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ON EQUATIONAL CLASSES OF ALGEBRAIC VERSIONS OF LOGIC I

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In this paper we study equational classes of one- and two-dimensional polyadic algebras. We intend to extend this study to higher dimensions, and to other kinds of algebraic logics, such as relation and cylindric algebras.

For one-dimensional polyadic algebras the lattice of equational classes forms a chain of type $\omega+1$, and we give explicit, simple, equations characterizing each class. We do not have such complete results for two-dimensional polyadic algebras, but we answer some fundamental questions about their lattice of equational classes: there are $\aleph_0$ classes; each class is finitely based; each class is determined by its finite members; each class has decidable equational theory.

The methods of this paper are elementary, but we make essential use of the fundamental article Jónsson [3]. Representation theory does not play a role here, although we may mention that every two dimensional polyadic algebra is representable, and every countable simple two-dimensional polyadic algebra can be embedded in the algebra of all subsets of $\omega \times \omega$ (cf. Everett and Ulam [1], McKinsey [5], Halmos [2, pp. 97–166]. G.-C. Rota suggested the problem of determining equational classes of polyadic algebras. Several conversations with James S. Johnson were useful.

We use standard set-theoretical language. The letters $m,n,\ldots$ range over ω, the set of natural numbers. We use $f\colon A \to B$, $f\colon A \rightarrowtail B$, $f\colon A \twoheadrightarrow B$ to indicate that f is a function mapping A into B, one-one into B, or one-one onto B respectively. We denote by ${}^{A}B$ the set of all functions mapping A into B. The f-image of a set X is denoted by f^*X. We use f_{xyz} interchangably with $f(x,y,z)$. By $\boldsymbol{I}$ we denote the identity function: $\boldsymbol{I}x=x$ for all x. If E is an equivalence relation on a set A, a subset of A is *E-closed* if it is a union of E-classes. The equivalence class

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of an element $a \in A$ is denoted by a/E. If $X \subseteq A$ we let $X/E = \{x/E : x \in X\}$. Further, $SA = \{X : X \subseteq A\}$.

Symmetric difference is denoted by $\oplus$. If $\mathfrak{A}$ is a Boolean algebra with operators, $\mathrm{At}\,\mathfrak{A}$ is the set of all atoms of $\mathfrak{A}$; for $a \in A$, $\mathrm{At}\,a$ is the set of all atoms $\leq a$.

If $\boldsymbol{K}$ is a class of similar algebras, then $\boldsymbol{HK}$, $\boldsymbol{SK}$, $\boldsymbol{PK}$ are the classes of homomorphic images of members of $\boldsymbol{K}$, isomorphs of subalgebras of members of $\boldsymbol{K}$, and isomorphs of direct products of members of $\boldsymbol{K}$. Thus $\boldsymbol{HSPK}$ is the equational closure of $\boldsymbol{K}$. An equational class $\boldsymbol{K}$ is *determined by* $\boldsymbol{L}$ if $\boldsymbol{K} = \boldsymbol{HSPL}$. If $\boldsymbol{L} = \{\mathfrak{A} : \mathfrak{A} \in \boldsymbol{K},\ |A| < \omega\}$, we say that $\boldsymbol{K}$ is *determined by its finite members*. A set Γ of equations *characterizes* $\boldsymbol{L}$ *relative* to $\boldsymbol{K}$ if $\boldsymbol{L}$ consists of all $\mathfrak{A} \in \boldsymbol{K}$ in which all equations of Γ hold. For algebras $\mathfrak{A}, \mathfrak{B}$ we write $f\colon \mathfrak{A} \to \mathfrak{B}$ to indicate that f is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$; $\mathfrak{A} \to \mathfrak{B}$ means that $f\colon \mathfrak{A} \to \mathfrak{B}$ for some f. Similarly $f\colon \mathfrak{A} \rightarrowtail \mathfrak{B}$, $\mathfrak{A} \rightarrowtail\!\!\!\twoheadrightarrow \mathfrak{B}$ (or $\mathfrak{A} \cong \mathfrak{B}$), etc. $\mathrm{Eq}\boldsymbol{K}$ denotes the set of all equations holding in all members of $\boldsymbol{K}$. A set Γ of equations is decidable if, under some standard Gödel numbering, Γ is recursive.

We assume an elementary knowledge of polyadic algebras (see Halmos [2]). A one-dimensional, or *monadic*, polyadic algebra is a structure $\langle A, +, \cdot, -, c\rangle$ where $\langle A, +, \cdot, -\rangle$ is a Boolean algebra, and c is a quantifier on $\langle A, +, \cdot, -\rangle$. Further $\boldsymbol{PA}_1$ is the class of all monadic algebras. The following known theorem is useful in section 1.

THEOREM 0.1. *If $\mathfrak{A}$ is a $\boldsymbol{PA}_1$ generated by m elements, then* $|A| \leq 2^{2^m 2^{2^m - 1}}$.

To introduce the notion of a two-dimensional polyadic algebra, we define $(0,1)$, $(1/0)$, $(0/1) \in {}^2 2$:

$$\begin{aligned} (0,1)0 &= 1; & (0,1)1 &= 0; \\ (1/0)0 &= 1; & (1/0)1 &= 1; \\ (0/1)0 &= 0; & (0/1)1 &= 0. \end{aligned}$$

We treat a two-dimensional polyadic algebra, a $\boldsymbol{PA}_2$, as a structure

$$\mathfrak{A} = \langle A +, \cdot, -, c_0, c_1, S(0,1), S(0/1), S(1/0)\rangle;$$

for the appropriate axioms, see Halmos [2]. For the sake of reference, we take the following redundant set as axioms (where $\sigma \in {}^2 2$, and $S(\boldsymbol{I} \restriction 2)$ is the identity on A).

P1 $\langle A, +, \cdot, -\rangle$ is a Boolean algebra;

P2 $c_i 0 = 0$;

P3 $x \leq c_i x$;

P4 $c_i(x \cdot c_i y) = c_i x \cdot c_i y$;

P5 $c_i c_j x = c_j c_i x$;
P6 $S(\sigma)(x+y) = S(\sigma)x + S(\sigma)y$;
P7 $S(\sigma)(-x) = -S(\sigma)x$;
P8 $S(\sigma)S(\tau)x = S(\sigma \circ \tau)x$;
P9 $S(\sigma)c_i x = S(\tau)c_i x$ if $\sigma \restriction 2 \sim \{i\} = \tau \restriction 2 \sim \{i\}$;
P10 $S(\sigma)c_0 c_1 x = c_0 c_1 x$;
P11 $S(0,1)c_0 x = c_1 S(0,1)x$;
P12 $S(0,1)c_1 x = c_0 S(0,1)x$;
P13 $c_1 S(0/1)x = S(0/1)x$;
P14 $c_0 S(1/0)x = S(1/0)x$.

The following elementary theorem is useful.

THEOREM 0.2. *If a is an atom in a $\boldsymbol{PA}_2\mathfrak{A}$, then $c_0 a$ is an atom in the Boolean algebra $c_0{}^*A$, and $c_1 a$ is an atom in $c_1{}^*A$.*

PROOF. Suppose $0 \neq c_0 b \leqq c_0 a$. Then $a \cdot c_0 b \neq 0$ (otherwise $c_0 a \cdot b = 0$), so $a \leq c_0 b$, $c_0 a \leq c_0 b$, and $c_0 b = c_0 a$.

Note that a simple monadic algebra is just a Boolean algebra with the trivial c:

$$cx = 1, \quad x \neq 0\,,$$
$$= 0, \quad x = 0\,.$$

A subalgebra of a simple $\boldsymbol{PA}_1$ or $\boldsymbol{PA}_2$ is simple. By Jónsson [3] the lattice of equational subclasses of $\boldsymbol{PA}_1$ (or of $\boldsymbol{PA}_2$) is distributive.

1. Monadic algebras.

The lattice of equational classes of $\boldsymbol{PA}_1$'s is a chain of type $\omega+1$; in this section we prove this and give explicit equations which characterize each class relative to $\boldsymbol{PA}_1$.

For each $m \in \omega \sim 1$ let $\mathfrak{A}_m$ be a simple monadic algebra of power 2^m, and let $\boldsymbol{K}_m = \boldsymbol{HSP}\{\mathfrak{A}_m\}$. Let $\boldsymbol{K}_0$ be the class of all one-element monadic algebras. Thus $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_1 \subseteq \ldots \subseteq \boldsymbol{PA}_1$. By Jónsson [3, 3.4], all of these classes are distinct. Let $\boldsymbol{L}$ be any equational class of monadic algebras, $\boldsymbol{L} \neq \boldsymbol{K}_0$ and let $\boldsymbol{L}'$ be the class of all simple members of $\boldsymbol{L}$. We distinguish two cases.

