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SIDON SETS IN R^n

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0. Introduction.

Let A be a locally compact abelian group and let $E \subset A$ be a closed subset of A, we shall say that E is a Carleson-Helson set of A (C-H set in short) if for every $f \in C_0(E)$ there exists $\mu \in M(\hat{A})$ such that $f = \hat{\mu}|_E$. We shall say that $E \subset A$, an arbitrary subset of A, is a Sidon set (S set in short) if E is a C-H set of A_d, that is of A assigned with the discrete topology.

Let $X_1, X_2, \ldots, X_n$ be discrete spaces and let $V_0 = C_0(X_1) \hat{\otimes} \ldots \hat{\otimes} C_0(X_n)$ be the Tensor Algebra defined on these spaces. Then we shall say that $E \subset X = X_1 \times \ldots \times X_n$ is a V-Sidon set (V-S set in short) if for every $f \in C_0(E)$ there exists $\psi \in V_0$ such that $f = \psi|_E$.

For the above notions we refer the reader to [3, Ch. 11], [4, Ch. 5], [5], [6, p. 3].

In this paper we shall consider throughout R^n as a vector space over Q (the rationals) and we shall say that a subset $E \subset \mathsf{R}^n$ is independent if it is independent over Q.

Relative to independent subsets of R^n the following two results are well known and will be crucial in what follows.

(R) If $E_1, E_2, \ldots, E_s \subset \mathsf{R}^n$ are finitely many independent subsets then the set $E = E_1 \cup E_2 \cup \ldots \cup E_s$ is a Sidon set of R^n [4, 5·7·5], [6, p. 10].

(H) Let $E \subset \mathsf{R}^n$ be such that for every $V \subset \mathsf{R}^n$ vector subspace over Q of R^n we have $|E \cap V| \leqq k \dim_{\mathsf{Q}} V$ where k denotes a natural integer [$|X| =$ card$X$ for any set X]. Then there exist k independent subsets $E_1, E_2, \ldots, E_k \subset \mathsf{R}^n$ such that

$$E = E_1 \cup E_2 \cup \ldots \cup E_k ,$$

cf. [2], [6].

We shall finally adopt the following terminology due to J.-P. Kahane.

Definition. We shall say that a certain property "P" that depends on a parameter x, where x runs through a topological space T ($x \in T$), is verified for *quasiall* $x \in T$ if there exists $\Sigma \subset T$ a subset of 1st Baire category such that the property "P" is verified for all $x \in T \setminus \Sigma$ (in what follows T will be most of the time a Banach space).

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We shall finally denote by $C_\nu(\mathsf{R}^m)$, $m=1,2,\ldots$; $0 \leqq \nu \in \mathsf{R}$, the Banach space of the real bounded continuous functions on R^m which have bounded continuous partial derivatives up to the $[\nu]$th order and who's partial derivatives of order $[\nu]$ belong to $\lambda_{\nu-[\nu]}$, when $[\nu]<\nu$, cf. [8, p. 42].

For any $f \in C_0(\mathsf{R}^m)$, $m \geqq 1$, we shall denote by $\Gamma(f)$ the set

$$\Gamma(f) = \{(\underline{x},y) \in \mathsf{R}^m \times \mathsf{R} \, ; \, \underline{x} \in \mathsf{R}^m, \; y=f(\underline{x})\} \subset \mathsf{R}^{m+1} .$$

In this paper we shall prove first the following combinatorial

THEOREM 1. *For any $\nu \geqq 0$, any $m=1,2,\ldots$, and quasiall $f \in C_\nu(\mathsf{R}^m)$ the set $\Gamma(f)$ can be decomposed into $m+1$ independent subsets*

$$E_0, E_2, \ldots, E_m \subset \mathsf{R}^{m+1}, \quad \Gamma(f) = E_0 \cup E_1 \cup \ldots \cup E_m .$$

An immediate corollary of this theorem and of (R) is the following

THEOREM 2. *For any $\nu \geqq 0$, any $m=1,2,\ldots$, and quasiall $f \in C_\nu(\mathsf{R}^m)$ the set $\Gamma(f)$ is a Sidon subset of R^{m+1}.*

