MATH. SCAND. 27 (1970), 24–38

EQUATIONAL BASES FOR LATTICE THEORIES

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This paper is based on the systematic treatment of equational logic in Tarski [7]. The results obtained solve some fundamental problems concerning systems of axioms for equational theories of lattices, problems that were formulated in [7] for equational theories of arbitrary finitary algebras.

The theories we discuss are defined in a formal system of equational logic with primitive symbols as follows: an equality symbol, $=$; a denumerably infinite (ordered) set of variables $v_0, v_1, \ldots$, represented for convenience by small latin letters x, y, z, ... (with or without subscripts); and two binary operation symbols, $+$ and $\cdot$. Loosely speaking, the formal system is a certain fragment of first order logic in which the only admitted formulas are equations, treated as universal sentences. Within this framework, theories commonly arise in two ways. Each algebra A with two binary operations corresponding to $+$ and $\cdot$ determines a theory ΘA, the equational theory of A, defined as the set of all equations $\sigma = \tau$ which are valid in A (for all values of the variables). Each set of equations Σ determines a theory $\Theta[\Sigma]$, defined as consisting of all equations which can be derived from Σ and the tautologous equations, $\tau = \tau$, by repeated applications of the operations of substitution and replacement of equals by equals. A set Θ of equations is called an (*equational*) *theory* if $\Theta = \Theta A$ for some algebra A or, what is known to be equivalent, if $\Theta = \Theta[\Sigma]$ for some set of equations Σ. Such a set Σ is called an (*equational*) *base* for Θ, or for A.

Lattices are defined to be algebras satisfying the two equations $x = x\cdot(x+y)$, $x = x + x\cdot y$, and the commutative and the associative law for each operation. The equational theories of lattices — or more briefly, *lattice theories* — are the theories containing these six equations. Thus the smallest lattice theory is the theory Λ, which has a base the set of lattice axioms; the largest is, of course, Ω, based on the single equation $x = y$.

Received May 18, 1968; in revised form April 29, 1969.

Research supported by U.S. National Science Foundation Grant GP 7578.

It was shown by Padmanabhan [5] that every finitely based lattice theory has a base of two equations. In § 1 we show that Λ *has a base consisting of one equation* — i.e. is one-based. It turns out that besides Λ and Ω no other lattice theory is one based. We note that Tarski has proved a theorem [7; Thm. 8] from which it follows that if the lattice theory Θ is finitely based, then the set of cardinalities of the independent (i.e. minimal) bases of Θ coincides with an interval $[\varkappa, \omega)$. If $\Theta = \Lambda$ or Ω we have $\varkappa = 1$; otherwise $\varkappa = 2$. The results of this section are extended to apply to arbitrary theories in which certain operations much weaker than the lattice operations can be defined.

In § 2 we prove that *every finite lattice has a finite base of equations.* (Schützenberger [6] asserted as much, but he did not indicate a proof and his statement appears to have gone unnoticed.) This result is valid for every finite algebra (with a finite number of basic operations) in which the operations of a lattice are definable.

The final section of this paper contains some examples, the most important being a set of equations Γ which is infinite, and independent relative to Λ. In other words, for each $\gamma \in \Gamma$ we have that $\gamma \notin \Theta[\Lambda \cup (\Gamma - \{\gamma\})]$. This construction yields immediately two interesting corollaries: *There are* $2^{\aleph_0}$ *lattice theories*; *there is a non finitely based lattice theory* (e.g. $\Theta[\Lambda \cup \Gamma]$). These results were proved by Baker [1], independently of this author, but his method was less elementary. In conclusion, we have a simple planar diagram (Figure 2) describing a lattice which has no finite base.

In addition to standard set-theoretic terminology, some special notations and abbreviations referring to equations or equational theories will be needed. For example, given a finite set of equations $\Sigma = \{\varepsilon_0, \ldots, \varepsilon_\varkappa\}$, $\Theta[\Sigma]$ may be written $\Theta[\varepsilon_0, \ldots, \varepsilon_\varkappa]$. We denote the equation $\sigma = \sigma \cdot \tau$ by the expression $\sigma \leq \tau$. Often we write $\sigma \sim_\Theta \tau$ to indicate that an equation $\sigma = \tau$ belongs to the theory Θ, and $\sigma \leq_\Theta \tau$ to indicate that $\sigma \leq \tau \in \Theta$. Thus $\sim_\Theta$ is an equivalence relation; and if $\Theta \supseteq \Lambda$, then $\leq_\Theta$ is a quasi partial ordering — reflexive and transitive on the set of terms — and $\sim_\Theta$ is the equivalence relation derived from it: $\sigma \sim_\Theta \tau$ if and only if both $\sigma \leq_\Theta \tau$ and $\tau \leq_\Theta \sigma$.

Given a non-void, finite set of terms T, $\sum T$ will designate a term ν having the property: for each term ϱ, $\nu \leq_\Lambda \varrho$ if and only if $\tau \leq_\Lambda \varrho$ for every $\tau \in T$. Obviously, any two terms with this property are Λ-equivalent; we can take for ν any term constructed from all elements of T using only $+$. Similarly, a product of T is designated by $\prod T$.

