>r,\ s \text{ even} . \end{aligned}$$

Consider first the estimate from above. Let $\eta \in C^\infty$ with $|\eta| \leqq 1$ be equal to 1 on $|y| \leqq \frac{1}{2}$ and vanish for $|y| \geqq \frac{3}{4}$. Under these assumptions we shall estimate $M_p(\eta a^n)$. We have in the support of η,

$$|a(y)| \leqq \exp(-c|y|^s) ,$$
$$|Da(y)| \leqq C|y|^{r-1} \exp(-c|y|^s) .$$

Hence

$$\|\eta a^n\|_2^2 \leqq \int \exp(-2cn|y|^s)\, dy \leqq Cn^{-s^{-1}} ,$$
$$\|D(\eta a^n)\|_2^2 \leqq Cn^2 \int |y|^{2(r-1)} \exp(-2cn|y|^s)\, dy \leqq Cn^{2-(2r-1)s^{-1}} .$$

By Lemmas 2.7 and 2.5 this gives

$$\|E_k^n\|_p \leqq CM_p(\eta a^n) \leqq Cn^{(\frac{1}{2}-p^{-1})(1-rs^{-1})} .$$

We now turn to the estimate from below. Let $\eta \in C_0^\infty$, $\eta \not\equiv 0$ be a function with support not containing 0, and set $\eta_n(y) = n^{\frac{1}{2}s^{-1}} \eta(n^{s^{-1}} y)$. We have

$$\lim_{n\to\infty} \|\eta_n a^n\|_2^2 = \lim_{n\to\infty} \int |\eta(y)|^2\, |a(n^{-s^{-1}} y)|^{2n}\, dy$$
$$= \int |\eta(y)|^2 \exp(-2\gamma|y|^s)\, dy > 0 .$$

Therefore, as in the proof of Theorem 4.1, for large n,

$$(4.5) \qquad c \leqq \|\eta_n a^n\|_2^{2(1-p^{-1})} \leqq C\|\mathscr{F}(\eta_n a^n)\|_{p'} \|\mathscr{F}(\eta_n a^n)\|_\infty^{1-2p^{-1}}$$
$$\leqq CM_p(a^n)\, \|\hat{\eta}_n\|_{p'}\, \|\mathscr{F}(\eta_n a^n)\|_\infty^{1-2p^{-1}} .$$

A trivial calculation gives

$$(4.6) \qquad \|\hat{\eta}_n\|_{p'} = \|\hat{\eta}\|_{p'}\, n^{(\frac{1}{2}-p^{-1})s^{-1}} .$$

With $x' = n^{-s^{-1}} x$ we have

$$(4.7) \qquad \mathscr{F}(\eta_n a^n)(x) = n^{-\frac{1}{2}s^{-1}} \int \exp(ix'y)\, \eta(y)\, a(n^{-s^{-1}} y)^n\, dy .$$

For y in the support of η we have

$$a(n^{-s^{-1}} y)^n = \exp\big(in^{1-rs^{-1}} \psi_n(y)\big)\, \chi_n(y) ,$$

where ψ_n is real and where $(\psi_n'')^{-1}$ and χ_n' are bounded, uniformly in n. Therefore, by (4.7) and Lemma 2.4 we obtain

$$(4.8) \qquad \|\mathscr{F}(\eta_n a^n)\|_\infty \leqq Cn^{-\frac{1}{2}s^{-1}-\frac{1}{2}(1-rs^{-1})} .$$

Together, (4.5), (4.6), and (4.8) now complete the proof.

We notice for later use that during the course of the proofs of the estimates from below in Theorems 4.1 and 4.2 we have actually proved the following stronger result.

Lemma 4.2. *Assume that the assumptions of Theorems* 4.1 *or* 4.2 *are satisfied. Then, if* $\chi \in C_0^\infty$, $\chi \not\equiv 0$, *is a function with support not containing the origin, we have*

$$M_p(\chi a(n^{-s^{-1}} \cdot)^n) \geqq c n^{|\frac{1}{2}-p^{-1}|(1-rs^{-1})} .$$

5. The rate of convergence.

In this section we shall prove the $L_{p\alpha}^*$ analogue of Theorem 1.2. We shall assume that condition (a) or (b) in the introduction is satisfied; r, s and $\beta(\alpha)$ will have the same meaning as there.

Theorem 5.1. *Under the above assumptions, we have for* $0 \leqq \alpha \leqq r$, $\alpha \neq r|\frac{1}{2}-p^{-1}|$, *and* $v \in L_{p,\alpha}^*$,

$$\|(E_k^n - E(nk))v\|_p \leqq C h^{\beta(\alpha)} \|v\|_{p,\alpha}^*, \quad nk \leqq T .$$

In view of Lemmas 2.9 and 2.10 this also proves Theorem 1.2.

