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MISBEHAVIOUR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'=f(x,y)+\varepsilon$ WHEN THE RIGHT HAND SIDE IS DISCONTINUOUS

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1. Purpose.

It is well known that by consideration of the corresponding integral equation, most qualitative theorems concerning initial-value problems for the first order ordinary differential equation $y'=f(x,y)$ can be extended to the case where the right side is no longer continuous (see e.g., [2, §§ 45–46], [3], [4]). In this note, however, we shall show by example that more than one widely used theorem in the continuous case cannot be so extended, at least not in a form which would preserve its most useful feature, as soon as the right side of the equation fails to be jointly continuous at just a single point, even though it remains bounded and continuous there in each variable separately.

2. Some theorems which cannot be extended.

THE APPROXIMATION THEOREM FOR THE MAXIMAL SOLUTION ([8], [9], [7], [5]). *Let $f(\cdot,\cdot)$ be a real-valued function which is jointly continuous in x and y for all (x,y) in $\mathsf{R}\times\mathsf{R}$. Choose any point (ξ,η) in $\mathsf{R}\times\mathsf{R}$ and, among all the solutions of the differential equation*

$$\mathrm{E}_\varepsilon \qquad y' = f(x,y)+\varepsilon \qquad (\varepsilon>0 \textit{ fixed})$$

that pass through the point $(\xi,\eta+\varepsilon)$, select any one and call it $y_\varepsilon(\cdot)$. Denote by $y_0(\cdot)$ the maximal solution of the differential equation

$$\mathrm{E}_0 \qquad y' = f(x,y)$$

issuing to the right from the point (ξ,η), and suppose that $\beta>\xi$ is so chosen that $y_0(\cdot)$ exists throughout the interval $[\xi,\beta]$. Then, if α is such that $\xi<\alpha<\beta$, the solutions $y_\varepsilon(\cdot)$ will exist on $[\xi,\alpha]$ for all ε sufficiently small and will converge uniformly on $[\xi,\alpha]$ to $y_0(\cdot)$ as ε tends to zero.

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An alternate version of this theorem (cf. [8], [9], [7]) has $\beta > \xi$ so chosen that all the solutions $y_\varepsilon(\cdot)$ exist on $[\xi,\beta]$ for ε sufficiently small, and then the conclusion is that the $y_\varepsilon(\cdot)$ converge uniformly on $[\xi,\beta]$, as ε tends to zero, to a function $y_0(\cdot)$ which turns out to be the maximal solution of E_0 on $[\xi,\beta]$. When ε is restricted to a sequence of values tending to zero, the theorem provides a constructive proof of the existence of the maximal solution, in contrast with the usual proof ([3, p. 45]) which requires the consideration of a *continuum* of solutions of the original equation E_0.

Both versions of the approximation theorem rely heavily upon a comparison theorem to allow the solutions $y_\varepsilon(\cdot)$ of the equations E_ε to be chosen so freely. The following one ([7], [5, §§ 44–45]) is typical.

THE COMPARISON THEOREM. *Suppose $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are both continuous and real-valued on* $\mathsf{R} \times \mathsf{R}$, *with*

$$f(x,y) < g(x,y)$$

for all (x,y). Then no solution curve of the differential equation E_0 issuing to the right from an arbitrary point (ξ,η) will meet any solution curve of the differential equation

$$\text{G} \qquad y' = g(x,y)$$

that issues to the right from a point $(\xi,\eta+\delta)$, $\delta > 0$.

3. Formulation as an integral equation.

As an integral equation, the initial-value problem for the differential equation E_0 takes the following form:

$$\text{IE}_0 \qquad y(x) = \eta + \int_\xi^x f(u,y(u))\,du\,.$$

This equation will have a solution as long as f satisfies the *Carathéodory hypotheses* [2, §§ 576–582]: f should be measurable in x, continuous in y, and have its absolute value dominated by a summable function of x which is independent of y. Under these circumstances, every solution $y(\cdot)$ of IE_0 will be absolutely continuous and, by the fundamental theorem of calculus for Lebesgue integrals, will satisfy the differential equation E_0 a.e. In particular, a solution $y(\cdot)$ of IE_0 will satisfy E_0 at every value of x for which $(x,y(x))$ is a point of joint continuity of the function f, as follows, e.g., from [10, p. 107].

4. Example one.

Suppose that

$$\begin{aligned} f(x,y) &= 2\,y/x && \text{for } |y| \leqq |x|,\ x \neq 0\,, \\ &= 2\,\mathrm{sign}\,xy && \text{for } 0 < |x| \leqq |y| \leqq 2|x|\,, \\ &= 4\,\mathrm{sign}\,xy - y/x && \text{for } 0 < 2|x| \leqq |y| \leqq 4|x|\,, \\ &= 0 && \text{otherwise}\,. \end{aligned}$$

The function f is continuous on the circle $x^2+y^2=1$, and therefore, by its homogeneity, is jointly continuous everywhere except at the origin. There it remains continuous in x and y separately.