We shall represent the set of all variables which occur in a term σ, or in an equation ε, by $V(\sigma)$ or $V(\varepsilon)$, respectively. If $V(\sigma) \subseteq \{x_1, \ldots, x_\varkappa\}$,

σ may be written as $\sigma(x_1,\ldots,x_\varkappa)$. Then $\sigma(\nu_1,\ldots,\nu_\varkappa)$ designates the term derived from σ by substituting for each variable x_μ, everywhere it occurs in σ, the term ν_μ. In using this notation we assume, of course, that the sequence $x_1,\ldots,x_\varkappa$ contains no repetitions. In each algebra A, $\tau = \tau(y_1,\ldots,y_\varkappa)$ gives rise to a $\varkappa$-place operation $\tau^{(A)}$, defined in an obvious fashion. Actually, many operations are defined by a given term, but, having written τ as $\tau(y_1,\ldots,y_\mu)$, a definite operation is specified; the context will thus determine what operation is referred to by $\tau^{(A)}$.

Notions pertaining to lattices and used without comment, e.g. "homomorphism", "subdirectly irreducible lattice" and "diagram of a lattice", are well known; consult [2].

1. One-based lattice theories.

We first show that, with the exception of Ω and (possibly) Λ, there are no one-based lattice theories.

Assume that Θ is a one-based theory, say $\Theta = \Theta[\varepsilon]$, and that $\Lambda \subseteq \Theta$ while $\Theta \neq \Omega$. Obviously, one side of ε is a solitary variable; the equation $x \equiv x \cdot x$ could not be derived from it otherwise. Assume therefore that ε has the form $x \equiv \tau$. Then we proceed to construct terms τ_0, τ_1 by replacing, everywhere in τ, each of its variables other than x by $\prod V(\varepsilon)$, or $\sum V(\varepsilon)$ respectively. Now it is clear that $\tau_0 \leq_\Lambda \tau \leq_\Lambda \tau_1$; and as the equations $x \equiv \tau_0$ and $x \equiv \tau_1$ are directly derivable form ε, we easily conclude that these two equations, combined with Λ, form a base for Θ.

As regards τ_0, it is obvious that either $\tau_0 \sim_\Lambda x$ or $\tau_0 \sim_\Lambda \prod V(\varepsilon)$ (since τ_0 is constructed from these two terms and $\prod V(\varepsilon) \leq_\Lambda x$). Even the latter case requires that $x \equiv \tau_0 \in \Lambda$, or equivalently $V(\varepsilon) = \{x\} - x \equiv y$, would be derivable from $\Lambda \cup \{x \equiv \tau_0\}$ otherwise (forcing $\Theta = \Omega$). Thus $x \equiv \tau_0 \in \Lambda$; similarly $x \equiv \tau_1 \in \Lambda$; and consequently $\Theta = \Lambda$, which is the desired result.

As mentioned above Λ is, in fact, one-based. The argument for Theorem 1.1, which proves this, depends on the following simple observation:

(I) If $G, G_0,\ldots,G_\varkappa$ is a system of functions mapping a set Y into itself, then all functions of the system are permutations of Y if and only if the function

$$G_0 \circ G_1 \circ \ldots \circ G_\varkappa \circ G \circ G_\varkappa \circ G_{\varkappa-1} \circ \ldots \circ G_0$$

is a permutation.

Aided by this observation, we obtain a proof of Theorem 1.1 that is algebraic and short; although the resulting equation to characterize lattices (equation λ below) is so long, containing over three hundred thousand symbols, that we can only represent it in an abbreviated form.

This leaves open the problem of finding a really elegant equation to characterize lattices.

For the proof we set down a system of four equations:

λ_0: $x = x+y\cdot(x\cdot z)$,
λ_1: $x = x\cdot(y+(x+z))$,
λ_2: $x = (y\cdot x+x\cdot z)+x$,
λ_3: $x = [(y+x)\cdot(x+z)]\cdot x$;

and one further equation less easy to express. Let us denote by $\tau(x,y,z)$ the term $x\cdot y+(y\cdot z+x\cdot z)$; and by $\sigma_\varkappa(x,y,z)$, for $0\leq\varkappa\leq 3$, the term on the right side of the equality symbol in equation $\lambda_\varkappa$. Then for any four terms $\nu_0, \nu_1, \nu_2, \nu_3$ we put

$$P_0\nu_0\ldots\nu_3 = \tau(\nu_1,\nu_0,\nu_0), \quad P_1\nu_0\ldots\nu_3 = \tau(\nu_0,\nu_0,\nu_1);$$

and for $0\leq\varkappa\leq 3$ we put

$$P_{\varkappa+2}\nu_0\ldots\nu_3 = \tau(\sigma_\varkappa(\nu_0,\nu_1,\nu_2),\nu_3,\nu_0).$$

With these symbols, the required equation can be written as

λ: $x = P_0P_1\ldots P_4P_5P_4P_3\ldots P_0xx_1\ldots x_{33}$.

We can also say that λ is the equation $x=\pi_{10}$, with the term π_{10} formed by applying $P_0, P_1, \ldots$ in the order indicated to construct consecutively

$$\pi_0 = P_0xx_1x_2x_3, \quad \pi_1 = P_1\pi_0x_4x_5x_6, \quad \ldots$$

and eventually $\pi_{10} = P_0\pi_9x_{31}x_{32}x_{33}$.

LEMMA. $\Theta[\lambda_0,\lambda_1,\lambda_2,\lambda_3] = \Lambda$.

THEOREM 1.1. $\Theta[\lambda] = \Lambda$.

PROOF OF THE LEMMA. Setting $\Theta = \Theta[\lambda_0,\lambda_1,\lambda_2,\lambda_3]$, we have to show that $\Theta = \Lambda$. It is obvious that each of the equations $\lambda_0,\ldots,\lambda_3$ can be derived from the lattice axioms, whence $\Theta\subseteq\Lambda$. Conversely, it must be shown that the lattice axioms belong to Θ. It simplifies matters to note that Θ is *self-dual*: if an equation ε belongs to Θ, the dual equation, formed by exchanging $+$ and $\cdot$, will also belong to Θ; this because the dual of each axiom of Θ belongs to Θ.