Proof. The operator $E_k^n - E(nk)$ corresponds on the Fourier transform side to multiplication by

$$a(hy)^n - \exp(-ink\varrho y) = \exp(-ink\varrho y)\,(a_\varrho(hy)^n - 1) ,$$

where $a_\varrho(y) = \exp(i\lambda\varrho y)\, a(y)$. Hence by Lemma 2.11 we have to prove that

$$M_p\big(\omega_\alpha^{-1}(a_\varrho(h\ \cdot)^n - 1)\big) \leqq C h^{\beta(\alpha)}, \quad nk \leqq T ,$$

or, after changing variables and setting $\sigma_{\alpha,h,n} = \omega_\alpha^{-1}\big(a_\varrho(h^{r^{-1}}\cdot)^n - 1\big)$, that

$$M_p(\sigma_{\alpha,h,n}) \leqq C h^{\beta_0(\alpha)}, \quad nk \leqq T , \tag{5.1}$$

where

$$\beta_0(\alpha) = \beta(\alpha) - \alpha(1-r^{-1}) = \min\big(0, (\alpha - r|\tfrac{1}{2}-p^{-1}|)(r^{-1}-s^{-1})\big) .$$

Again it is sufficient to consider $p \geqq 2$. Let φ be the function in Lemma 2.8 and set

$$\varphi_j(y) = \varphi(2^{-j}y), \quad j = 1, 2, \dots ,$$

$$\varphi_0(y) = 1 - \sum_{j=1}^{\infty} \varphi_j(y) ,$$

$$\Phi_J(y) = \varphi_0(y) + \sum_{j=1}^{J} \varphi_j(y), \quad J = 1, 2, \dots .$$

By our assumptions we have

$$\begin{aligned} |a_\varrho(h^{r^{-1}}y)^n - 1| &\leqq C \min(|y|^r, 1) , \\ |D(a_\varrho(h^{r^{-1}}y)^n - 1)| &\leqq C |y|^{r-1} . \end{aligned} \tag{5.2}$$

Hence we obtain for $0 \leqq \alpha \leqq r$

$$\|\varphi_j \sigma_{\alpha,h,n}\|_\infty \leqq C 2^{-\alpha j}, \quad j \geqq 0 ,$$
$$\|\varphi_j \sigma_{\alpha,h,n}\|_2 \leqq C 2^{(-\alpha+\frac{1}{2})j}, \quad j \geqq 0 ,$$
$$\|D(\varphi_j \sigma_{\alpha,h,n})\|_2 \leqq C 2^{(-\alpha+r-\frac{1}{2})j}, \quad j > 0 .$$

By (5.2) the last inequality still holds for $j=0$ and $0 \leqq \alpha \leqq r-1$. For $j=0$ and $r-1<\alpha<r$ the function $D(\varphi_j \sigma_{\alpha,h,n})$ has a singularity at $y=0$, but not for $\alpha=r$, as a simple computation proves. Hence by Lemma 2.5 (ii), we have for $j>0$, $0 \leqq \alpha \leqq r$, and for $j=0$, if $0 \leqq \alpha \leqq r-1$ or if $\alpha=r$ the estimate,

$$M_p(\varphi_j \sigma_{\alpha,h,n}) \leqq C 2^{(-\alpha+r|\frac{1}{2}-p^{-1}|)j} . \tag{5.3}$$

For $r-1<\alpha<r$ we write $r=\alpha+\beta$, $0<\beta<1$. Let $\chi \in C_0^\infty$ be 1 in the support of φ_0. We then have $\varphi_0 \omega_\alpha^{-1} = \chi |y|^\beta \varphi_0 \omega_r^{-1}$. But for $0<\beta<1$,

$$\chi(y)|y|^\beta \in FL_1 \subseteqq M_p .$$

Hence (5.3) is proved for $j \geqq 0$, $0 \leqq \alpha \leqq r$.