Case 1. All members of $\boldsymbol{L}'$ are finite. Then, by an ultraproduct or compactness argument, $\boldsymbol{L}'$ has a member of maximum cardinality 2^m. Obviously then $\boldsymbol{L} = \boldsymbol{K}_m$.

Case 2. Some member of $\boldsymbol{L}'$ is infinite. Then all finite simple monadic algebras are in $\boldsymbol{L}'$. Since every algebra is the union of its finitely generated subalgebras, and since every finitely generated monadic algebra is finite, $\boldsymbol{L}=\boldsymbol{PA}_2$.

Each of the classes $\boldsymbol{K}_0, \boldsymbol{K}_1, \ldots, \boldsymbol{PA}_1$ is characterized relative to $\boldsymbol{PA}_1$ by a single equation. For $\boldsymbol{K}_0$ and $\boldsymbol{K}_1$ we can take $v_0 \simeq v_1$ and $cv_0 \simeq v_0$ respectively. For $\boldsymbol{K}_m$, $m \in \omega \sim 2$, we can take

$$(1) \qquad \prod_{i<j<n} c(v_i \oplus v_j) \simeq 0 ,$$

where $n=2^m+1$; finally, for $\boldsymbol{PA}_1$ take $v_0 \simeq v_0$. In the natural equation (1), 2^m+1 variables occur. It is natural to ask how few variables can appear in equations relatively characterizing $\boldsymbol{K}_m$. The number 2^m+1 can easily be reduced to m; for $m>1$, $\boldsymbol{K}_m$ is relatively characterized by the equation

$$\prod_{i<j<m} c(v_i \oplus v_j) \cdot \prod_{i<j<m} -c(v_i \cdot v_j) \cdot \prod_{i<m} -v_i \cdot \prod_{i<m} cv_i \simeq 0 .$$

On the other hand, if $2^{2^p} \leqq 2^m$, it is clear that an equation with p or fewer variables cannot relatively characterize $\boldsymbol{K}_m$. An exact value for the number of variables needed is not known. A reformulation of this problem is: for $m>1$, what is the least n such that $\mathfrak{A} \in \boldsymbol{K}_m$ whenever every subalgebra of $\mathfrak{A}$ generated by $\leqq n$ elements is in $\boldsymbol{K}_m$?

Since each class $\boldsymbol{K}_m$, as well as $\boldsymbol{PA}_1$, is finitely axiomatizable, by Theorem 0.1 it follows that each $\mathrm{Eq}\boldsymbol{K}_m$, as well as $\mathrm{Eq}\boldsymbol{PA}_1$ is decidable. It seems to be a difficult problem whether the full elementary theory of $\boldsymbol{PA}_1$'s is decidable.

2. Algebraic lemmas for $\boldsymbol{PA}_2$'s.

In this section we collect some algebraic facts which will be needed in discussing equational classes of $\boldsymbol{PA}_2$'s.

If $\mathfrak{A}$ and $\mathfrak{B}$ are two $\boldsymbol{PA}_2$'s and $F \subseteq A$, an *$\mathfrak{A}$-isomorphism of F into $\mathfrak{B}$* is a one-one map f of F into B such that if $x,y \in F$, then $f(x+y)=fx+fy$, $f(xy)=fx\,fy$, $f(-x)=-fx$, $fc_0x=c_0fx$, $fc_1x=c_1fx$, $fS(0,1)x=S(0,1)fx$, $fS(1/0)x=S(1/0)fx$, and $fS(0/1)x=S(0/1)fx$; these conditions under the assumptions that, respectively, $x+y \in F, \ldots, S(0/1)x \in F$. The following theorem is analogous to an unpublished theorem of Leon Henkin concerning two-dimensional cylindric algebras and representable two-dimensional cylindric algebras. The present theorem, with its immediate corollary that $\mathrm{Eq}\boldsymbol{PA}_2$ is decidable, was independently proved by Henkin.

THEOREM 2.1. *If $\mathfrak{A}$ is a simple $\boldsymbol{PA}_2$ and F is a finite subset of A, then there is a finite simple $\boldsymbol{PA}_2\mathfrak{B}$ with at most 2^{2^m} elements, where $m=12|F|$, such that there is an $\mathfrak{A}$-isomorphism from F into $\mathfrak{B}$.*

PROOF. Let

$$F_0 = F \cup c_0{}^*F \cup c_1{}^*F, \qquad F_1 = F_0 \cup S(0,1)^*F_0 \cup S(1/0)^*F_0 \cup S(0/1)^*F_0 ,$$

and let B be the Boolean subalgebra of $\mathfrak{A}$ generated by F_1. Thus $|B| \leqq 2^{2^m}$, as desired. For the Boolean operations on B we take the restrictions of those of $\mathfrak{A}$. Clearly B is closed under $S(0/1)$, $S(1/0)$ and $S(0,1)$, and we let these operations on B also be the restrictions of those of $\mathfrak{A}$. Now for any $x \in A$ and $i < 2$ let

$$(1) \qquad c_i{}'x = \prod\nolimits_{x \leq c_i t \in B} c_i t .$$

Clearly B is closed under $c_0{}'$ and $c_1{}'$, and we let the restrictions of these to B be the remaining operations on B, giving an algebraic structure $\mathfrak{B}$. The following properties of the $c_i{}'$ are clear for any $x \in A$, $i < 2$:

$$(2) \qquad c_i c_i{}' x = c_i{}' x \in B ,$$

$$(3) \qquad c_i x \leq c_i{}' x ,$$

$$(4) \qquad c_i x \in B \Rightarrow c_i x = c_i{}' x .$$

From (4) it follows that the identity map on F is an $\mathfrak{A}$-isomorphism of F into $\mathfrak{B}$. It remains only to show that $\mathfrak{B}$ is a simple $\boldsymbol{PA}_2$. Conditions P1, P6, P7, P8 and P9 are immediate. P2 follows from (4), and (1) yields P3. Turning to P4, suppose $x,y \in B$ and $i < 2$. By P3, $xc_i{}'y \leq c_i{}'x\, c_i{}'y$; hence $c_i{}'(xc_i{}'y) \leq c_i{}'x\, c_i{}'y$. Next, let t be any element of A such that $xc_i{}'y \leq c_i t \in B$. Then $x \leq c_i t + -c_i{}'y$, and the latter is a c_i-closed member of B. Hence $c_i{}'x \leq c_i t + -c_i{}'y$ by (1), that is, $c_i{}'x\, c_i{}'y \leq c_i t$. Using (1) again, this shows that $c_i{}'x\, c_i{}'y \leq c_i{}'(xc_i{}'y)$. Hence P4 holds. Using P1–P4 and (3) it is easily seen that $c_0{}'c_1{}'x = c_1{}'c_0{}'x = 1$ whenever $0 \neq x \in B$. Hence P5 holds (and $\mathfrak{B}$ is hence shown simple if the remaining postulates are verified). Condition P10 also easily follows now. It remains only to check one of the two symmetric conditions P11 and P12, say P11; and one of the two symmetric conditions P13 and P14, say P13. For P11, we have for $x \in B$

$$(5) \qquad S(0,1)c_0{}'x = \prod\nolimits_{x \leq c_0 t \in B} S(0,1)c_0 t ,$$

$$(6) \qquad c_1{}'S(0,1)x = \prod\nolimits_{S(0,1)x \leq c_1 t \in B} c_1 t .$$

If $x \leq c_0 t \in B$, then $S(0,1)x \leq c_1 S(0,1)t \in B$ and hence by (6)

$$c_1{}'S(0,1)x \leq c_1 S(0,1)t = S(0,1)c_0 t .$$

It follows that $c_1{}'S(0,1)x \leq S(0,1)c_0{}'x$ by (5). The converse inclusion is similar.

As to P 13, if $x \in B$, then $c_1 S(0/1)x = S(0/1)x$, and P 13 results by (4). This completes the proof.

The proofs of other algebraic facts which we need are facilitated by the introduction of the notion of an *atomic structure*. This notion was introduced and studied for arbitrary Boolean algebras with operators in Jónsson and Tarski [4], and its theory was extensively developed for cylindric algebras by Henkin. An *atomic structure* is a quadruple $\mathfrak{A} = \langle A, E, f, B \rangle$ satisfying the following conditions:

A 1 A is a non-empty set;
A 2 E is an equivalence relation on A;
A 3 f is a permutation of A;
A 4 $B \subseteq A$;
A 5 $f \circ f = \boldsymbol{I} \restriction A$;
A 6 for all $x, y \in A/E$ there is an $a \in x$ such that $fa \in y$;
A 7 $|B \cap x| = 1$ for every $x \in A/E$;
A 8 $fb = b$ for each $b \in B$.