In contrast with theorem 2 we shall also prove

THEOREM 3. *Let $Q_1, Q_2, \ldots, Q_k \in \mathsf{R}[u]$ be arbitrary real polynomials of one variable, let*

$$\Sigma = \{(x_1, x_2, \ldots, x_{k+1}) \in \mathsf{R}^{k+1} \, ; \; 0 \leqq x_1 \leqq 1, \, x_{j+1} = Q_j(x_1), \, j=1,2,\ldots,k\}$$

be the algebraic variety they define over the unit interval. Then Λ is never a Sidon set of R^{k+1}.

Theorem 3 shows in particular that if $f \in C_\nu(\mathsf{R}^{m+1})$ coincides with some polynomial over some open subset Ω of R^m, then f belongs to the exceptional set of theorem 1.

The above theorems will be further commented in § 3.

1. Notations.

For the proof of theorem 1 we shall need to introduce a fairly complicated array of notations and remarks which will be enumerated below with capital letters (A)–(F).

(A) Let us denote for $n, s = 1,2,\ldots$

$$\mathsf{Z}_s^n = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathsf{Z}^n \, ; \, \sup_j |\alpha_j| \leqq s\}$$

and let $(j) = (0,0,\ldots,0,1,0,\ldots,0) \in \mathsf{Z}^n$, where the 1 lies on the jth place, be the "basis" vectors. Futher let $E = \mathsf{R}^N$ be some euclidean space, let

$x_1, x_2, \ldots, x_n \in E$ be n vectors of E, and let us denote by M the net of order s defined over the set $\{x_1, x_2, \ldots, x_n\}$, that is

$$M = M_s(x_1, x_2, \ldots, x_n) = \left\{\sum_{j=1}^{n} \alpha_j x_j \; ; \; (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathsf{Z}_s^{\,n}\right\} \subset E .$$

We shall call n the (formal) dimension of M ($\dim M = n$) and we shall denote by $\mathscr{N}_n(E)$ the set of all nets of dimension n in E.

(B) Let $\boldsymbol{\alpha} \in \mathsf{Z}_s^{\,n}$ and let us set

$$L_\alpha(u_1, u_2, \ldots, u_n) = \sum_{j=1}^{n} \alpha_j u_j$$

which is a linear form of n variables, let us also set

$$\mathscr{L}_s^{\,n} = \{L_\alpha \; ; \; \boldsymbol{\alpha} \in \mathsf{Z}_s^{\,n}\} .$$

We shall identify throughout an element of the m-fold cartesian product

$$\boldsymbol{L} = (L^{(1)}, L^{(2)}, \ldots, L^{(m)}) \in (\mathscr{L}_s^{\,n})^m$$

with a vector valued form

$$\boldsymbol{L} : \mathsf{C}^{nm} \to \mathsf{C}^m$$

defined by

$$\boldsymbol{L}(\underline{u}) = \left(L^{(1)}(\underline{u}^{(1)}), \ldots, L^{(m)}(\underline{u}^{(m)})\right) \in \mathsf{C}^m, \quad \underline{u} = (\underline{u}^{(1)}, \ldots, \underline{u}^{(m)}) \in \mathsf{C}^n \times \ldots \times \mathsf{C}^n .$$

(C) The substitution $u_j \to x_j \in E$, $j = 1, \ldots, n$, induces a mapping from $\mathscr{L}_s^{\,n}$ on $M = M_s(x_1, \ldots, x_n)$. When this mapping is 1–1, we shall say that the net M is independent.

The thing to observe here is that for an arbitrary net $M = M_s(x_1, x_2, \ldots, x_n)$ if we choose $x_1', \ldots, x_n'$ an appropriate basis over Q of the subspace $\mathrm{Vec}_{\mathsf{Q}}\{x_1, x_2, \ldots, x_n\} \subset E$, then we have

$$M \subset M' = M_{s'}(x_1', \ldots, x_{n'}'), \quad n' \leqq n ,$$

for an appropriate $s' \geqq 1$ and where now M' is independent.