Together the estimates (5.3) give by addition

$$M_p(\Phi_J \sigma_{\alpha,h,n}) \leqq \begin{cases} C 2^{(-\alpha+r|\frac{1}{2}-p^{-1}|)J}, & 0 \leqq \alpha < r|\frac{1}{2}-p^{-1}| , \\ C, & r|\frac{1}{2}-p^{-1}| < \alpha \leqq r . \end{cases} \tag{5.4}$$

For $r|\frac{1}{2}-p^{-1}|<\alpha \leqq r$ we may let J tend to infinity to prove (5.1) in this case. For the case $0 \leqq \alpha < r|\frac{1}{2}-p^{-1}|$, we notice that

$$M_p((1-\Phi_J)\omega_\alpha^{-1}) \leqq C 2^{-\alpha J} ,$$

and consequently, using Theorem 4.1 or 4.2 depending on whether s is infinite or finite,

$$\begin{aligned} M_p((1-\Phi_J)\sigma_{\alpha,h,n}) &\leqq M_p((1-\Phi_J)\omega_\alpha^{-1})(M_p(a^n)+1) \\ &\leqq C 2^{-\alpha J} n^{|\frac{1}{2}-p^{-1}|(1-rs^{-1})} \end{aligned} \tag{5.5}$$

Adding (5.4) and (5.5) with J chosen so that $2^J \leqq n^{r^{-1}-s^{-1}} < 2^{J+1}$ now completes the proof.

We shall also prove that the estimate in Theorem 5.1 is best possible in the following sense.

Theorem 5.2. *Under the same assumptions as above, if $T>0$ and $0 \leqq \alpha \leqq r$, there is a positive constant h_0 such that*

$$\sup\{\|(E_k^n - E(nk))v\|_p ; \|v\|_{p,\alpha}^* \leqq 1, nk \leqq T\} \geqq c h^{\beta(\alpha)}, \quad 0 \leqq h \leqq h_0 .$$

Proof. We shall prove that there are positive constants h_0 and c such that for $T>0$,

$$M_p\big(\omega_\alpha^{-1}(a_\varrho(h\,\cdot)^n - 1)\big) \geqq c h^{\beta(\alpha)}, \quad h \leqq h_0, \ nk = T . \tag{5.6}$$

In a neighborhood of $y=0$ we have

$$|a_\varrho(h^{r^{-1}}y)^n-1| \geqq c|y|^r+O(h^{r^{-1}}) \quad \text{as } h \to 0\,,$$

uniformly in y. Hence there are positive constants h_0 and c such that

$$0 < c \leqq \|\sigma_{\alpha,h,n}\|_\infty \leqq M_p(\sigma_{\alpha,h,n}), \quad h \leqq h_0,\ nk=T\,.$$

After a change of variables this proves that

$$M_p\big(\omega_\alpha^{-1}(a_\varrho(h\ \cdot)^n-1)\big) \geqq ch^{\alpha(1-r^{-1})}, \quad h\leqq h_0,\ nk=T\,,$$

and thus proves (5.6) for $r|\frac{1}{2}-p^{-1}|\leqq\alpha\leqq r$.

Let now $\chi \in C_0^\infty$, $\chi \not\equiv 0$, be a function with support not containing the origin. Then $\chi\omega_\alpha \in M_p$ and hence

$$M_p\big(\chi(a(h^{s^{-1}}\cdot)^n-1)\big) \leqq C\,M^p\big(\omega_\alpha^{-1}(a(h^{s^{-1}}\cdot)^n-1)\big)\,.$$

On the other hand, by Lemma 4.2,

$$M_p\big(\chi(a(h^{s^{-1}}\cdot)^n-1)\big) \geqq C\,h^{-|\frac{1}{2}-p^{-1}|(1-rs^{-1})}, \quad h\leqq h_0,\ nk=T\,.$$

Altogether, after a change of variables this proves

$$M_p\big(\omega_\alpha^{-1}(a(h\ \cdot)^n-1)\big) \geqq ch^{\alpha(1-s^{-1})-|\frac{1}{2}-p^{-1}|(1-rs^{-1})}, \quad h\leqq h_0,\ nk=T\,,$$

and thus completes the proof of (5.6).

6. Smoothing operators.

In this section we shall prove that in the case $0\leqq\alpha<r|\frac{1}{2}-p^{-1}|$ in Theorem 5.1 where the nonstability of E_k effects the rate of convergence, one can get rid of this nonstable behavior by applying certain averaging operators to the initial data and thereby obtain the same order of convergence as in the stable case.

Thus let $\psi \in M_1$ be analytic on the extended real line and let G_h be the operator with symbol $\psi(h^{1-r^{-1}}y)$ so that

$$\mathscr{F}(G_h v)(y) = \psi(h^{1-r^{-1}}y)\,\hat{v}(y)\,.$$

We shall assume that for certain natural numbers μ and ν, ψ satisfies

$$\psi(y) = 1+O(y^\mu) \quad \text{as } y\to 0\,, \tag{6.1}$$

$$\psi(y) = O(y^{-\nu}) \quad \text{as } y\to\infty\,. \tag{6.2}$$

The first of these assumptions means that G_h approximates the identity operator with a certain accuracy; in particular, if $\mu=r$ this accuracy is of order $r-1$ just as for the operator E_k. The second assumption is the one that guarantees the smoothing effect of G_h. We shall exhibit at the end of this section specific operators satisfying these assumptions for different μ and ν.