Given an atomic structure $\mathfrak{A}$, we let

$$\boldsymbol{PA}_2 \mathfrak{A} = \langle SA, \cup, \cap, \sim, c_0, c_1, S(0,1), S(0/1), S(1/0) \rangle ,$$

where, for any $X \subseteq A$,

A 9 $c_0 X = \{a \in A : \exists b \in X(aEb)\}$;
A 10 $c_1 X = \{a \in A : \exists b \in X(faEfb)\}$;
A 11 $S(0,1)X = \{a \in A : fa \in X\}$;
A 12 $S(1/0)X = \bigcup_{a \in X \cap B} a/E$;
A 13 $S(0/1)X = S(0,1)S(1/0)X$.

Conversely, if $\mathfrak{B}$ is a simple complete atomic $\boldsymbol{PA}_2$ we let

$$\mathfrak{A}\mathrm{t}\, \mathfrak{B} = \langle \mathrm{At}\, \mathfrak{B}, E, f, C \rangle ,$$

where

A 14 $E = \{(a,b) : a, b \in \mathrm{At}\, \mathfrak{B}, c_0 a = c_0 b\}$;
A 15 $f = S(0,1) \restriction \mathrm{At}\, \mathfrak{B}$;
A 16 $C = \{a \in \mathrm{At}\, \mathfrak{B} : S(1/0)a \neq 0\}$.

The basic theorem on atomic structures is as follows.

THEOREM 2.2. (i) *If* $\mathfrak{A}$ *is a finite atomic structure, then* $\boldsymbol{PA}_2 \mathfrak{A}$ *is a simple finite* $\boldsymbol{PA}_2$, *and* $\mathfrak{A}\mathrm{t}\, \boldsymbol{PA}_2 \mathfrak{A} \cong \mathfrak{A}$.

(ii) *If* $\mathfrak{B}$ *is a simple finite* $\boldsymbol{PA}_2$, *then* $\mathfrak{A}\mathrm{t}\, \mathfrak{B}$ *is a finite atomic structure, and* $\boldsymbol{PA}_2 \mathfrak{A}\mathrm{t}\, \mathfrak{B} \cong \mathfrak{B}$.

PROOF. (i). Clearly $\boldsymbol{PA}_2\mathfrak{A}$ satisfies P1–P4 and P14. Axiom P5 will follow from simplicity, which we proceed to prove. Suppose $0 \neq X \subseteq A$ and $a \in A$; we wish to show that $a \in c_0c_1X$ and $a \in c_1c_0X$. Choose $b \in X$. Choose $x \in fb/E$ such that $fx \in a/E$ (using A6). Thus $aEfx$, $ffx = xEfb$, and $b \in X$. It follows that $a \in c_0c_1X$. The proof that $a \in c_1c_0X$ is analogous.

P6 and P7 are obvious for $\sigma = (0,1)$. If $a \in S(1/0)(X \cup Y)$, with $X, Y \subseteq A$, choose $b \in (X \cup Y) \cap B$ so that aEb; this clearly implies that $a \in S(1/0)X \cup S(1/0)Y$. The converse is similar, so P6 holds for $\sigma = (1/0)$. Suppose $a \in S(1/0)(\sim X)$. Choose $b \in \sim X \cap B$ such that aEb. Condition A7 rules out the possibility that $a \in S(1/0)X$. Thus $S(1/0)(\sim X) \subseteq \sim S(1/0)X$, and the converse is obvious. Hence P7 holds for $\sigma = (1/0)$. Finally, from A13 it is now obvious that P6 and P7 hold for $\sigma = (0/1)$.

We take only one representative case for P8. Suppose $a \in S(1/0)S(0,1)X$. Choose $b \in S(0,1)X \cap B$ such that aEb. Thus $fb \in X$; but $fb = b$ by A8. Hence $b \in X \cap B$, so $a \in S(1/0)X$. The converse is similar.

As to P9, we also treat only one case: $S(0,1)c_1X = S(1/0)c_1X$. Indeed, suppose $a \in S(0,1)c_1X$. Thus $fa \in c_1X$; choose $x \in X$ such that $aEfx$. Choose $b \in (fx/E) \cap B$. Then $fb = bEfx$, and so $b \in B \cap c_1X$. Since aEb, it follows that $a \in S(1/0)c_1X$. The converse is similar.

P10 is obvious by simplicity, P11 is easy, and P12 follows from P11. Finally, P13 easily follows from the remaining conditions. We have now established that $\boldsymbol{PA}_2\mathfrak{A}$ is a simple finite $\boldsymbol{PA}_2$. It is obvious that $\mathfrak{At}\,\boldsymbol{PA}_2\mathfrak{A} \cong \mathfrak{A}$.

(ii). Clearly $\mathfrak{At}\,\mathfrak{B}$ satisfies A1–A5. Let $x, y \in \mathrm{At}\,\mathfrak{B}/E$. Choose $d \in x$, $b \in y$. Then $c_0c_1fb = 1$, so $d \leq c_0c_1fb$. It follows that there is an $a \leq c_1fb$ such that $d \leq c_0a$. Hence dEa,

$$S(0,1)a \;\leq\; S(0,1)c_1fb \;=\; c_0S(0,1)S(0,1)b \;=\; c_0b\,.$$

Hence dEa, $faEb$, so $a \in x$ and $fa \in y$, as desired–A6 holds. For A7, let $x \in \mathrm{At}\,\mathfrak{B}/E$. Choose $a \in x$. Then $S(1/0)c_0a = c_0a$, so

$$0 \;\neq\; c_0a \;=\; \textstyle\sum_{b \in At\mathfrak{B}, b \leq c_0a} S(1/0)b\,.$$

Choose $b \in \mathrm{At}\,\mathfrak{B}$ with $b \leq c_0a$ and $S(1/0)b \neq 0$. Thus $b \in x \cap C$. Suppose also that $d \in x \cap C$ and $b \neq d$. Thus $d \leqq c_0a$ and $S(1/0)d \neq 0$. Now

$$c_0S(1/0)d \;=\; S(1/0)d \;\leq\; S(1/0)c_0d \;=\; c_0d \;=\; c_0a;$$

hence $S(1/0)d = c_0a$ by Theorem 0.2. Similarly $S(1/0)b = c_0a$. Hence

$$c_0a \;=\; S(1/0)b \cdot S(1/0)d \;=\; S(1/0)(b \cdot d) \;=\; 0\,,$$

a contradiction. Hence A7 holds. Note that we have shown that $S(1/0)b=c_0b$ for any $b \in C$. To verify A8, suppose $b \in C$. Then

$$S(1/0)fb = S(1/0)S(0,1)b = S(1/0)b \neq 0;$$

hence $fb \in C$. Further, $c_0fb=S(1/0)fb=S(1/0)b=c_0b$. Hence $fb \in C \cap (b/E)$, so $fb=b$ by A7. We have now proved that $\mathfrak{At}\,\mathfrak{B}$ is an atomic structure.

It is easily verified that the function g such that $gx=\{a \in \mathrm{At}\,\mathfrak{B}: a \leq x\}$ for any $x \in B$ is an isomorphism from $\mathfrak{B}$ onto $\boldsymbol{PA}_2\,\mathfrak{At}\,\mathfrak{B}$. This completes the proof.

THEOREM 2.3. *Let* $\mathfrak{A}=\langle A,E,f,B\rangle$ *and* $\mathfrak{C}=\langle C,F,g,D\rangle$ *be two finite atomic structures. Suppose* $G\colon C \to \boldsymbol{S}A$ *such that*

(i) *G is a one-one map of C onto a partition of A;*
(ii) *if $u,v \in C$ and uFv, then $Gu/E=Gv/E$;*
(iii) *if $u,v \in C$ and $u\not{F}v$, then $Gu/E \cap Gv/E=0$;*
(iv) *if $u \in C$ then $Ggu=f^*Gu$;*
(v) *if $d \in D$, then $B \cap \bigcup_{a \in Gd} a/E \subseteq Gd$.*

For $X \subseteq C$ *define* $G'X=\bigcup_{x \in X} Gx$. *Then* $G'\colon \boldsymbol{PA}_2\,\mathfrak{C} \rightarrowtail \boldsymbol{PA}_2\,\mathfrak{A}$.