(D) Let E_1, E_2 be two euclidean spaces and let $x_j = (x_j^{(1)}, x_j^{(2)}) \in E_1 \times E_2 = E$, $j = 1, \ldots, n$; we have then trivially

$$M_s(x_1, x_2, \ldots, x_n) \subset M_s(x_1^{(1)}, \ldots, x_n^{(1)}) \times M_s(x_1^{(2)}, \ldots, x_n^{(2)}) \subset E_1 \times E_2 .$$

(E) Let us denote now by $\mathscr{M}_s^{\,n} \subset \mathscr{N}_n(\mathsf{R})$ the set of all independent nets of order s and dimension n of R. We can then find $F_1, F_2, \ldots, F_\alpha \subset \mathsf{R}^n$, a finite number of hyperplanes (subspaces of co-dimension 1 over R), such that

$$(1.1) \qquad M_s(x_1, \ldots, x_n) \in \mathscr{M}_s^{\,n} \Leftrightarrow (x_1, \ldots, x_n) \in \complement \bigcup_{j=1}^{\alpha} F_j .$$

The set $\complement\bigcup_{j=1}^{\alpha} F_j$ being a K_σ can be decomposed as a countable union of closed cubes $C_1, C_2, \ldots$ of R^n, $\complement\bigcup_{j=1}^{\alpha} F_j = \bigcup_{k=1}^{\infty} C_k$, and the identification (1.1) allows us then to write for any such decomposition

$$\mathscr{M}_s{}^n = \bigcup_{k=1}^{\infty} C_k . \tag{1.2}$$

(F) It is easy to see that for every $n, s, k \geqq 1$ we can choose a closed cube $C_k{}^{ns} \subset \mathsf{R}^n$ and a finite family of disjoint compact intervals

$$\Omega_k{}^{ns} = \{\omega_\alpha \subset \mathsf{R};\ \alpha \in \mathsf{Z}_s{}^n\} \tag{1.3}$$

such that the cubes $C_k{}^{ns}$ form a decomposition as in (1.2),

$$\mathscr{M}_s{}^n = \bigcup_{k=1}^{\infty} C_k{}^{ns} , \tag{1.4}$$

and such that

$$M = M_s(x_1, x_2, \ldots, x_n) \in C_k{}^{ns} \iff x_j \in \omega_{(j)}, \quad 1 \leqq j \leqq n ,$$

$$x_j \in \omega_{(j)},\ 1 \leqq j \leqq n \Rightarrow \begin{cases} M \subset \bigcup_{\alpha \in \mathsf{Z}_s{}^n} \omega_\alpha \subset \mathsf{R} , \\ |x \in M;\ x \in \omega_\alpha| \leqq 1 \ \ \forall \alpha \in \mathsf{Z}_s{}^n . \end{cases}$$

The choice of the cubes (1.4) and of the intervals (1.3) will be done once and for all here and will be kept fixed throughout the next section.

2. Proof of theorem 1.

If we use (H) of the introduction, we see that theorem 1 is an immediate consequence of the following

THEOREM 1′. *Let ν, m with $0 \leqq \nu \in \mathsf{R}$, $1 \leqq m \in \mathsf{Z}$ be given. Then for quasiall $f \in \boldsymbol{C}_\nu(\mathsf{R}^m)$ we have*

$$|\Gamma(f) \cap M| \leqq (m+1)n \quad \forall M \in \mathscr{N}_n(\mathsf{R}^{m+1}) , \quad n \geqq 1 .$$

Using (C), (D) and (E) of § 1 we see that theorem 1′ follows from

LEMMA 2.1. *Let ν and m be as in theorem 1′, let $n, s, k_1, \ldots, k_m$ be fixed positive integers, and let us denote by $S_1 \subset \boldsymbol{C}_\nu(\mathsf{R}^m)$ the set of those functions f for which there exist nets $M_r \in C_{k_r}^{ns}$, $r = 1, \ldots, m$, and distinct points $a_p \in M_1 \times M_2 \times \ldots \times M_m$, $p = 1, \ldots, n(m+1)+1$, such that*

$$\operatorname{rank}_{\mathsf{Q}} \{f(a_p)\ ;\ p = 1, \ldots, n(m+1)+1\} \leqq n .$$