Substituting the term $y+(x+z)$ for z, and $(y+x)\cdot(x+z)$ for y, in λ_0 and applying λ_1 and λ_3, we have that

(1) $$x \sim_\Theta x+x, \quad \text{and (therefore)} \quad x \sim_\Theta x\cdot x\,.$$

Substituting on the one hand $x+z$ for y, on the other x for z in λ_1 we have by (1) and by duality that

(2) $$x \sim_\Theta x\cdot(x+z), \quad x \sim_\Theta x\cdot(y+x),$$
and
$$x \sim_\Theta x+x\cdot z, \quad x \sim_\Theta x+y\cdot x\,.$$
From (2) we infer
$$(x+z)+x \sim_\Theta (x+z)+x\cdot(x+z) \sim_\Theta x+z$$
and hence $(x+z)+x \sim_\Theta x+z$. Thus, by replacing y by $x+z$ in λ_3 and applying (1), we obtain

(3) $$x \sim_\Theta (x+z)\cdot x;$$

which leads rather shortly to the commutative laws:

(4) $$x+y \sim_\Theta y+x, \quad \text{and dually} \quad x\cdot y \sim_\Theta y\cdot x\,.$$

In fact, by (2), (3) and λ_2,
$$x+y \sim_\Theta [y\cdot(x+y)+(x+y)\cdot x]+(x+y) \sim_\Theta (y+x)+(x+y);$$
which, combined with (2), (3), gives
$$(y+x)\cdot(x+y) \sim_\Theta (x+y)\cdot(y+x) \sim_\Theta y+x;$$
and we easily obtain (4) from this by interchanging x and y and comparing the resulting formula with the original.

The proof is concluded by deriving the associative laws:

(5) $$x+(y+z) \sim_\Theta (x+y)+z, \quad x\cdot(y\cdot z) \sim_\Theta (x\cdot y)\cdot z\,.$$

Observe that for three terms, χ, φ_0, φ_1, if $\chi\cdot\varphi_0 \sim_\Theta \varphi_0$ and $\chi\cdot\varphi_1 \sim_\Theta \varphi_1$ then also $\chi\cdot(\varphi_0+\varphi_1) \sim_\Theta \varphi_0+\varphi_1$; this follows from (3), since by (4) and λ_2
$$\chi \sim_\Theta (\varphi_0\cdot\chi+\chi\cdot\varphi_1)+\chi \sim_\Theta (\varphi_0+\varphi_1)+\chi\,.$$
We easily conclude from this fact — by taking $\chi = x+(y+z)$ and deriving $\chi\cdot\varphi \sim_\Theta \varphi$ for each one of the terms $\varphi = x,y,z$, using (2), (4) and λ_1 — that $\chi\cdot\varphi \sim_\Theta \varphi$ where $\chi = x+(y+z)$ and $\varphi = (x+y)+z$. Now $z+(y+x) \sim_\Theta \varphi$, by two applications of (4); hence interchanging x and z converts χ into φ, and φ into χ. So we also have $\varphi\cdot\chi \sim_\Theta \chi$ and
$$\chi \sim_\Theta \varphi\cdot\chi \sim_\Theta \chi\cdot\varphi \sim_\Theta \varphi$$
follows, proving (5).

Proof of Theorem 1.1. We first recall the definition of the terms τ, $\sigma_\varkappa$ ($\varkappa \le 3$) and $P_\mu v_0v_1v_2v_3$ ($\mu \le 5$) employed to express λ. From the form in which λ is written it is obvious that it can be derived from the system of equations $x = P_\mu xx_1x_2x_3$, $0 \le \mu \le 5$. These equations clearly belong to Λ, whence we infer that $\lambda \in \Lambda$.

The assertion that $\Lambda \subseteq \Theta[\lambda]$ is the less obvious half of the theorem. To prove this by the direct method, as used for the lemma, would be depressingly tedious; we will prove instead the equivalent statement that every algebra in which λ is valid is a lattice. Assume therefore that $B = \langle |B|, +, \cdot \rangle$ is an algebra which satisfies λ.

For simplification we introduce some special notation for the polynomial operations in B corresponding to formal terms mentioned above. For $0 \leq \varkappa \leq 3$, $S_\varkappa$ is the three-place function $\sigma_\varkappa^{(B)}$ [for example $S_0(a,b,c) = a + b \cdot (a \cdot c)$]; T is the three-place function $\tau^{(B)}$

$$T(a,b,c) = a \cdot b + (b \cdot c + a \cdot c);$$

and for $0 \leq \mu \leq 5$ and $\mathbf{s} = \langle s^1, s^2, s^3 \rangle \in {}^3|B|$, $R_\mu[\mathbf{s}]$ is the one-place function derived from the term $\varrho_\mu = P_\mu x x_1 x_2 x_3$ by setting

$$R_\mu[\mathbf{s}](b) = \varrho_\mu^{(B)}(b, s^1, s^2, s^3) .$$

(Thus, for example $R_2[\mathbf{s}](b) = T(S_0(b, s^1, s^2), s^3, b)$.)

We have to show that $\Lambda \subseteq \Theta B$ or, in view of the lemma, that $\{\lambda_0, \ldots, \lambda_3\} \subseteq \Theta B$. Another way to say this is

(1) Assume that $\varkappa \leq 3$ and that $a, b, c \in |B|$. Then $S_\varkappa(a,b,c) = a$.