PROOF. From (i) it easily follows that G' is a Boolean isomorphism into. *G' preserves c_0.* Let $u \in G'c_0X$. Choose $x \in c_0X$ such that $u \in Gx$. Say $xFy \in X$. Then $Gx/E=Gy/E$ by (ii). Choose $z \in Gy$ such that $u/E=z/E$. Thus $uEz \in Gy$, $z \in G'X$, $u \in c_0G'X$.

Conversely, suppose that $u \in c_0G'X$. Say $uEv \in G'X$, $v \in Gx$, $x \in X$. By (i) choose $w \in C$ such that $u \in Gw$. Thus $v \in Gx$, $v \in u/E$, so $Gx/E \cap Gw/E \neq 0$; by (iii), xFw. Hence $w \in c_0X$, $u \in G'c_0X$.

G' preserves $S(0,1)$. Let $u \in G'S(0,1)X$. Say $u \in Gx$, $x \in S(0,1)X$. Thus $gx \in X$. By (iv), $fu \in Ggx$, so $fu \in G'X$ and $u \in S(0,1)G'X$. The converse is similar.

G' preserves c_1: follows from the preceding two cases.

G' preserves $S(1/0)$. Let $u \in G'S(1/0)X$. Say $u \in Gx$, $x \in S(1/0)X$. Choose $d \in X \cap D$ such that xFd. Thus $Gx/E=Gd/E$ by (ii). Hence choose $w \in Gd$ such that $u/E=w/E$. Choose $b \in (w/E) \cap B$. By (v), $b \in Gd$. Thus $b \in G'X \cap B$, so, since uEb, $u \in S(1/0)G'X$.

Conversely, suppose $u \in S(1/0)G'X$. Choose $b \in G'X \cap B$ such that uEb. Say $b \in Gx$, $x \in X$. By (i) choose $y \in C$ such that $u \in Gy$. Thus $Gx/E \cap Gy/E \neq 0$, so by (iii) xFy. Choose $d \in x/F \cap D$. Thus xFd, so by (ii) $Gx/E=Gd/E$. Hence

$$b \in B \cap \bigcup_{a \in Gd} a/E \subseteq Gd$$

by (v). Thus $b \in Gx \cap Gd$, so $x = d$. Hence $u \in Gy$, yFx, $x \in D \cap X$, so $u \in G'S(1/0)X$.

It is obvious that G' *preserves* $S(0/1)X$.

The following notation will be found useful in what follows. If $\mathfrak{A} = \langle A, E, f, B \rangle$ is an atomic structure and u and v are E-closed subsets of A, we set

$$\begin{aligned} N_u^{\mathfrak{A}} &= \{a \in u : fa = a\}, \\ P_u^{\mathfrak{A}} &= \{a \in u : a \neq fa \in u\}, \\ M_{uv}^{\mathfrak{A}} &= \{a \in u : fa \in v\} \end{aligned}$$

Note that if u is an E-class, then $N_u^{\mathfrak{A}} \neq 0$ (since $|B \cap u| = 1$), and $|P_u^{\mathfrak{A}}|$ is infinite or else finite and even. If u and v are distinct E-classes, then $M_{uv}^{\mathfrak{A}} \neq 0$ by A6; also note that $|M_{uv}^{\mathfrak{A}}| = |M_{vu}^{\mathfrak{A}}|$. The limiting possibilities $|N_u^{\mathfrak{A}}| = 1$, $P_u^{\mathfrak{A}} = 0$, $|M_{uv}^{\mathfrak{A}}| = 1$ can be realized.

THEOREM 2.4. *Let $\mathfrak{A} = \langle A, E, f, B \rangle$ and $\mathfrak{C} = \langle C, F, g, D \rangle$ be finite atomic structures, and suppose given $H : A/E \rightarrowtail\!\!\!\twoheadrightarrow C/F$ such that for all distinct $x, y \in A/E$,*

(i) $|P_{Hx}^{\mathfrak{C}}| \leq |P_x^{\mathfrak{A}}|$,
(ii) $|N_{Hx}^{\mathfrak{C}}| - \frac{1}{2}(|P_x^{\mathfrak{A}}| - |P_{Hx}^{\mathfrak{C}}|) \leq |N_x^{\mathfrak{A}}|$,
(iii) $|M_{Hx,Hy}^{\mathfrak{C}}| \leq |M_{xy}^{\mathfrak{A}}|$.

Then $\boldsymbol{PA}_2 \mathfrak{C} \rightarrowtail \boldsymbol{PA}_2 \mathfrak{A}$.

PROOF. Let $x_0, \ldots, x_{m-1}$ be an enumeration of A/E without repetitions. For each $i < m$ and each j with $i < j < m$ choose $Q_{0i}, Q_{1i}, Q_{2i}, Q_{3ij}$ such that:

(1) $Q_{0i} \subseteq N_{x_i}^{\mathfrak{A}}$, $Q_{0i} \cap B = 0$, and $|Q_{0i}| = \max(0, |N_{Hx_i}^{\mathfrak{C}}| - \frac{1}{2}(|P_{x_i}^{\mathfrak{A}}| - |P_{Hx_i}^{\mathfrak{C}}|) - 1)$;

(2) Q_{1i} is a collection of pairwise disjoint doubletons $\{y, z\} \subseteq P_{x_i}^{\mathfrak{A}}$ such that $fy = z$, with
$$|Q_{1i}| = |N_{Hx_i}^{\mathfrak{C}}| - |Q_{0i}| - 1;$$

(3) $Q_{2i} \subseteq P_{x_i}^{\mathfrak{A}}$, $(Q_{2i} \cup f^* Q_{2i}) \cap Q_{1i} = 0$, $Q_{2i} \cap f^* Q_{2i} = 0$, and $|Q_{2i}| = \frac{1}{2}|P_{Hx_i}^{\mathfrak{C}}|$;

(4) Q_{3ij} is a partition of $M_{x_i x_j}^{\mathfrak{A}}$ into $|M_{Hx_i, Hx_j}^{\mathfrak{C}}|$ pairwise disjoint non-empty sets.

To see that this is possible, note first that Q_{3ij} can be chosen to satisfy (4) by virtue of (iii). Concerning (1)–(3), we consider two cases.

Case 1. $|N_{Hx_i}^{\mathfrak{C}}| - \frac{1}{2}(|P_{x_i}^{\mathfrak{A}}| - |P_{Hx_i}^{\mathfrak{C}}|) - 1 \leq 0$. Then

$$2(|N^{\mathfrak{C}}_{Hx_i}|-1) \le |P^{\mathfrak{A}}_{x_i}|-|P^{\mathfrak{C}}_{Hx_i}| \,.$$

Thus Q_{1i} can be chosen as in (2), and then $|P^{\mathfrak{A}}_{x_i} \sim \bigcup_{\alpha \in Q_{1i}} \alpha| \ge |P^{\mathfrak{C}}_{Hx_i}|$. It is then clear that Q_{2i} can be chosen as in (3). Of course in the present Case 1 we let $Q_{0i}=0$.

Case 2. $|N^{\mathfrak{C}}_{Hx_i}|-\frac{1}{2}(|P^{\mathfrak{A}}_{x_i}|-|P^{\mathfrak{C}}_{Hx_i}|)-1>0$. By (ii) we can choose Q_{0i} as in (1). Then

$$|N^{\mathfrak{C}}_{Hx_i}|-|Q_{0i}|-1 = \tfrac{1}{2}(|P^{\mathfrak{A}}_{x_i}|-|P^{\mathfrak{C}}_{Hx_i}|)$$

Then we can choose Q_{1i} as in (2), and we will then have

$$|P^{\mathfrak{A}}_{x_i} \sim \bigcup_{\alpha \in Q_{1i}} \alpha| = |P^{\mathfrak{C}}_{Hx_i}| \,,$$

which makes possible the choice of Q_{2i} satisfying (3).

We now define a function G by recursion. If $G \restriction \bigcup_{j<i} Hx_j$ has been defined, let G on Hx_i be such that G maps

$N^{\mathfrak{C}}_{Hx_i} \sim D$ one-one onto $\{\{a\}:\ a \in Q_{0i}\} \cup Q_{1i}$,

$N^{\mathfrak{C}}_{Hx_i} \cap D$ one-one onto $\{(N^{\mathfrak{A}}_{x_i} \cup P^{\mathfrak{A}}_{x_i}) \sim (Q_{0i} \cup Q_{1i} \cup Q_{2i} \cup f^*Q_{2i})\}$,

$P^{\mathfrak{C}}_{Hx_i}$ one-one onto $\{\{a\}:\ a \in Q_{2i}\} \cup \{\{fa\}:\ a \in Q_{2i}\}$

in such a way that $Ggd = f^*Gd$ for all $d \in P^{\mathfrak{C}}_{Hx_i}$;

for every $j<i$ and for every $d \in M^{\mathfrak{C}}_{Hx_i,Hx_j}$, $Gd = f^*Ggd$;

and for every $j>i$, G maps $M^{\mathfrak{C}}_{Hx_i,Hx_j}$ one-one onto Q_{3ij} .