With these assumptions about G_h and the same assumptions as in Theorem 5.1 on E_k we shall now prove the following $L^*_{p,\alpha}$ analogue of Theorem 1.3. By Lemmas 2.9 and 2.10, Theorem 1.3 is then a consequence of this result.

THEOREM 6.1. *Under the above assumptions we have, $\tilde{\beta}(\alpha)$ defined by* (1.5),

$$\|(E_k^n G_h - E(nk))v\|_p \leqq Ch^{\tilde{\beta}(\alpha)} \|v\|^*_{p,\alpha}, \quad nk \leqq T ,$$

for $0 \leqq \alpha \leqq \min(\mu, r)$, $\alpha \neq r|\frac{1}{2}-p^{-1}|-\nu$.

PROOF. After a change of variables and with

$$\tilde{\sigma}_{\alpha,h,n} = \omega_\alpha^{-1}(\psi a_\varrho(h^{r^{-1}} \cdot)^n - 1)$$

we want to prove that

$$M_p(\tilde{\sigma}_{\alpha,h,n}) \leqq Ch^{\tilde{\beta}_0(\alpha)} ,$$

where for $0 \leqq \alpha \leqq \min(\mu, r)$,

$$\tilde{\beta}_0(\alpha) = \tilde{\beta}(\alpha) - \alpha(1-r^{-1}) = \min(0, (\alpha+\nu-r|\tfrac{1}{2}-p^{-1}|)(r^{-1}-s^{-1})) .$$

Let φ_j and Φ_J be as before. Using (6.1) we see as in the proof of (5.3) that for $\alpha \leqq \mu$,

$$M_p(\varphi_0 \tilde{\sigma}_{\alpha,h,n}) \leqq C, \quad nk \leqq T .$$

We then notice that for y bounded away from 0, multiplication by ψ has the same effect as to change ω_α into $\omega_{\alpha+\nu}$ and hence as in (5.3) we get that

$$M_p(\varphi_j \tilde{\sigma}_{\alpha\ h,n}) \leqq C 2^{j(r|\frac{1}{2}-p^{-1}|-(\alpha+\nu))}, \quad j \geqq 0 .$$

Also, in analogy with (5.5) we obtain

$$M_p((1-\Phi_J)\tilde{\sigma}_{\alpha,h,n}) \leqq C 2^{-J(\alpha+\nu)} n^{(1-rs^{-1})|\frac{1}{2}-p^{-1}|} ,$$

and the proof is completed as before.

We shall display some special operators G_h corresponding to different μ and ν. Let $p_{\mu,\nu}(\sin z)$ be the polynomial in $\sin z$ of lowest degree such that

$$p_{\mu,\nu}(\sin z) = z^\nu + O(z^{\mu+\nu}) \quad \text{as } z \to 0 .$$

Then (6.1) and (6.2) are satisfied if we choose for ψ the function

$$\psi_{\mu,\nu}(y) = (\pi y)^{-\nu} p_{\mu,\nu}(\sin \pi y) .$$

In particular

$$\psi_{2,1}(y) = (\pi y)^{-1} \sin \pi y, \quad \psi_{2,2}(y) = (\pi y)^{-2} \sin^2 \pi y .$$

The corresponding operators G_h are $(h_r = h^{1-r^{-1}})$

$$G_h^{2,1} v(x) = h_r^{-1} \int_{-\frac{1}{2}h_r}^{\frac{1}{2}h_r} v(x-t)\, dt ,$$

$$G_h^{2,2} v(x) = h_r^{-1} \int_{-h_r}^{h_r} \left(1 - \frac{|t|}{h_r}\right) v(x-t)\, dt .$$

For the two operators E_k defined by (1.6) and (1.7) we have $r=3$, and thus in order to get the full rate of convergence by Theorem 6.1 in the whole range $0 \leqq \alpha \leqq r$ we have to take $\mu=3$, $\nu=2$. Possible choices of the function ψ and the corresponding operator G_h are then

$$\psi_{4,2}(y) = (\pi y)^{-2} (\sin^2 \pi y + \tfrac{1}{3} \sin^4 \pi y) ,$$

$$G_h^{4,2} v(x) = \tfrac{7}{6} G_h^{2,2} v(x) - \tfrac{1}{12} [G_h^{2,2} v(x+h_r) + G_h^{2,2} v(x-h_r)] .$$

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