To prove (1), we first notice that the validity of λ is equivalent to the following system of functional relations:

(2) For every $\mathbf{s}_0, \ldots, \mathbf{s}_5, \mathbf{t}_0, \ldots, \mathbf{t}_4 \in {}^3|B|$,
$$R_0[\mathbf{s}_0] \circ \ldots \circ R_5[\mathbf{s}_5] \circ R_4[\mathbf{t}_4] \circ \ldots \circ R_0[\mathbf{t}_0] = \iota_B ,$$
the identity function on $|B|$.

These relations in turn imply

(3) Assume that $\mu \leq 5$ and that $\mathbf{s}, \mathbf{s}' \in {}^3|B|$. Then $R_\mu[\mathbf{s}]$ is a permutation of $|B|$, moreover $R_\mu[\mathbf{s}] = R_\mu[\mathbf{s}']$.

In fact we easily obtain the first statement in (3) by placing $\mathbf{t}_\nu = \mathbf{s}_\nu$ in (2) for $\nu \leq 4$, and by referring to our earlier observation (I) concerning permutations. The second statement then follows if we permit $\mathbf{s}_\mu$ to vary in (2) while keeping all the other parameters fixed, and subsequently cancel permutations from the resulting functional relations.

To finish the proof we choose $\varkappa \leq 3$ and $a, b, c \in |B|$, and set $d = S_\varkappa(a,b,c)$. Looking briefly at the definitions involved, we see that

$$\begin{aligned} R_1[\langle a,a,a \rangle](a) &= R_0[\langle a,a,a \rangle](a) & (= T(a,a,a)) , \\ R_0[\langle d,a,a \rangle](a) &= R_{\varkappa+2}[\langle b,c,a \rangle](a) & (= T(d,a,a)) , \\ R_{\varkappa+2}[\langle b,c,d \rangle](a) &= R_1[\langle a,a,a \rangle](d) & (= T(d,d,a)) . \end{aligned}$$

From this we infer, with two applications of the second statement in (3), that $R_1[\langle a,a,a\rangle](a)=R_1[\langle a,a,a\rangle](d)$. As $R_1[\langle a,a,a\rangle]$ is a permutation, $a=d$ follows — that is, $S_\varkappa(a,b,c)=a$. This concludes the proof of (1) and completes the proof of Theorem 1.1.

The proof of Theorem 1.1 exemplifies a line of argument which permits many refinements showing that various equational theories are one-based. Without making any attempt to discover the best result obtainable by the method, we can easily prove the following theorem, which yields as one special case the result obtained for Λ. The proof does not differ essentially from the foregoing argument.

Theorem 1.2. *Let Θ be a finitely based equational theory formalized in an arbitrary system of equational logic. Assume that for some term $\tau(x_1,\ldots,x_\varkappa)$ $(\varkappa\geq 2)$ Θ contains each one of the equations*

$$x\equiv\tau(y,x,\ldots,x),\ x\equiv\tau(x,y,x,\ldots,x),\ \ldots,\ x\equiv\tau(x,\ldots,x,y)\ .$$

Assume also that Θ has a base composed solely of equations of the form $x\equiv\sigma$. Under these assumptions Θ is one-based.

Theorem 1.2 has important implications for every theory Θ which is definitionally equivalent, in the sense defined in [7], to the theory of a lattice with operators (i.e. a lattice having possibly other fundamental operations besides the lattice theoretic sum and product). Such a theory Θ automatically has a term τ with the property stated in Theorem 1.2. Each such theory is one-based iff it has a finite base consisting of equations of the form $x\equiv\sigma$, where x is a variable. (Moreover, on the strength of Theorem 1.1 alone it follows that every such theory having a finite base can be based on two equations. The proof for this may be summarized in a few words: Θ has two terms $\sigma(x,y)$ and $\pi(x,y)$ which define, so to speak, the lattice operations. By Theorem 1.1, Θ then contains a certain equation $x\equiv\varrho$ which is equivalent to the lattice axioms with σ and π replacing $+$ and $\cdot$. The proof is concluded by observing that if $\{\chi_\mu\equiv\varphi_\mu\}$ is a finite set of equations of Θ, then each of the equations is derivable from $x\equiv\varrho$ combined with the equation of Θ which — treating σ and π as $+$ and $\cdot$ and choosing distinct variables y_ν not occuring in any equation $\chi_\mu\equiv\varphi_\mu$ — can be written as $\sum\chi_\mu\cdot y_\mu\equiv\sum\varphi_\mu\cdot y_\mu$.)

We may, in particular, take for Θ a finitely based theory definitionally equivalent to the theory of a boolean algebra with operators. The assumptions of Theorem 1.2 are readily verified in this case, and it follows that Θ is one-based. This result is also a corollary of [7; Theorem 3]; in fact, it seems to be the only interesting application the two theorems have in common.

For theories definitionally equivalent with the equational theory of a lattice (without operators), it turns out that one-based theories have a simple characterization. The results of this section are largely summarized in Theorem 1.2 and in the following theorem. The proof is rather trivial, requiring an application of Theorem 1.2 and one new idea.

THEOREM 1.3. *Let Θ' be a finitely based equational theory definitionally equivalent with a lattice theory Θ. Then Θ' is two-based. Furthermore, an equivalent condition for Θ' to be one-based is that either $\Theta = \Lambda$ or $\Theta = \Omega$, and in case $\Theta = \Lambda$, that every lattice term which serves as the definition in Θ of an operation symbol of Θ' be Λ-equivalent to a monomial, i.e. to a sum or product of variables.*

As a corollary, we easily infer that every theory definitionally equivalent with Λ and having just one operation symbol — e.g. the theory of lattices treated as algebras $\langle X, D\rangle$ with the operation D defined by $Dxyuv = x \cdot y + u \cdot v$ — fails to be one-based.