Then S_1 is a set of the 1st *category in $\boldsymbol{C}_\nu(\mathsf{R}^m)$.*

But using (B) and (F) of § 1 we see that lemma 2.1 in its turn follows from

LEMMA 2.2. *Let* v *and* m *be as in theorem* 1′, *let* $n, s, k_1, \ldots, k_m$ *be positive integers, let* $L_p \in (\mathscr{L}_s{}^n)^m$, $p = 1, 2, \ldots, n(m+1)+1$, *be distinct forms, and let*

$$\{\lambda_{ij} \in \mathbf{Q}\ ;\ i = 1, 2, \ldots, n\ ; j = 1, 2, \ldots, nm+1\}$$

be a fixed rational matrix. Let us define then $S_2 \subset C_v(\mathsf{R}^m)$, *the set of those functions* $f \in C_v(\mathsf{R}^m)$ *for which there exist vectors*

$$(2.1) \qquad \underline{u} = (\underline{u}^{(1)}, \ldots \underline{u}^{(m)}) \in \mathsf{R}^n \times \ldots \times \mathsf{R}^n, \quad \underline{u}^{(r)} = (u_1{}^{(r)}, \ldots, u_n{}^{(r)}), \quad r = 1, \ldots, m\,,$$

such that

$$(2.2) \qquad u_i{}^{(r)} \in \omega_{(i)} \in \Omega_{k_r}^{ns}, \quad r = 1, \ldots, m, \ i = 1, \ldots, n\,,$$

$$\sum_{i=1}^{n} \lambda_{ij} f(L_i(\underline{u})) = f(L_{n+j}(\underline{u})), \quad j = 1, 2, \ldots, nm+1\,.$$

Then S_2 *is a closed subset of* $C_v(\mathsf{R}^m)$ *with an empty interior.*

The fact that S_2 is closed is easy to verify. Indeed, let $f_q \in S_2$, $q = 1, 2, \ldots$, be a sequence such that $f_q \to f \in C_v(\mathsf{R}^m)$ uniformly and let $\underline{u}_q \in \mathsf{R}^{nm}$ be the associated sequence of vectors (2.1) that together with the function f_q satisfy (2.2); using then the compactness of the intervals ω we see that we may suppose that $\underline{u}_q \to \underline{u} \in \mathsf{R}^{nm}$ as $q \to \infty$, and then it is clear that $\underline{u}$ satisfies (2.2) with the function f, which function therefore belongs to S_2.

The rest of this section will be devoted to showing that S_2 is of empty interior.

Let P_h, $h \geqq 1$, denote the space of real polynomials of m variables and degree at most h, we shall assign this finite dimensional space with its natural topology and we shall denote by $P_h{}^t$ ($t \geqq 1$) the tth cartesian power of P_h assigned with the product topology. We shall prove then the following two lemmas.

LEMMA 2.3. *Let* m, v, n, s, $L_p \in (\mathscr{L}_s{}^n)^m$, *and the matrix* $\{\lambda_{ij} \in \mathbf{Q}\}$ *be as in lemma* 2.2. *Then for every* $h \geqq 1$ *there exists an everywhere dense subset* $E_h \subset P_h{}^{n(m+1)+1}$ *such that for arbitrary*

$$(2.3) \qquad (\varphi_p)_{p=1}^{n(m+1)+1} \in E_h$$

the system of algebraic equations

$$(2.4)\qquad \sum_{i=1}^{n} \lambda_{ij}\varphi_i(L_i(\underline{u})) = \varphi_{n+j}(L_{n+j}(\underline{u})), \quad j=1,2,\ldots,nm+1,$$