It is routine to check that the conditions of Theorem 2.3 are met, so $\boldsymbol{PA}_2\mathfrak{C} \rightarrowtail \boldsymbol{PA}_2\mathfrak{A}$.

In order to generalize 2.4 we need the following theorem, which is of independent interest.

THEOREM 2.5. *Let $\mathfrak{A}$ be a simple infinite $\boldsymbol{PA}_2$ such that $c_0{}^*A$ is finite. Then every finitely generated subalgebra of $\mathfrak{A}$ is finite. In fact, if $X \subseteq A$ then the subalgebra generated by X has at most 2^{2^m} elements, where $m= 8|c_0{}^*A|+4|X|$.*

PROOF. Let $B_0 = X \cup c_0{}^*A \cup c_1{}^*A$. Note that $|c_0{}^*A| = |c_1{}^*A|$ because of the Boolean automorphism $S(0,1)$. Let

$$B_1 = B_0 \cup S(0,1)^*B_0 \cup S(0/1)^*B_0 \cup S(1/0)^*B_0 \,.$$

The Boolean subalgebra generated by B_1 clearly coincides with the $\boldsymbol{PA}_2$-subalgebra generated by X, and the theorem follows.

In Halmos [2, pp. 92–93] it is shown that Theorem 2.5 cannot be extended to all $\boldsymbol{PA}_2$'s.

Our next result is a generalization of Theorem 2.4 and plays a crucial role in the next section. We recall that for any $\boldsymbol{PA}_2$ $\mathfrak{A}$, the c_0-closed elements of A, form a Boolean algebra; we denote this Boolean algebra by $c_0{}^*\mathfrak{A}$.

THEOREM 2.6. *Let $\mathfrak{A}$ be a simple finite $\boldsymbol{PA}_2$, and $\mathfrak{B}$ any simple atomic $\boldsymbol{PA}_2$. Assume that $H\colon \mathrm{At}\, c_0{}^*\mathfrak{A} \rightarrowtail\!\!\twoheadrightarrow \mathrm{At}\, c_0{}^*\mathfrak{B}$. For distinct atoms x,y of $c_0{}^*\mathfrak{A}$ let*

$$\begin{aligned} f_x &= |\{a \in \mathrm{At}x\colon S(0,1)a = a\}| \,, \\ g_x &= |\{a \in \mathrm{At}x\colon a \neq S(0,1)a \in \mathrm{At}x\}| \,, \\ h_{xy} &= |\{a \in \mathrm{At}x\colon S(0,1)a \in \mathrm{At}y\}| \,; \end{aligned}$$

and similarly define $f_{Hx}, g_{Hx}, h_{Hx,Hy}$. Then H extends to an isomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ iff for any two distinct atoms x,y of $c_0{}^*\mathfrak{A}$ the following conditions hold:

(i) $g_x \leq g_{Hx}$;
(ii) *g_{Hx} is infinite, or g_{Hx} is finite and* $f_x - \frac{1}{2}(g_{Hx} - g_x) \leq f_{Hx}$;
(iii) $h_{xy} \leq h_{Hx,Hy}$.

PROOF. $\Rightarrow$: both (i) and (iii) are obvious. Now assume that g_{Hx} is finite. If $a \in \mathrm{At}x$ and $a = S(0,1)a \in \mathrm{At}x$, then

$$Ha \cap \{b \in \mathrm{At}Hx\colon b \neq S(0,1)b \in \mathrm{At}Hx\}$$

is either empty or has a positive even number of elements. From this observation the inequality (ii) follows easily.

$\Leftarrow$: By Theorem 2.5 we may assume that $\mathfrak{B}$ is finite. Passing to the associated atomic structures we see that the conditions of Theorem 2.4 are satisfied, and the proof of Theorem 2.4 gives the desired result.

THEOREM 2.7. *If $\mathfrak{A}$ is a simple $\boldsymbol{PA}_2$ with $|c_0{}^*A| = 2^{m+1}$, then $\mathfrak{A}$ has a subalgebra $\mathfrak{B}$ with $|c_0{}^*B| = 2^m$.*

PROOF. By Theorem 2.6 we may assume that for distinct atoms x,y of $c_0{}^*A$ we have

$$|\{a \in \mathrm{At}x\colon S(0,1)a = a\}| \;=\; 1 \,,$$

$$|\{a \in \mathrm{At}x\colon a \neq S(0,1)a \in \mathrm{At}x\}| = 0\,,$$
$$|\{a \in \mathrm{At}x\colon S(0,1)a \in \mathrm{At}y\}| = 1\,.$$

Now let x and y be distinct atoms of $c_0{}^*\mathfrak{A}$. Let B consist of all $a \in \mathrm{At}\,\mathfrak{A}$ such that $a \nleq x+y$ and $S(0,1)a \nleq x+y$, together with all elements

$$a+b \text{ with } a \leq x,\ b \leq y,\ S(0,1)a+S(0,1)b \leq z$$

for some $z \in \mathrm{At}c_0{}^*A$, $z \neq x,y$;

$$a+b \text{ with } a \leq x,\ b \leq y,\ S(0,1)a \leqq y,\ S(0,1)b \leq x;$$
$$a+b \text{ with } a \leq x,\ b \leq y,\ S(0,1)a = a,\ S(0,1)b = b\,.$$

It is easily checked that the subalgebra generated by B is the desired subalgebra of $\mathfrak{A}$.

We now turn to the case of $\boldsymbol{PA}_2$'s with $c_0{}^*A$ infinite. Here a central role is played by the following lemma.

LEMMA 2.8. *Let $\mathfrak{A} = \langle A,E,f,B\rangle$ and $\mathfrak{C} = \langle C,F,g,D\rangle$ be two finite atomic structures, with $9\cdot|C|^2 \leq |A/E|$. Then $\boldsymbol{PA}_2\mathfrak{C} \rightarrowtail \boldsymbol{PA}_2\mathfrak{A}$.*

PROOF. Let $n = |C|$ and $m = |C/F|$. By Theorem 2.7 we may assume that $|A/E| = 9mn$. Let $\langle z_{ijk}\colon i<m, j<3, k<3n\rangle$ be an enumeration of A/E. For $i<m$ let $Z_i = \{z_{ijk}\colon j<3,\ k<3n\}$. An *i-selector* is a subset X of A such that $X \cap z_{ijk} \neq 0$ for all $j<3$, $k<3n$, while $X \subseteq \bigcup_{j<3,k<3n} z_{ijk}$. Now we claim:

(1) For each $i<m$ there is a family T of $3n$ pairwise disjoint i-selectors such that $X \subseteq M_{Z_iZ_i}$ for each $X \in T$, and $X \cap f^*Y = 0$ for any $X, Y \in T$.

For let α be a rotation of $3n$: $\alpha l = l+1$ for $l+1<3n$ and $\alpha(3n-1) = 0$; and let β be a rotation of 3: $\beta 0 = 1$, $\beta 1 = 2$, $\beta 2 = 0$. For $j<3$, $k<3n$, $l<3n$ choose

$$x_{jkl} \in M\big(z(i,j,k), z(i,\beta j, \alpha^l k)\big);$$

and let $X_l = \{x_{jkl}\colon j<3,\ k<3n\}$. Clearly $\langle X_l\colon l<3n\rangle$ is a system of $3n$ pairwise disjoint i-selectors all $\subseteq M_{Z_iZ_i}$ with $X_l \cap f^*X_{l'} = 0$ for each $l, l' < 3n$. Thus (1) holds. Similarly:

(2) If $i<i'<m$, then there is a family T of $3n$ pairwise disjoint i-selectors such that $X \subseteq M_{Z_iZ_{i'}}$ for each $X \in T$.