2. Bases for finite lattices.

Though it lacks profundity, it is an interesting fact about finite lattices that each has a finite equational base. In this section we establish that fact. The argument we shall use is quite elementary and rather trivial compared with an argument to establish the analogous result for finite groups [4]. Theories of the form $\boldsymbol{\Theta} L$, where L is a finite lattice, are large in one sense [3]: There exist only a finite number of lattice theories extending such a theory and each is the theory of a finite lattice; no such theory is identical with Λ. Consequently it follows from Theorem 1.3 and 2.1 below that every such theory $\boldsymbol{\Theta} L$ is two-based, but fails to be one-based unless the universe of L has just one element.

We preface our theorem with a few remarks. The substance of the remarks is well known, and applies as well to any finite algebra whose set of basic operations also is finite. Let A be a finite lattice. For each integer $\alpha > 0$ we put

$$\boldsymbol{\Theta}^{(\alpha)} A = \boldsymbol{\Theta}[\Psi^{(\alpha)}] \quad \text{where} \quad \Psi^{(\alpha)} = \{\varepsilon \in \boldsymbol{\Theta} A : {}^{\#}(V(\varepsilon)) \leq \alpha\};$$

in other words the theory generated by the valid equations of A in which at most α variables occur. Now it should be clear that if $\boldsymbol{\Theta} A$ is finitely based then $\boldsymbol{\Theta} A = \boldsymbol{\Theta}^{(\alpha)} A$ for some α; but the converse is also true: it is not difficult to see that each of the theories $\boldsymbol{\Theta}^{(\alpha)} A$ will have a finite base, and even one that can be constructed by a mechanical procedure.

To see this, set $\Theta = \boldsymbol{\Theta} A$ and consider the relation of Θ-equivalence $\sim_{\Theta}$.

Restricted to terms τ containing only the first α variables, $\sim_\Theta$ coincides with the kernel of the function $\tau \rightsquigarrow \tau^{(A)}$, and therefore it divides the set of these terms into a finite number of classes, or cosets, of equivalent terms — the number of cosets being bounded by the number of α place operations on $|A|$. We choose from these cosets a system of representative terms $\pi_0,\ldots,\pi_\beta$, one from each coset, taking in particular each of the first α variables as the representative of its class. (This can be done unless $^{\#}(|A|)=1$ and that case is trivial. Notice that given A and α a system of representatives can be effectively found.) One can then prove that the following equations form a base for $\Theta^{(\alpha)}A$: for $\mu,\nu \leq \beta$, the equations

$$\pi_\mu + \pi_\nu = \pi_\zeta \quad \text{and} \quad \pi_\mu \cdot \pi_\nu = \pi_\delta\,,$$

where π_ζ and π_δ are the respective representatives of the cosets of $\pi_\mu + \pi_\nu$ and $\pi_\mu \cdot \pi_\nu$. In fact, letting $\Theta_0^{(\alpha)}$ denote the theory based on these equations, an easy argument by induction on the length of a term τ built from the first α variables shows that if τ is in the $\sim_\Theta$ coset of π_μ then $\tau = \pi_\mu \in \Theta_0^{(\alpha)}$. If then a certain equation $\tau = \sigma$ relating two such terms belongs to Θ — that is $\tau \sim_\Theta \sigma$ — then it also belongs to $\Theta_0^{(\alpha)}$. As every equation involving at most α variables is equivalent to one of these equations, $\Theta^{(\alpha)}A = \Theta_0^{(\alpha)}$ follows immediately.

THEOREM 2.1. *Let L be a finite lattice. Then ΘL has a finite base. Moreover, if the universe of L has $\varkappa$ elements, ξ is the greatest integer μ satisfying $2^\mu \leq \varkappa$, and $\eta = 1 + \varkappa^\xi$, then $\Theta L = \Theta^{(\eta)}L$.*

PROOF. From the above remarks the theorem will follow if, setting $H = \Theta^{(\eta)}L$, we can prove that $H = \Theta L$. Letting $V^\#$ denote a fixed, but arbitrary, finite set of variables, it is sufficient to show that every valid equation of L whose variables all belong to $V^\#$ is a member of H. This will obviously be an immediate consequence of the following two propositions (which concern a certain syntactical transformation, S, defined in the fourth paragraph below).

(I) Let the equation ε, or $\varphi = \chi$, satisfy $V(\varepsilon) \subseteq V^\#$. If $\varepsilon \in \Theta L$, then the equation $S(\varphi) = S(\chi)$ is a member of H.

(II) Let the term φ satisfy $V(\varphi) \subseteq V^\#$. Then the equation $\varphi = S(\varphi)$ is a member of H.

After defining S, the remainder of the argument will be directed to proving (I) and (II).