has no solution $\underline{u} \in \mathsf{C}^{nm}$ *(i.e. is incompatible).*

LEMMA 2.4. *Let* $F_0, F_1, \ldots F_l \in \mathsf{C}[x_1, x_2, \ldots, x_l]$ *be* $l+1$ *complex polynomials of* l *variables, and let*

$$V_\lambda(\alpha) = \{\underline{u} \in C^l \,;\, F_\lambda(\underline{u}) = \alpha\}, \quad \alpha \in \mathsf{C}, \quad \lambda = 0,1,\ldots,l,$$

be the associated algebraic variety. Then for any $\lambda = 0,1,\ldots,l$ *there exists* $D_\lambda \subset \mathsf{R}^{\lambda+1}$ *an everywhere dense subset such that for every* $(\alpha_0, \alpha_1, \ldots, \alpha_\lambda) \in D_\lambda$ *we have*

$$\dim\left(V_0(\alpha_0) \cap V_1(\alpha_1) \cap \ldots \cap V_\lambda(\alpha_\lambda)\right) \leqq l-\lambda-1 .$$

PROOF OF LEMMA 2.4. The proof is done by induction on λ. For $\lambda=0$ the result is evident, so suppose that lemma 2.4 holds for $\lambda=\mu<l$ let $\boldsymbol{\alpha}=(\alpha_0,\ldots,\alpha_\mu) \in D_\mu$ be some fixed point and let

$$V_0(\alpha_0) \cap V_1(\alpha_1) \cap \ldots \cap V_\mu(\alpha_\mu) = V = V^{(1)} \cup V^{(2)} \cup \ldots \cup V^{(\varrho)}$$

be the decomposition of V into its irreducible components [7, § 94–95]. Let further

$$\begin{aligned} X_\sigma(\boldsymbol{\alpha}) &= \{r \in \mathsf{R} \,;\, \dim(V^{(\sigma)} \cap V_{\mu+1}(r)) = l-\mu-1\} \\ &= \{r \in \mathsf{R} \,;\, V_{\mu+1}(r) \supset V^{(\sigma)}\}, \qquad \sigma = 1,2,\ldots,\varrho, \end{aligned}$$

cf. [7, § 94–95]. Since now for $r_1 \neq r_2$, two distinct real numbers, the two varieties $V_{\mu+1}(r_1)$ and $V_{\mu+1}(r_2)$ are disjoint we conclude that the set $X_\sigma(\boldsymbol{\alpha})$, $\sigma=1,2,\ldots,\varrho$, consists of at most one point and that therefore the set

$$E = \bigcup\{(\boldsymbol{\alpha},r) \,;\, \boldsymbol{\alpha} \in E_\mu,\ r \in \mathsf{R},\ r \notin X_\sigma(\boldsymbol{\alpha}),\ \sigma=1,2,\ldots,\varrho\}$$

is everywhere dense. Also by the definition of D we can set $D_{\mu+1}=D$ in our lemma and complete the inductive step.

PROOF OF LEMMA 2.3. Let us suppose that

$$\varphi_p^{(0)} \in P_h, \quad p=1,2,\ldots,n(m+1)+1,$$

is a choice of polynomials for which the system (2.4) is compatible. Using then lemma 2.4 with $\lambda=l=nm$ we see that there exists an everywhere dense subset $\Omega \subset \mathsf{R}^{nm+1}$ such that for every $\boldsymbol{\alpha}=(\alpha_j\,;\, j=1,2,\ldots,nm+1) \in \Omega$ the choice

$$\varphi_i = \varphi_i^{(0)}, \quad i=1,2,\ldots,n; \qquad \varphi_{n+j} = \varphi_{n+j}^{(0)} + \alpha_j, \quad j=1,2,\ldots,nm+1\,,$$

in (2.3) makes the system (2.4) incompatible. This proves lemma 2.3.