Now let $\{x_0,\ldots,x_{m-1}\} = C/F$. By (1) choose for each $i<m$ sets G_i, H_i such that

(3) G_i and H_i are families of pairwise disjoint i-selectors such that $X \subseteq M_{Z_iZ_i}$ for each $X \in G_i \cup H_i$ and $X \cap f^*Y = 0$ for any $X, Y \in G_i \cup H_i$,

(4) If $X \in G_i \cup H_i$ and $Y \in G_i \cup H_i$ and $X \neq Y$, then $X \cap Y = 0$,

(5) $G_i \cap H_i = 0$,

(6) $|G_i| = |N_{x_i}| - 1$,

(7) $|H_i| = \frac{1}{2}|P_{x_i}|$.

By (2), if $i < i' < m$ choose a family $K_{ii'}$ of pairwise disjoint i-selectors all $\subseteq M_{Z_iZ_i}$ such that

(8) $|K_{ii'}| = |M_{x_ix_{i'}}|$ and $\bigcup K_{ii'} = M_{Z_iZ_{i'}}$.

We now define $L: C \to SA$; we define L on $x_0, \ldots, x_{m-1}$ by recursion. Suppose that $i < m$ and $L \restriction (x_0 \cup \ldots \cup x_{i-1})$ has been defined. Let L map $N_{x_i} \sim D$ one-one onto $\{X \cup f^*X : X \in G_i\}$, and let L map P_{x_i} one-one onto $H_i \cup \{f^*X : X \in H_i\}$ in such a way that

$$Lgu = f^*Lu \text{ for each } u \in P_{x_i}.$$

Furthermore:

$$\text{for } u \in x_i \cap D \text{ let } Lu = M_{Z_iZ_i} \sim \bigcup L^*(M_{x_ix_i} \sim D);$$

$$\text{for } i' < i \text{ and } u \in M_{x_ix_{i'}} \text{ let } Lu = f^*Lgu;$$

$$\text{for } i < i' \text{ let } L \text{ map } M_{x_ix_{i'}} \text{ one-one onto } K_{ii'}.$$

Now by Theorem 2.3 the desired result follows.

THEOREM 2.9. *If $\mathfrak{A}$ is a simple $\boldsymbol{PA}_2$ with $c_0{}^*A$ infinite, then every simple finite $\boldsymbol{PA}_2$ can be embedded in $\mathfrak{A}$.*

PROOF. By Theorem 2.8 it suffices to prove

(1) for every $m \in \omega \sim 1$ there is a finite subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with $m \leq |c_0{}^*B|$.

Let X be any subset of $c_0{}^*A$ with m members. Let $B_0 = X \cup S(0,1)^*X$, and let B_1 be the Boolean subalgebra of B_0 generated by B_0. Clearly B_1 is a subalgebra of $\mathfrak{A}$ with the desired property.

For the proof of Theorem 2.11 and for later purposes it is convenient to expand on some previous notation. If $\mathfrak{A}$ is an atomic $\boldsymbol{PA}_2$ with $c_0{}^*A$ finite, a *signature* of $\mathfrak{A}$ is a quadruple $t = (m,f,g,h)$ such that for some $x: m \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathrm{At}\, c_0{}^*\mathfrak{A}$ and for $i,j \in m$,

$$f_i = |\{a \in \mathrm{At}x_i\colon S(0,1)a = a\}| \cap \omega;$$
$$g_i = |\{a \in \mathrm{At}x_i\colon a \neq S(0,1)a \in \mathrm{At}x_i\}| \cap \omega;$$
$$h_{ij} = |\{a \in \mathrm{At}x_i\colon S(0,1)a \in \mathrm{At}x_j\}| \cap \omega\ .$$

We say then that t is *compatible with* x. Given two signatures $s_i = (m, f_i, g_i, h_i)$ for $i \in 2$, we write $s_0 \lesssim s_1$ provided that $m_0 = m_1$, $g_{0i} \leq g_{1i}$, g_{1i} is infinite or $f_{0i} - \frac{1}{2}(g_{1i} - g_{0i}) \leq f_{1i}$, and $h_{0ij} \leq h_{1ij}$ for all distinct $i,j < m$. Clearly $\lesssim$ is a partial ordering on signatures. Further, by Theorem 2.6,

THEOREM 2.10. *If $\mathfrak{A}$ is a simple finite $\boldsymbol{PA}_2$ and $\mathfrak{B}$ is a simple atomic $\boldsymbol{PA}_2$ with $|c_0{}^*A| = |c_0{}^*B|$, then $\mathfrak{A}$ can be imbedded in $\mathfrak{B}$ iff there are signatures s,t of $\mathfrak{A}, \mathfrak{B}$ respectively such that $s \lesssim t$.*

THEOREM 2.11. *Let $i < \omega$, and let L be a finite set of simple atomic $\boldsymbol{PA}_2$'s $\mathfrak{A}$ with $|c_0{}^*A| = 2^i$. Let M be the class of all finite simple $\boldsymbol{PA}_2$'s $\mathfrak{B}$ with $|c_0{}^*B| = 2^i$ such that $\mathfrak{B}$ cannot be imbedded in any $\mathfrak{A} \in L$. Finally, let N consist of one algebra from each isomorphism type of the minimal members of M under the subalgebra relation. Then N is finite, and for each finite simple $\boldsymbol{PA}_2\mathfrak{C}$ with $|c_0{}^*C| = 2^i$ the following two conditions are equivalent:*

(i) *$\mathfrak{C}$ cannot be imbedded in any $\mathfrak{A} \in L$;*
(ii) *for some $\mathfrak{B} \in N$, $\mathfrak{B}$ can be imbedded in $\mathfrak{C}$.*

PROOF. Let S be the set of all signatures of members of L; note that S is finite. Let T be the set of all signatures $s' = (i, f', g', h')$ of finite simple $\boldsymbol{PA}_2$'s such that $s' \not\lesssim t$ for all $t \in S$ and such that the following conditions hold, where $s'' = (i, f'', g'', h'')$:

(1) if $j < i$, $f_j' > 1$, and s'' differs from s' only in that $f_j'' = f_j' - 1$, then $s'' \lesssim t$ for some $t \in S$;
(2) if $j < i$, $g_j' > 2$, *and* s'' differs from s' only in that $g_j'' = g_j' - 2$, then $s'' \lesssim t$ for some $t \in S$;
(3) if $j,k < i$, $j \neq k$, $h'_{jk} > 1$, and s'' differs from s' only in that $h''_{jk} = h'_{jk} - 1$ and $h''_{kj} = h'_{kj} - 1$, then $s'' \lesssim t$ for some $t \in S$.

It is then clear that the following conditions hold for each $s' \in T$, where $t = (i, f^t, g^t, h^t)$ for each $t \in S$:

(4) $\forall j < i\ f_j' \leq \max\{f_j{}^t + \frac{1}{2}g_j{}^t + 1 : f_j{}^t, g_j{}^t < \omega,\ t \in S\}$
(5) $\forall j < i\ g_j' \leq \max\{g_j{}^t + 2 : g_j{}^t < \omega, t \in S\}$
(6) $\forall j,k < i\ (j \neq k \Rightarrow h'_{jk} \leq \max\{h^t_{jk} + 1 : h^t_{jk} < \omega, t \in S\})$.

From (4)–(6) it follows that T is finite. However, it is clear that any signature of a member of N is in T; hence N is finite. The equivalence of (i) and (ii) is obvious.

THEOREM 2.12. *For every n there is an $m > n$ such that every finite simple $\boldsymbol{PA}_2$ of power $\geq m$ has a proper subalgebra of power $> n$.*

PROOF. Let $m = 2^{2^p}$, where $p = 12n + 5$. Let $\mathfrak{A}$ be a finite simple $\boldsymbol{PA}_2$ of power $\geq m$. We consider two cases.

Case 1. $|c_0{}^*A| \leq n$. Let X be any subset of A of power $n+1$. Then, by 2.5, the subalgebra of $\mathfrak{A}$ generated by X has at most $12n+4$ elements, and is, hence, proper.

Case 2. $n < |c_0{}^*A|$. Let $X \subseteq c_0{}^*A$ have $n+1$ elements, and choose B_1 as in the proof of Theorem 2.9. The desired conclusion again follows.

3. Equational classes of $\boldsymbol{PA}_2$'s.

In this section we apply the lemmas of section 2 to obtain information about the lattice $\mathscr{L}$ of equational classes of $\boldsymbol{PA}_2$'s.

THEOREM 3.1. *If an equational class $\boldsymbol{K} \subseteq \boldsymbol{PA}_2$ has a simple member $\mathfrak{A}$ with $c_0{}^*A$ infinite, then $\boldsymbol{K} = \boldsymbol{PA}_2$.*

PROOF. By Theorems 2.1 and 2.9.

The following theorem results at once from Theorems 2.1, 2.5, and 2.9.