To simplify matters we now make the assumption that $\varkappa \geq 4$. This excludes certain trivial cases that are easily handled and, in particular, implies that $\varkappa^2 \leq \varkappa^\xi$, a fact we shall need. Notice that $\Lambda \subseteq H$. Moreover, for $\nu > \xi$ the equation

$$(1) \qquad x \cdot \sum_{\mu<\nu} y_\mu = \sum_{\alpha \in \xi^\nu} x \cdot \sum_{\mu<\xi} y_{\alpha(\mu)}$$

belongs to H. In fact, each of these equations can be derived from the one equation where $\nu = \xi + 1$ (combined with Λ). That equation has $\xi+2$ variables and $\xi + 2 \leq \eta$. To see that it belongs to ΘL, it suffices to observe that for any $a_0, \ldots, a_\xi \in |L|$, there is a $\delta \leq \xi$ such that

$$\sum_\mu a_\mu = \sum_{\mu \neq \delta} a_\mu .$$

If this were not the case then the sums of the various subsets of the a_μ would be distinct elements of L, and we should have $2^{\xi+1} \leq \varkappa$, contradicting the definition of ξ.

In the reasoning that follows, the letters C, D, E and C', D', E' are reserved for equivalence relations on finite sets of variables. We use $\operatorname{dom} C$ to designate the domain of C; Cxy to express that x, y are C-related; $\mathscr{E}(V)$ for the set of all equivalence relations with V for domain; and to each positive integer δ, $\mathscr{E}_\delta(V)$ for the subset of those having at most δ cosets.

Assume now that the term σ and the relation C satisfy $V(\sigma) \subseteq \operatorname{dom} C$. Then $\mathbf{M}_C(\sigma)$ will designate the term obtained from σ by means of a simultaneous substitution $x \rightsquigarrow \prod\{y : Cxy\}$, $x \in V(\sigma)$, and $\mathbf{M}_C{}^0(\sigma)$ will designate the term obtained by substituting instead $x \rightsquigarrow x_C$, where x_C is the least variable y (in the preassigned ordering of all the variables) such that Cxy. If $C \in \mathscr{E}_\delta(\operatorname{dom} C)$ then $\mathbf{M}_C{}^0(\sigma)$ contains no more than δ variables. We list below some further obvious consequences of these definitions.

(2) Let V be a finite set of variables, C, D be members of $\mathscr{E}(V)$ and σ, τ be terms such that $V(\sigma) \cup V(\tau) \subseteq V$.

- (i) $\mathbf{M}_C(\sigma \cdot \tau) = \mathbf{M}_C(\sigma) \cdot \mathbf{M}_C(\tau)$.
- (ii) $\mathbf{M}_C(\sigma + \tau) = \mathbf{M}_C(\sigma) + \mathbf{M}_C(\tau)$.
- (iii) $\mathbf{M}_C(\sigma) = \mathbf{M}_C(\mathbf{M}_C{}^0(\sigma))$.
- (iv) $\mathbf{M}_C{}^0(\sigma) \sim_\Lambda \mathbf{M}_C{}^0(\mathbf{M}_C(\sigma))$.
- (v) If $C \subseteq D$, then
$$\mathbf{M}_D(\sigma) \leq_\Lambda \mathbf{M}_C(\sigma) \leq_\Lambda \sigma .$$
- (vi) If $C \in \mathscr{E}_\delta(V)$ and $D \in \mathscr{E}_{\delta'}(V)$, then $C \cap D \in \mathscr{E}_{\delta \cdot \delta'}(V)$.
- (vii) If $E \in \mathscr{E}(\{x_C : x \in V\})$, and $E' \in \mathscr{E}(V)$ is defined by $E'xy \leftrightarrow Ex_C y_C$, then
$$\mathbf{M}_{E'}(\sigma) \sim_\Lambda \mathbf{M}_C\big(\mathbf{M}_E(\mathbf{M}_C{}^0(\sigma))\big) .$$

We now define the function S, referring to the previously given set of variables $V^\#$, by setting

$$\mathrm{S}(\sigma) = \sum \{\mathbf{M}_D(\sigma): D \in \mathscr{E}_\varkappa(V^\#)\} ,$$

for every term σ for which $V(\sigma) \subseteq V^\#$.

It remains to prove that S has the properties (I) and (II), as claimed at the beginning. (I) presents no difficulty. Indeed, assume that $\varphi = \chi$ is an equation valid in L, whose variables belong to $V^\#$. Let then $D \in \mathscr{E}_\varkappa(V^\#)$. From the given equation we derive by substitution the equation $\mathbf{M}_D{}^0(\varphi) = \mathbf{M}_D{}^0(\chi)$, which has at most $\varkappa$ variables and thus belongs to H. If we apply to this equation the substitution operator $\mathbf{M}_D$, then, by (2;iii), the result is the equation $\mathbf{M}_D(\varphi) = \mathbf{M}_D(\chi)$. Therefore, this equation also belongs to the theory H. Summing over $D \in \mathscr{E}_\varkappa(V^\#)$ we get $\mathrm{S}(\varphi) \sim_H \mathrm{S}(\chi)$, the desired result.

Statement (II) is proved by induction on the length of the term φ constructed from the variables $V^\#$. It follows from (2;v) that $\mathrm{S}(\varphi) \leq_H \varphi$, and it must be shown that $\varphi \leq_H \mathrm{S}(\varphi)$. The only non-trivial step in this induction is

(II′) If $\varphi = \chi \cdot \psi$, $\chi \leq_H \mathrm{S}(\chi)$ and $\psi \leq_H \mathrm{S}(\psi)$, then $\varphi \leq_H \mathrm{S}(\varphi)$.

Thus, we are reduced to proving (II′); the next two statements contain the heart of the matter.

(3) Assume that $C \in \mathscr{E}_{\varkappa^\xi}(V^\#)$ and $V(\sigma) \subseteq V^\#$. Let x be any variable not belonging to $V^\#$. Then

(i) $x \cdot \mathbf{M}_C(\sigma) \leq_H \sum\{x \cdot \mathbf{M}_D(\sigma): D \in \mathscr{E}_\varkappa(V^\#)\}$;

(ii) $\mathbf{M}_C(\sigma) \leq_H \mathrm{S}(\sigma)$.