PROOF OF $\mathring{S}_2=\emptyset$. To do that we first observe that from the definition in § 1 (F) it follows that there exist compact disjoint cubes

$$\{K_p \subset \mathsf{R}^m\,;\, p=1,2,\ldots,n(m+1)+1\}$$

such that if

$$u_i^{(r)} \in \omega_{(i)} \in \Omega_{k_r}^{ns}, \quad r=1,\ldots,m,\, i=1,\ldots,n\,,$$

as in (2.2), then

$$L_p(\underline{u}) \in K_p, \quad p=1,\ldots,n(m+1)+1\,.$$

Let then $E_h \subset P_h{}^{n(m+1)+1}$ be the everywhere dense subset constructed in lemma 2.3, and let us define

$$H = \{f \in C_\nu(\mathsf{R}^m)\,;\, \exists h \geqq 1,\, (\varphi_p)_{p=1}^{n(m+1)+1} \in E_h \text{ such that } \varphi_p|_{K_p}=f|_{K_p}\ \forall p\}\,.$$

Now if we use the well known fact that for every compact cube $K \subset \mathsf{R}^m$ the space of restrictions of polynomials is dense in $C_\nu(\mathsf{R}^m)/I(K)$, where

$$I(K) = \{f \in C_\nu(\mathsf{R}^m)\,;\, f^{-1}(0) \supset K\}\,,$$

and if we separate the cubes K_p, $p=1,2,\ldots,n(m+1)+1$, using well known techniques of partitions of unity, we see that H is everywhere dense in $C_\nu(\mathsf{R}^m)$.

Also from the definition of H it follows that for every $f \in H$ the system (2.2) is incompatible and therefore that $H \cap S_2=\emptyset$. This proves our assertion and completes the proof of theorem 1′.

3. Applications to harmonic analysis and theorem 2.

As we have already observed in the introduction, theorem 1 implies at once theorem 2 on the Sidon character of $\Gamma(f)$. The interest of theorem 2 lies in the fact that the sets $\Gamma(f)$ in general are not C-H sets of R^{m+1}.

Indeed, using the results in [1] it is easy to see that for any $m \geqq 1$ and any $\nu \geqq m+2$ there exists $\Omega \subset C_\nu(\mathsf{R}^m)$ an open subset such that for every $f \in \Omega$ we have $M_0(\Gamma(f)) \neq \{0\}$, that is, there exists $0 \neq \mu \in M(\Gamma(f))$ some measure whose Fourier transform tends to zero at infinity ($\hat{\mu}(\underline{x}) \to 0$ as $\underline{x} \to \infty$). But this implies of course [4, Ch. 5] that then $\Gamma(f)$ is not a C-H set. The following two remarks are relevant on this point.

REMARKS. (i) We can set $\nu = +\infty$ in theorems 1 and 2. More precisely we can replace $C_\nu(\mathsf{R}^m)$ by the space of the real functions on R^m that are bounded and have bounded derivatives of all orders. That space assigned with the family of seminorms

$$p_\alpha(f) = \sup_{x \in \mathsf{R}^m} \left| \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right|, \quad \alpha = (\alpha_1, \ldots, \alpha_m),$$

where α runs through all positive multiindices, is complete and metric.

(ii) Let us consider the space $\prod_\nu^m = (C_\nu(\mathsf{R}^m))^\infty$, the countable cartesian power of $C_\nu(\mathsf{R}^m)$ assigned with the product topology, and $B_\nu{}^m = L^\infty(\mathsf{Z}^+ ; C_\nu(\mathsf{R}^m))$, the space of bounded sequences $b = (f_1, f_2, \ldots, f_n, \ldots)$ with the norm $\|b\| = \sup_j \|f_j\|$. We have then

THEOREM 1″. *For any* $0 \leqq \nu \in \mathsf{R}$ *and* $1 \leqq m \in \mathsf{Z}$ *and quasiall* $(f_1, f_2, \ldots, f_n, \ldots) \in \prod_\nu^m$ *(resp.* $\in B_\nu{}^m$*) the set* $\Gamma = \bigcup_{j=1}^\infty \Gamma(f_j) \subset \mathsf{R}^{m+1}$ *admits a decomposition* $\Gamma = E_0 \cup E_1 \cup \ldots \cup E_m$ *into* $m+1$ *independent subsets* $E_0, E_1, \ldots, E_m \subset \mathsf{R}^{m+1}$.