THEOREM 3.2. *Every equational class of $\boldsymbol{PA}_2$'s is determined by its finite members.*

We now discuss *covers* in $\mathscr{L}$. By the argument of Jónsson [3, Corollary 4.4], we have:

THEOREM 3.3. *If $K \subset \boldsymbol{PA}_2$, then $\boldsymbol{K}$ has a cover in $\mathscr{L}$.*

THEOREM 3.4. *$\boldsymbol{PA}_2$ does not cover any $\boldsymbol{K} \in \mathscr{L}$.*

PROOF. Let n be maximal such that $\boldsymbol{K}$ has a simple member $\mathfrak{A}$ with $|c_0{}^*A| = n$. Let $\mathfrak{B}$ be a simple $\boldsymbol{PA}_2$ with $\omega > |c_0{}^*B| > n$. Then $\boldsymbol{K} \subset \boldsymbol{HSP}(\boldsymbol{K} \cup \{\mathfrak{B}\}) \subset \boldsymbol{PA}_2$.

THEOREM 3.5. *Suppose $\boldsymbol{L}$ and $\boldsymbol{K}$ are equational classes of $\boldsymbol{PA}_2$'s such* *that* $\boldsymbol{L}$ *covers* $\boldsymbol{K}$. *Then there is a finite simple* $\mathfrak{A} \in \boldsymbol{L} \sim \boldsymbol{K}$ *such that* $\mathfrak{B} \in \boldsymbol{K}$ *for each proper subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$, *and* $\boldsymbol{L} = \boldsymbol{HSP}(\boldsymbol{K} \cup \{\mathfrak{A}\})$.

PROOF. Let $\mathfrak{C}$ be a simple algebra in $\boldsymbol{L}$ but not in $\boldsymbol{K}$. By 2.5 there is some finite subalgebra $\mathfrak{A}$ of $\mathfrak{C}$ which is in $\boldsymbol{L}$ but not in $\boldsymbol{K}$. Thus

$$\boldsymbol{K} \subset \boldsymbol{HSP}(\boldsymbol{K} \cup \{\mathfrak{A}\}) \subseteq \boldsymbol{L} ,$$

so $\boldsymbol{HSP}(\boldsymbol{K} \cup \{\mathfrak{A}\}) = \boldsymbol{L}$. By Jónsson [3, 3.6 and 3.2], the simple members of $\boldsymbol{L}$ are those of $\boldsymbol{K}$ together with members of $\boldsymbol{S}\{\mathfrak{A}\}$. If $\mathfrak{B} \subset \mathfrak{A}$, the simple members of $\boldsymbol{HSP}(\boldsymbol{K} \cup \{\mathfrak{B}\})$ are those of $\boldsymbol{K}$ together with members of $\boldsymbol{S}\{\mathfrak{B}\}$; $\mathfrak{B} \notin \boldsymbol{K}$ would yield $\boldsymbol{HSP}(\boldsymbol{K} \cup \{\mathfrak{B}\}) = \boldsymbol{L}$ and so $\mathfrak{A} \in \boldsymbol{K} \cup \boldsymbol{S}\{\mathfrak{B}\}$, contradiction.

To complete our discussion of covers, we first need the following important result about equational classes.

THEOREM 3.6. *If* $\boldsymbol{K}$ *is an equational class properly contained in* $\boldsymbol{PA}_2$, *then there exist* $m \in \omega \sim 1$ *and* $\boldsymbol{L}_0, \ldots, \boldsymbol{L}_m \subseteq \boldsymbol{K}$ *such that the following conditions hold*:

(i) $\boldsymbol{K} = \boldsymbol{HSP}(\boldsymbol{L}_0 \cup \ldots \cup \boldsymbol{L}_m)$;
(ii) $\forall i \leq m(\boldsymbol{L}_i$ *is a finite set of simple* (*if* $0 < i$) *atomic* $\boldsymbol{PA}_2$*'s* $\mathfrak{A}$ *with* $|c_0{}^*A| = 2^i)$;
(iii) $\forall i \in m \sim 1\ \forall \mathfrak{A}$ (*If* $\mathfrak{A}$ *is a simple* $\boldsymbol{PA}_2$ *with* $|c_0{}^*A| = 2^i$, *then* $\mathfrak{A} \in \boldsymbol{K}$ *iff every finite subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$ *with* $|c_0{}^*B| = 2^i$ *can be imbedded in some member of* $\boldsymbol{L}_i$).

PROOF. Let $\boldsymbol{L}_0$ be a singleton of a one-element $\boldsymbol{PA}_2$. By Theorem 3.2, $\boldsymbol{K}$ is determined by its finite members. Hence $\boldsymbol{K} = \boldsymbol{HSPM}$, where $\boldsymbol{M}$ is the class of all simple atomic members of $\boldsymbol{K}$. Since $\boldsymbol{K} \neq \boldsymbol{PA}_2$, there is an $m \in \omega \sim 1$ such that for any $\mathfrak{A} \in \boldsymbol{M}$, $|c_0{}^*A| \leq 2^m$. Now let $i \in m \sim 1$ be fixed. Let T_i be the set of all signatures of members $\mathfrak{A}$ of $\boldsymbol{M}$ with $|c_0{}^*A| = 2^i$, and T_i' be the set of all $\lesssim$-maximal members of T_i. For each $s \in T_i'$ let $\mathfrak{A}_s \in \boldsymbol{M}$ have signature s, and let $\boldsymbol{L}_i = \{\mathfrak{A}_s : s \in T_i'\}$.

To verify (iii), let $\mathfrak{A}$ be a simple $\boldsymbol{PA}_2$ with $|c_0{}^*A| = 2^i$. First suppose $\mathfrak{A} \in K$, and $\mathfrak{B}$ is a finite subalgebra of $\mathfrak{A}$ with $|c_0{}^*B| = 2^i$. Thus $\mathfrak{B} \in \boldsymbol{M}$. Let t be a signature of $\mathfrak{B}$. By Zorn's lemma, $t \lesssim s$ for some $s \in T_i'$. Then by Theorem 3.6, $\mathfrak{B}$ can be imbedded in $\mathfrak{A}_s$, as desired. The converse follows from the equational character of $\boldsymbol{K}$. Condition (i) clearly follows from (iii). It remains only to show that $\boldsymbol{L}_i$ is finite.

Suppose $\boldsymbol{L}_i$ is infinite. Let F be a non-principal ultrafilter on T_i', and let $\mathfrak{B} = \boldsymbol{P}_{s \in T_i'} \mathfrak{A}_s / F$. Then $\mathfrak{B}$ is again a simple atomic member of $\boldsymbol{M}$

with $|c_0{}^*B|=2^i$. Let t be a signature of $\mathfrak{B}$, say $t=(i,f,g,h)$. There is a formula ψ with the following properties:

(1) the free variables of ψ are $v_0,\ldots,v_{i-1}$;

(2) If $\mathfrak{C}$ is a simple atomic $\boldsymbol{PA}_2$ with $|c_0{}^*C|=2^i$ and with $\langle x_0,\ldots,x_{i-1}\rangle$ an enumeration of the atoms of $c_0{}^*\mathfrak{C}$, then $\mathfrak{C}\models\psi[x_0,\ldots,x_{i-1}]$ iff for all distinct $j,k<i$,

$$(f_j \text{ finite} \Rightarrow |\{a\in \mathrm{At}\,x_j\colon S(0,1)a=a\}| = f_j)\,,$$
$$(g_j \text{ finite} \Rightarrow |\{a\in \mathrm{At}\,x_j\colon a\neq S(0,1)a\in \mathrm{At}\,x_j\}| = g_j)\,,$$
$$(h_{jk} \text{ finite} \Rightarrow |\{a\in \mathrm{At}\,x_j\colon S(0,1)a\in \mathrm{At}\,x_k\}| = h_{jk})\,.$$

Let φ be the sentence

$$\exists v_0\ldots\exists v_{i-1}\big[\textstyle\bigwedge_{j<i}(v_i \text{ is an atom of the } c_0\text{-closed Boolean part})\wedge\psi\big].$$

Thus φ holds in $\mathfrak{B}$, so $U=\{s\in T_i'\colon \varphi \text{ holds in } \mathfrak{A}_s\}\in F$; hence U is infinite. For each $s\in U$ there is a signature u_s of $\mathfrak{B}$ such that $s\precsim u_s$. Since U is infinite and $\mathfrak{B}$ has only finitely many signatures, there is an infinite subset V of U and a signature v of $\mathfrak{B}$ such that $s\precsim v$ for all $s\in V$. Since $V\subseteq T_i'$ and $v\in T_i$, this is a contradiction. The proof is complete.