(4) Assuming that $V(\sigma) \subseteq V^\#$ and $x \notin V^\#$ then we have

$$x \cdot \mathrm{S}(\sigma) \leq_H \sum \{x \cdot \mathbf{M}_D(\sigma): D \in \mathscr{E}_\varkappa(V^\#)\} .$$

To prove (3), we first put $\tau = \mathbf{M}_C{}^0(\sigma)$ and $V' = \{y_C: y \in V^\#\}$ and notice that the equation

(i′) $x \cdot \tau \leq \sum\{x \cdot \mathbf{M}_E(\tau): E \in \mathscr{E}_\varkappa(V')\}$

belongs to H. In fact, under the assumptions of (3), this equation certainly has at most η $(= 1 + \varkappa^\xi)$ variables. To see that (i′) is a valid equation of L we use, for the first time, the simple observation motivating all of our definitions, viz: to every function $\mathbf{a}$, correlating elements of L with the variables of V', there corresponds a relation $E \in \mathscr{E}_\varkappa(V')$ such that τ and $\mathbf{M}_E(\tau)$ take the same value in L when evaluated at $\mathbf{a}$ — E is defined, of course, by $Ex_1x_2 \leftrightarrow \mathbf{a}(x_1)=\mathbf{a}(x_2)$. Next we apply to (i′) the substitution operator $\mathbf{M}_C$, leaving x untouched, and infer, by (2;iii) and (2;vii), that H contains the equation

(i″) $x\bullet\mathbf{M}_C(\sigma) \leq \sum\{x\bullet\mathbf{M}_{E'}(\sigma) : E \in \mathscr{E}_\varkappa(V')\}$;

in which the relation E' matched with E is defined by $E'yz \leftrightarrow Ey_Cz_C$. Clearly each $E' \in \mathscr{E}_\varkappa(V^\sharp)$, so (3;i) follows immediately. (3;ii) follows directly from (3;i).

To prove (4), let θ be the term represented by the right hand expression in (4), claimed to satisfy $x\bullet\mathbf{S}(\sigma) \leq_H \theta$. We notice that in view of (1) and the definition of $\mathbf{S}$, (4) will be established if we can prove that

$$x\bullet\sum\{\mathbf{M}_D(\sigma) : D \in \mathscr{T}\} \leq_H \theta ,$$

where $\mathscr{T}$ is any non-empty subset of $\mathscr{E}_\varkappa(V^\sharp)$ with at most ξ members. We let $C=\bigcap\mathscr{T}$ and simply observe that, by (2;vi), $C \in \mathscr{E}_{\varkappa^\xi}(V^\sharp)$ and consequently by (3;i),

$$x\bullet\mathbf{M}_C(\sigma) \leq_H \theta ;$$

which when combined with the relation

$$x\bullet\sum\{\mathbf{M}_D(\sigma) : D \in \mathscr{T}\} \leq_A x\bullet\mathbf{M}_C(\sigma) ,$$

easily inferred from (2;v), gives the desired result.

Returning now to (II′), let $\varphi=\chi\bullet\psi$, $\chi \leq_H \mathbf{S}(\chi)$ and $\psi \leq_H \mathbf{S}(\psi)$. Then by repeated application of (4) we obtain

$$\varphi \leq_H \mathbf{S}(\chi)\bullet\mathbf{S}(\psi) \leq_H \sum\{\mathbf{M}_D(\chi)\bullet\mathbf{M}_{D'}(\psi) : D, D' \in \mathscr{E}_\varkappa(V^\sharp)\} .$$

To prove (II′), it therefore suffices to prove that each term in this sum is $\leq_H \mathbf{S}(\varphi)$. Let thus $D, D' \in \mathscr{E}_\varkappa(V^\sharp)$. We set $C=D\cap D'$ and infer by (2;vi) — recalling that $\varkappa^2 \leq \varkappa^\xi$ was assumed at the outset — that $C \in \mathscr{E}_{\varkappa^\xi}(V^\sharp)$. We can thus apply (3;ii), as well as (2;v) and (2;i), and obtain

$$\mathbf{M}_D(\chi)\bullet\mathbf{M}_{D'}(\psi) \leq_A \mathbf{M}_C(\chi)\bullet\mathbf{M}_C(\psi) = \mathbf{M}_C(\varphi) \leq_H \mathbf{S}(\varphi) .$$

This establishes (II′), and the proof of Theorem 2.1 is now complete.

A slightly stronger result may easily be established by examining the foregoing argument. Given $\varkappa$ and defining η just as before, every lattice L of cardinality $\varkappa$ has as a base the set $\Theta^{(\varkappa)}L$, combined with the set of all equations of $\Theta^{(\eta)}L$ which are valid in every $\varkappa$-element lattice.

It would certainly be interesting to know whether Theorem 2.1 is true for some smaller values of η (depending recursively on $\varkappa$). In this connection, we define $\eta(\varkappa)$ to be the least integer α such that the theory of any lattice of cardinality $\varkappa$ can be based on equations containing no more than α variables. It is known then that

$$\eta(1) = 2, \quad \eta(2) = \eta(3) = \eta(4) = 3, \quad \eta(5) = 4;$$

while Theorem 2.1 states that

$$\eta(\varkappa) \leq 1+\varkappa^{\xi} \quad \text{if} \quad \varkappa < 2^{\xi+1}.$$

We have not been able to determine whether the function η is effectively computable (recursive).