The proofs of the above facts are easy modifications of the ones already given.

4. *V*-Sidon sets and theorem 3.

Let us denote by $A \subset \mathsf{R}$ the set of real algebraic numbers, and let us observe that the set

$$V = \{\log x \,;\, x \in A,\ x > 0\} \subset \mathsf{R}$$

is a vector subspace of R over Q.

We shall prove first that theorem 3 is a consequence of

THEOREM 3′. *Let* $P_1, P_2, \ldots, P_k \in \mathsf{R}[u,v]$ *be real polynomials of two variables and let*

$$\Lambda = \{(x_1, x_2, \ldots, x_{k+2}) \in \mathsf{R}^{k+2} \,;\, x_1, x_2 \in (0,1) \cap A,\ x_{j+2} = P_j(x_1, x_2),\ j = 1, 2, \ldots, k\}.$$

Then Λ *is not a Sidon set of* R^{k+2}.

Indeed, let $Q_1, Q_2, \ldots, Q_k$ be the polynomials of theorem 3, let τ be some transcendental number and let us define

$$P_j(u,v) = Q_j(u + \tau v), \quad j = 1, 2, \ldots, k;$$

It is immediate that if theorem 3 were false, then theorem 3′ with the above choice of P_j (modulo a rational change of variables to get the range in $[0,1]$) would also be false.

The proof of theorem 3′ is based on the following lemma (4.1) that uses the notions of tensor algebras, and whose proof is trivial and left to the reader.

LEMMA 4.1. *Let A, B be two discrete abelian groups, let $m \geqq 1$ be a positive integer and let $E_1, E_2, \ldots, E_s$, be discrete spaces. Let also*

$$g_j : A^m \to E_j, \quad h_j : E_j \to B, \quad j = 1,2,\ldots,s,$$
$$g(a) = (g_1(a),\ldots,g_s(a)) \in E = E_1 \times \ldots \times E_s, \quad a \in A^m,$$
$$h(e_1,\ldots,e_s) = \sum_{j=1}^{s} h_j(e_j) \in B, \quad e_j \in E_j, \quad j = 1,\ldots,s,$$

be fixed mappings, and let finally $I \subset A$ a be fixed subset of A. Let us then denote

$$G = \{(a,e) \in A^m \times E \; ; \; a \in I^m \subset A^m, \; e = g(a)\},$$
$$L = \{(a,b) \in A^m \times B \; ; \; a \in I^m \subset A^m, \; b = h \circ g(a)\},$$

and let us suppose that L is a Sidon set of the discrete group $A^m \times B$. Then G is a V-Sidon set for the algebra

$$V_0 = C_0(A)^{\hat{\otimes} m} \,\hat{\otimes} \hat{\bigotimes_{1 \leqq j \leqq s}} C_0(E_j),$$

where $C_0(A)^{\hat{\otimes} m} = C_0(A) \hat{\otimes} \ldots \hat{\otimes} C_0(A)$ indicates the m-th tensor power of $C_0(A)$.

We shall now apply lemma 4.1 by setting $m = 2$, $A = \mathsf{R}$, $B = \mathsf{R}^k$ (both with the discrete topology), $I = (0,1) \cap A$, and by choosing the remaining parameters in such a way that

$$L = \Lambda \subset A^2 \times B = \mathsf{R}^{k+2}.$$

To do this it suffices to choose each $E_j = \mathsf{R}$ and each $g_j : A^2 = \mathsf{R}^2 \to \mathsf{R}$ an appropriate monomial mapping

$$g_j(x_1,x_2) = a_j x_1^{\alpha_1} x^{\alpha_2}, \quad a_j \in \mathsf{R}, \; \alpha_1, \alpha_2 \text{ positive integers},$$

and also to choose each $h_j : E_j = \mathsf{R} \to B = \mathsf{R}^k$ of the form

$$h_j(r) = (0,0,\ldots,r,0,\ldots,0) \in \mathsf{R}^k, \quad r \in \mathsf{R},$$

the r being in the β_jth place where β_j is appropriately chosen and depends on j, and j runs of course through an appropriate set, $j = 1,2,\ldots,s$.