Corollary 3.7. *Any equational class of $\boldsymbol{PA}_2$'s is determined by a finite set of its simple members.*

Our discussion of covers is now completed by the following result:

Theorem 3.8. *Each member of $\mathscr{L}$ has only finitely many covers.*

Proof. Let $\boldsymbol{K}\in\mathscr{L}$, $\boldsymbol{K}\neq\boldsymbol{PA}_2$. Choose $m,\boldsymbol{L}_0,\ldots,\boldsymbol{L}_m$ in accordance with 3.6. Then choose $\boldsymbol{N}_0,\ldots,\boldsymbol{N}_m$ in accordance with Theorem 2.11. Suppose $\boldsymbol{K}'$ covers $\boldsymbol{K}$. By 3.5, $\boldsymbol{K}'=\boldsymbol{HSP}(\boldsymbol{K}\cup\{\mathfrak{A}\})$, where $\mathfrak{A}$ is finite and simple, and $\mathfrak{B}\in\boldsymbol{K}$ for each proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. By Theorem 2.7 we see that either $\mathfrak{A}$ is a minimal $\boldsymbol{PA}_2$ such that $|c_0{}^*A|=2^{m+1}$ or else $\mathfrak{A}$ is isomorphic to some member of $\bigcup_{i\le m}\boldsymbol{L}_i$. At any rate, there are only finitely many choices for $\mathfrak{A}$, up to isomorphism, as desired.

Theorem 3.6 is also crucial in proving the following theorem, which is one of the main results of this paper.

Theorem 3.9. *Each member of $\mathscr{L}$ is finitely based.*

Proof. Suppose $\boldsymbol{K}\in\mathscr{L}$, $\boldsymbol{K}\neq\boldsymbol{PA}_2$. Choose $m,\boldsymbol{L}_0,\ldots,\boldsymbol{L}_m$ as in 3.6. Let $i\le m$. Let $\boldsymbol{N}_i$ be chosen in accordance with 2.11, and let n_i $\max\{|B|\colon \mathfrak{B}\in\boldsymbol{N}_i\}$. Now there is an open formula ψ_i with variables $v_0,\ldots,v_{n_i-1}$, $w_0,\ldots,w_{i-1}$ such that:

(1) for any $PA_2\mathfrak{B}$, $\mathfrak{B} \models \psi_i[b_0,\ldots,b_{n_i-1},d_0,\ldots,d_{i-1}]$ iff $\{b_0,\ldots,b_{n_i-1}\}$ is a subalgebra of $\mathfrak{B}$, and $d_0,\ldots,d_{i-1}$ are all of the distinct atoms of the Boolean algebra of c_0-closed elements of $\{b_0,\ldots,b_{n_i-1}\}$.

Now let S be the set of all signatures of members of L_i; note that S is finite. For any $j<i$ and $l<\omega$ let φ_{ijl} be the open formula

$$\bigwedge\nolimits_{k\leq l}\left[0 \neq u_k \wedge u_k \leq w_j \wedge S(0,1)u_k \neq u_k \wedge S(0,1)u_k \leq w_j \wedge \bigvee\nolimits_{s<n_i}(u_k = v_s)\right] \wedge \wedge \bigwedge\nolimits_{s<t\leq k}(u_s \cdot u_t = 0)$$

For $s=(i,f,g,h)\in S$, $j<i$, $l<g_j$, and $f_j,g_j<\omega$, we let

$$p(i,j,s) = f_j + \tfrac{1}{2}(g_j - l - 1) .$$

Then let φ_i be the following sentence:

$$\forall v_0 \ldots \forall v_{n_i-1} \forall w_0 \ldots \forall w_{i-1} \Big\{ \psi_i \to \bigvee\nolimits_{s=(i,f,g,h)\in S} \Big[\Big(\bigwedge\nolimits_{j<i,g_j<\omega} \neg \exists u_0 \ldots \exists u_{g_j} \varphi_{ijg_j} \Big) \Big(\bigwedge\nolimits_{j<i,g_j<\omega,f_j<\omega} \bigwedge\nolimits_{l<g_j,\, l\,\mathrm{odd}} \forall u_0 \ldots \forall u_l$$
$$\Big(\varphi_{ijl} \wedge \bigwedge\nolimits_{k\leq n} \big[0 \neq v_k \wedge v_k \leq w_j \wedge S(0,1)v_k \neq v_k \wedge S(0,1)v_k \leq w_j \to \bigvee\nolimits_{t\leq l}(u_t \leq v_k) \big] \to \neg \exists x_0 \ldots \exists x_{p(s,j,l)}$$
$$\big[\bigwedge\nolimits_{t\leq p(s,j,l)} (0 \neq x_t \wedge x_t \leq w_j \wedge S(0,1)x_t = x_t) \wedge \bigwedge\nolimits_{t<r}(x_t \cdot x_r = 0) \big] \Big) \Big)$$
$$\wedge \Big(\bigwedge\nolimits_{j,k<i,j\neq k,h_{jk}<\omega} \neg \exists u_0 \ldots \exists u_{h_{jk}} \Big(\bigwedge\nolimits_{l\leq h_{jk}} \big[0 \neq u_l \wedge u_l \leq w_j \wedge S(0,1)u_l \leq w_k \wedge \bigvee\nolimits_{t<n}(u_l = v_t) \big] \wedge \bigwedge\nolimits_{t<r\leq h_{jk}}(u_t \cdot u_r = 0) \Big) \Big) \Big] \Big\} .$$

Clearly φ_i is logically equivalent to a universal sentence χ_i. We now claim:

(2) for any $i<m$, if $\mathfrak{B}$ is a simple PA_2 then $\mathfrak{B}$ satisfies χ_i iff each finite subalgebra $\mathfrak{C}$ of $\mathfrak{B}$ with $|c_0{}^*C| = 2^i$ can be imbedded in a member of L_i.

Indeed, the direction $\Leftarrow$ is obvious. Now suppose that $\mathfrak{C}$ is a finite subalgebra of $\mathfrak{B}$ with $|c_0{}^*C| = 2^i$ which cannot be imbedded in any member of L_i; let $\mathfrak{C}$ be minimal with this property. Then $\mathfrak{C}$ is isomorphic to some member of N_i, and hence $|C| \leq n_i$. It easily follows that χ_i fails in $\mathfrak{C}$, and hence also in $\mathfrak{B}$.

Now let θ_K be a universal sentence logically equivalent to the following sentence:

$$\neg \exists v_0 \ldots \exists v_{m+1} \left[\bigwedge_{i<j\leq m+1} (v_i \cdot v_j = 0) \wedge \bigwedge_{i\leq m+1} (c_0 v_i = v_i \wedge v_i \neq 0) \right] \wedge \bigwedge_{i\leq m} \chi_i .$$

From 3.6 and (2) we easily infer that:

(3) for any simple $\boldsymbol{PA}_2\mathfrak{B}$, $\mathfrak{B}$ satisfies θ_K iff $\mathfrak{B} \in \boldsymbol{K}$.

Now with each open formula ϱ we associate a term $\tau\varrho$:

$$\tau(\sigma \simeq \xi) = c_0 c_1(\sigma \oplus \xi), \quad \tau(\neg \varrho) = -\tau\varrho, \quad \tau(\varrho_0 \rightarrow \varrho_1) = -\tau\varrho_0 \cdot \tau\varrho_1$$

Then ϱ holds in a simple $\boldsymbol{PA}_2\mathfrak{A}$ iff $\tau\varrho \simeq 0$ holds in $\mathfrak{A}$. It follows that if the open part of θ_K above is ϱ, then $\tau\varrho \simeq 0$ characterizes $\boldsymbol{K}$ relative to $\boldsymbol{PA}_2$, as desired.

Corollary 3.10. $|\mathscr{L}| = \aleph_0$.

From Theorems 2.1, 2.5, and 3.9 we obtain

Corollary 3.11. *For each* $\boldsymbol{K} \in \mathscr{L}$, Eq$\boldsymbol{K}$ *is decidable.*

In conclusion we may mention the following unpublished result of Ralph McKenzie, which constitutes a far-reaching generalization of a part of 3.9:

Theorem. *If* $\mathfrak{A}$ *is a finite lattice with finitely many additional operations, then* $\boldsymbol{HSP}\{\mathfrak{A}\}$ *is finitely based.*

Added in proof. The question concerning the number of variables needed to characterize $\boldsymbol{K}_m$ relative to $\boldsymbol{PA}_1$ has been solved by Th. Lucas.

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