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ON A THEOREM OF N. TH. VAROPOULOS

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1. Introduction.

The following theorem is proved in [1]:

THEOREM OF VAROPOULOS. For any real number $\nu \geqq 0$, any integer $m \geqq 1$ and quasi all $f \in C_\nu(\mathsf{R}^m)$ (that is, all f outside some set of first category) the set $\Gamma(f)$ (the graph of f in R^{m+1}) is a Sidon set in the discrete abelian group R^{m+1}.

Here $C_\nu(\mathsf{R}^m)$ is the Banach space of all bounded continuous functions $f\colon \mathsf{R}^m \to \mathsf{R}$ which have bounded continuous partial derivatives up to order $[\nu]$ (entier of ν), and whose partial derivatives of order $[\nu]$ are of Lipschitz type $\lambda_{\nu-[\nu]}$. For the definition of λ_α, $0 \leqq \alpha < 1$, and Λ_α, $0 < \alpha \leqq 1$, we refer to [2, page 42].

To prove this theorem it is shown in [1] that all $f \in C_\nu$ whose graphs $\Gamma(f)$ are non-Sidon lie in the union of countably many closed subsets of C_ν with empty interiors. In this article we shall indicate an alternative method to prove that these subsets have no interior, which avoids the use of lemmas 2.3 and 2.4 in [1]. The method also gives analogous theorems for the Banach spaces Λ_ν, $\nu > 0$, of all bounded continuous functions $f\colon \mathsf{R}^m \to \mathsf{R}$ which have bounded continuous partial derivatives up to order $(\nu) = \sup_{\nu' < \nu} [\nu']$, and whose partial derivatives of order (ν) are of Lipschitz type $\Lambda_{\nu-(\nu)}$. These theorems could not be obtained by the method of [1] because polynomials are not dense (when restricted to compacta) in Λ_ν.

2. The method.

Instead of graphs $\Gamma(f) \subset \mathsf{R}^{m+1}$ we consider zero sets $Z(f) = f^{-1}(\{0\}) \subset \mathsf{R}^m$. The integer m in this article therefore corresponds to $m+1$ in [1]. Obviously the collection of graphs is contained in the collection of zero sets:

$$\{\Gamma(f)\,;\, f \in C_\nu(\mathsf{R}^{m-1})\} \subset \{Z(f)\,;\, f \in C_\nu(\mathsf{R}^m)\}\,.$$

The method is based on the following lemma.

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LEMMA. *If* $f: \mathsf{R}^q \to \mathsf{R}^r$, *q and r positive integers, is a function of Lipschitz type* Λ_α, *and if* $q < \alpha r$, *then* $f(\mathsf{R}^q)$ *is a subset of* R^r *with Lebesgue measure zero.*

The proof is easy and is left to the reader. The conclusion is no longer true if $q \geqq \alpha r$, as is shown for instance by the well known Peano function $f: \mathsf{R} \to \mathsf{R}^2$ which is of type $\Lambda_{\frac{1}{2}}$ and whose image of $[0,1]$ fills a whole square.

Now let $m \geqq 2$, $n \geqq 1$, $p \geqq 1$ be fixed positive integers, α a real number, $0 < \alpha \leqq 1$, and let $mn < \alpha p$. Consider R^m as a vector space over Q, the rational numbers, let $V \subset \mathsf{R}^m$ be an n-dimensional Q-linear subspace of R^m, and let $A = \{a_1, a_2, \ldots, a_p\} \subset V$ be a subset of V with $\operatorname{Card} A = p$. Further let $f: \mathsf{R}^m \to \mathsf{R}$ be a function of type Λ_α such that f vanishes in the points of A. We define $\Phi_f: \mathsf{R}^{mp} \to \mathsf{R}^p$ by

$$\Phi_f(x_1, \ldots, x_p) = (f(x_1), \ldots, f(x_p)) .$$

The map Φ_f vanishes in the point $(a_1, \ldots, a_p) \in \mathsf{R}^{mp}$. Let $C \subset \mathsf{R}^{nm}$ be a cube in R^{nm} and let $\Delta: \mathsf{R}^{nm} \to \mathsf{R}^{mp}$ be an R-linear mapping such that $(a_1, \ldots, a_p) \in \Delta(C)$.

Then $\Phi_f \circ \Delta: \mathsf{R}^{nm} \to \mathsf{R}^p$ is still a function of type Λ_α, hence by the lemma there are constants $x = (\xi_1, \ldots, \xi_p) \in \mathsf{R}^p$ with arbitrarily small norms $|x| = \sup |\xi_k|$ such that the functions $(\Phi_f \circ \Delta) + x$ do not vanish in any point of C (not even of R^{nm}).

A careful study of [1] should show how C and Δ have to be chosen depending on V and A. It follows then that for all $x \in \mathsf{R}^p$ there exists a function $g \in \bigcap_{\nu \geqq 0} C_\nu(\mathsf{R}^m)$ such that

$$(\Phi_{f+g} \circ \Delta)|_C = (\Phi_f \circ \Delta)|_C + x ,$$

and such that $\|g\|_{C_\nu} \leqq K|x|$, K a constant not depending on x.

This way to approximate f by functions $f+g$ such that the functions $\Phi_{f+g} \circ \Delta$ do not vanish in any point of C replaces the technique of lemmas 2.3 and 2.4 of [1].

We now take $\alpha = 1$ and $p = nm + 1$. The proof of the theorem of Varopoulos can then be completed as in [1], with the change indicated above. Instead of approximation by polynomials we can approximate by functions in the larger class $C_\nu \cap \Lambda_1$. We leave the details to the reader of [1].

3. New results.

The method of section 2 gives in the same way the following results. We remark that theorem 1 below is the analogue of theorem 1′ in [1].

Theorem 1. *If $m \geqq 2$ and $0 < \alpha \leqq 1$, then for quasi all $f \in \Lambda_\alpha(\mathsf{R}^m)$ and all Q-linear subspaces $V \subset \mathsf{R}^m$ we have*

$$\mathrm{Card}(Z(f) \cap V) \leqq m\alpha^{-1} \dim V .$$

Theorem 2. *If $m \geqq 2$ and $\nu > 0$, then for quasi all $f \in \Lambda_\nu(\mathsf{R}^m)$ the zero set $Z(f)$ is a Sidon set for the discrete group R^m.*

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LITERATURE

1. N. Th. Varopoulos, *Sidon sets in* R^n, Math. Scand. 27 (1970), 39–49.
2. A. Zygmund, *Trigonometric series* I, Cambridge University Press, 1959.

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