In the interval $[-1,1]$, the maximal solution $y_0(\cdot)$ of the differential equation E_0 (understood in the Carathéodory sense), issuing to the right from the point $(\xi,\eta)=(-1,0)$, consists of the broken line

$$\begin{aligned} y_0(x) &= 0 && \text{for } -1 \leqq x \leqq 0\,, \\ &= 2x && \text{for } 0 \leqq x \leqq 1\,. \end{aligned}$$

To confirm this, we may note that $f(x,y)$ is locally Lipschitzian in y for $x<0$, $|y|<2\,|x|$, so that E_0 possesses a (locally) unique solution through any point in this region. Since $y_0(x)=0$ *is* a solution of the equation E_0 in the half-open interval $[-1,0)$ with $y_0(-1)=0$, it is, by default, the maximal solution on $[-1,0)$ issuing from the point $(-1,0)$.

At the origin, uniqueness breaks down. Therefore, to see that $y_0(x)=2x$ is the maximal solution of E_0 issuing to the right from the origin, we verify, first, that it is a solution, and then note that $f(x,y)$ satisfies everywhere the inequality $|f(x,y)| \leqq 2$. Since every solution $y(\cdot)$ in the Carathéodory sense in the interval $[0,1]$, for which $y(0)=0$, is continuous in $[0,1]$ and differentiable in $(0,1]$, the mean-value theorem of differential calculus applies, and we conclude that, for any x in $(0,1]$,

$$y(x) \;=\; y(x)-y(0) \;\leqq\; \max\nolimits_{0<u<x} y'(u)(x-0) \;\leqq\; 2x\,,$$

which proves that $y_0(\cdot)$ is maximal on $[0,1]$. (We could, of course, have achieved the same result by estimating the integral in IE_0 (for $\xi=\eta=0$) directly.)

Combining this with the result of the previous paragraph gives us the maximal property of the solution $y_0(\cdot)$.

On the other hand, taking for the solutions $y_\varepsilon(\cdot)$ of the equations E_ε, for $0<\varepsilon\leqq 1$, the functions $y_\varepsilon(x)=-\varepsilon x$, we see that as ε tends to zero the functions $y_\varepsilon(\cdot)$ converge uniformly on $[-1,1]$ to the function $y(x)\equiv 0$, which is *not* the maximal solution of E_0 on this interval.

5. Example two.

In the above example, the maximal solution $y_0(\cdot)$ fails to be differentiable at the origin. However, if we modify the definition of f inside the first quadrant, we can produce an example where the maximal solution $y_0(\cdot)$, as well as the ε-solutions $y_\varepsilon(\cdot)$, are C^{n-1} functions, and still the convergence fails.

For $x>0$, $y>0$, set ([2, § 585])

$$\begin{aligned} f(x,y) &= ny/x \quad \text{for } y<x^n ,\\ &= nx^{n-1} \quad \text{for } y\geqq x^n , \end{aligned}$$

otherwise, let f be defined as in the previous example. The maximal solution on $[-1,-1]$, issuing to the right from the point $(-1,0)$, is now

$$\begin{aligned} y_0(x) &= 0 \quad \text{for } -1\leqq x<0 ,\\ &= x^n \quad \text{for } \quad 0\leqq x\leqq 1 . \end{aligned}$$

The function $y_0(\cdot)$ actually satisfies the differential equation E_0 at *every* point of the interval $[-1,1]$. Nevertheless, taking the solutions $y_\varepsilon(\cdot)$ as before, we see once again that they do not converge to $y_0(\cdot)$ on the interval $(0,1]$. Nor does the comparison theorem hold when $g(\cdot,\cdot)$ is taken to be $f(\cdot,\cdot)+\varepsilon$.

6. Final remarks.

When the discontinuity of $f(\cdot,\cdot)$ occurs only at the *left endpoint* of the interval $[\xi,\infty)$, the comparison theorem of § 2 still remains valid, cf. [6, § 58]. Our examples show that this form of the theorem is essentially best possible.

Under Carathéodory hypotheses on f and g, a somewhat weaker form of the comparison theorem is known to hold (cf. [1], [4]) if the conclusion is modified to restrict the various competing solutions of G to the maximal solution alone. With a similar restriction on the solutions $y_\varepsilon(\cdot)$, the approximation theorem will likewise hold [3, p. 47], but its principal virtue has now been sacrificed.

REFERENCES

1. F. Cafiero, *Su due teoremi di confronto relativi ad un'equazione differenziale ordinaria del primo ordine*, Boll. Un. Mat. Ital. (3) 3 (1948), 124–128.
2. C. Carathéodory, *Vorlesungen über reelle Funktionen*, 2. Aufl., B. G. Teubner, Leipzig, 1927.
3. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York · Toronto · London, 1955.

4. G. S. Goodman, *Subfunctions and the initial-value problem for differential equations satisfying Carathéodory hypotheses*, J. Differential Equations 7 (1970), 232–242.
5. E. Kamke, *Differentialgleichungen reeller Funktionen*, Leipzig, 1930.
6. E. Kamke, *Differentialgleichungen*, I. *Gewöhnliche Differentialgleichungen*, 5. Aufl., Leipzig, 1964.
7. P. Montel, *Sur l'intégrale supérieure et l'intégrale inférieure d'une équation différentielle*, Bull. Sci. Math. (2) 50 (1926), 205–217.
8. G. Peano, *Sull'integrabilità delle equazioni differenziali di primo ordine*, Atti R. Accad. Torino 21 (1885–86), 677–685.
9. O. Perron, *Ein neuer Existenzbeweis für die Integrale der Differentialgleichung* $y' = f(x,y)$, Math. Ann. 76 (1915), 471–484.
10. S. Saks, *Theory of the integral* (Translation of Monografic Mat. 7), Hafner Publ. Co., New York, 1937.

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