The integer $s \geqq 1$, the monomials g_j, and the β_j's are of course chosen so as to "build up the polynomials $P_1, P_2, \ldots, P_k$" and so as to have $L = \Lambda$ (cf. § 0, theorem 3) as required.

Let us now observe that we can write the above g_j's in the form

$$g_j = a_j \exp[L_j(\log x_1, \log x_2)] ,$$

where L_j, $j = 1, 2, \ldots, s$, are linear forms of two variables with integer coefficients. From this we see by performing an obvious change of variables in the carrier space of the tensor algebra

$$V_0 = C_0(\mathsf{R})^{\hat{\otimes}(m+s)}$$

that theorem 3 is then an immediate consequence of lemma 4.1 and

Lemma 4.2. *Let $L_j(u,v) = n_j u + m_j v$, $j = 1, 2, \ldots, s$, be s linear forms of two variables with integer coefficients and let*

$$G = \{(x_1, x_2, \ldots, x_{s+2}) \in \mathsf{R}^{s+2} ;\ x_1, x_2 \in V \cap (-\infty, 0),\\ x_{j+2} = L_j(x_1, x_2),\ j = 1, 2, \ldots, s\} \subset \mathsf{R}^{s+2} .$$

Then G is not a V-S set of the algebra $C_0(\mathsf{R})^{\hat{\otimes}(s+2)}$.

Lemma 4.2 is itself an immediate consequence of the following lemma (cf. [5], [6, p. 16]).

Lemma 4.3. *Let $L_j(u,v) = n_j u + m_j v$, $j = 1, 2, \ldots, p$, be p linear forms of two variables with integer coefficients. We can then find finite subsets $Z_1, Z_2, \ldots, Z_n, \ldots \subset V^2$ such that for all $n \geqq 1$*

$$(4.2) \qquad |Z_n| \geqq (p+1)^n, \quad \sup_{1 \leq j \leq p} |L_j(Z_n)| \leqq p^n .$$

Proof. We shall proceed by induction. The construction of Z_1 is trivial, it suffices to fix an arbitrary $z_0 \in V^2$ and then choose points $z_1, z_2, \ldots, z_p \in V^2$ distinct among themselves and from z_0 such that

$$L_j(z_j) = L_j(z_0), \quad j = 1, 2, \ldots, p .$$

Then we can set $Z_1 = \{z_0, z_1, \ldots, z_p\}$.

Let us now suppose that $Z_1, Z_2, \ldots, Z_n$ have been constructed satisfying (4.2), and let $0 < \alpha \in \mathsf{Q}$ be so large that for two distinct vectors $z_\nu, z_\mu \in Z_1$, $\nu \neq \mu$, $\nu, \mu = 0, 1, \ldots, p$, we have

$$(\alpha z_\nu + Z_n) \cap (\alpha z_\mu + Z_n) = \emptyset;$$

we can set then $Z_{n+1} = \alpha Z_1 + Z_n$ and it is trivial to verify that Z_{n+1} satisfies (4.2).

REMARK. The method used to prove theorem 3 can be generalized so as to show that for other special types of $f \in C_\nu(\mathsf{R}^n)$ the set $\Gamma(f)$ cannot be Sidon. For example, if $n = 2$ and

$$f(x,y) = \sum_{i=1}^{N} (\varphi_i(x))^{\varrho_i} (\psi_i(y))^{\sigma_i}, \tag{4.3}$$

where $N \geqq 1$, $\varrho_i, \sigma_i \in \mathsf{R}$, $\varphi_i, \psi_i \in C_\nu(\mathsf{R})$, $i = 1, 2, \ldots, N$, we can deduce that $\Gamma(f)$ is not Sidon. More general forms than (4.3) can also be treated in the same way.

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