By means of another argument, similar to the above but more involved, Theorem 2.1 can be extended to finite lattices with a finite but arbitrary set of additional operations. Examples have been constructed by several people to show that a finite algebra with finitely many operations need not have a finite equational base. The following theorem implies, on the other hand, that every such algebra in which the operations of a lattice can be defined does have a finite base.

THEOREM 2.2. *Let B be a finite lattice with operators, having altogether a finite number of basic operations. Then the equational theory of B is finitely based; in fact it can be based on two equations.*

3. Examples of non finitely based theories.

All theories mentioned in this section are understood to be lattice theories. We first consider some equations suggested by the lattices $B_\varkappa$

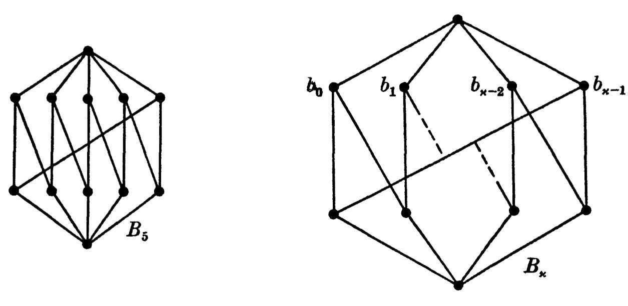

Fig. 1

($\varkappa \geq 5$) diagrammed in Fig. 1. For $\varkappa$ a positive integer and $\mu, \nu, \xi < \varkappa$ let us define $\mathfrak{d}_\varkappa(\mu,\nu)$ to mean

$$\nu \not\equiv \mu \pm 1 \ (\varkappa),$$

and $\mathfrak{f}_\varkappa(\mu,\nu,\xi)$ to mean that

$$\mu \not\equiv \nu(\varkappa) \wedge \mu \not\equiv \xi(\varkappa) \wedge \xi \equiv \nu+1 \ (\varkappa).$$

Then for $\varkappa > 4$ we have an equation

$$\gamma_\varkappa : \prod_{\mathfrak{f}_\varkappa(\mu,\nu,\xi)} (x_\mu + x_\nu \cdot x_\xi) \leq \sum_{\mathfrak{d}_\varkappa(\zeta,\eta)} x_\zeta \cdot x_\eta$$

with the variables $x_0, \ldots, x_{\varkappa-1}$. One can easily check that $\gamma_\varkappa$ holds in each lattice B_λ where $5 \leq \lambda \neq \varkappa$. However, $\gamma_\varkappa$ is not valid in $B_\varkappa$ — in fact, it clearly fails when the variables $x_0, \ldots, x_{\varkappa-1}$ take for their respective values the elements denoted by $b_0, \ldots, b_{\varkappa-1}$ in Fig. 1.

We infer from this that the infinite set of equations

$$\Gamma = \{\gamma_\varkappa : \varkappa \geq 5\}$$

is independent relative to Λ; it follows immediately that the theory $\Theta[\Gamma \cup \Lambda]$ has no finite base. It is not clear how the models of this theory

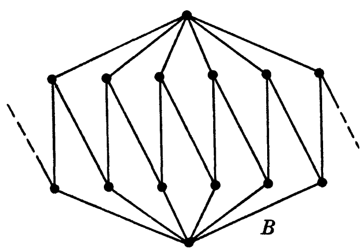

Fig. 2

may be described, but let us consider the infinite lattice of Fig. 2. We have

$$\Theta B = \bigvee_{\lambda > 4} \bigwedge_{\varkappa > \lambda} \Theta B_\varkappa;$$

in other words the equations of B are just those equations which hold in all but finitely many of the lattices $B_\varkappa$ — as can be proved by noting that B and $B_\varkappa$ have the same sublattices generated by λ elements, if $2\lambda < \varkappa$.

Consequently, ΘB has no finite base. Indeed, any finite subset of ΘB is contained in a $\Theta B_\varkappa$ and cannot imply the related equation $\gamma_\varkappa$ which is valid in B.

It proves rewarding to make a more critical study of theories connected with these lattices. The results of that investigation will be stated without proof. The results were discovered in part by A. Kostinsky. Let

$$\Xi = \bigwedge_{\varkappa > 4} \Theta B_\varkappa .$$

Then the theories in the interval $[\Xi, \Theta B]$ form a lattice, under the canonical operations which make the set of all lattice theories into a lattice. This lattice $[\Xi, \Theta B]$ is isomorphic to the lattice of all subsets of Γ.

For the isomorphism we map $\Sigma \rightsquigarrow \Theta[\Xi \cup \Sigma]$, $\Sigma \subseteq \Gamma$. The minimal elements in the lattice are the theories

$$\Theta[\Xi \cup \{\gamma_\lambda\}] = \bigwedge \{\Theta B_\varkappa : 4 < \varkappa \neq \lambda\};$$

and the maximal elements are the theories

$$\Theta[\Xi \cup (\Gamma - \{\gamma_\lambda\})] = \Theta B \cap \Theta B_\lambda .$$

Every theory Θ in the interval $\Xi \subseteq \Theta \subseteq \Theta B$ fails to have a finite base, although each one has an independent base. Actually, it is not very difficult to construct an independent base for $\Theta[\Xi \cup \Sigma]$, given $\Sigma \subseteq \Gamma$. (It is an open question whether every lattice theory has an independent base.)

The proof of these results makes essential use of [3; Cor. 3.2], from which it follows (indirectly) that every finitely generated, subdirectly irreducible model of the theory Ξ is monomorphic to one of the lattices $B_\varkappa$.

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