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ON FINITELY GENERATED FLAT MODULES II

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1. Introduction.

A ring A is said to be a left n-FGFP-ring (right n-FGFP-ring) if every flat left A-module (flat right A-module) generated by n elements is projective (cf. Sandomierski and Smith [12]).

For a commutative ring A we have that A is a left 1-FGFP-ring if and only if A is a left n-FGFP-ring for all n (cf. [9] and [7]).

In section 2 we shall construct a ring A such that A is a left and right n-FGFP-ring, but neither a left $(n+1)$-FGFP-ring nor a right $(n+1)$-FGFP-ring (for any given $n \geqq 1$). This construction is suggested by P. M. Cohn.

The main result in section 3 is the following:

Let A be a subring of B and M a flat and finitely generated left A-module. If $B \otimes_A M$ is projective considered as a left B-module, then M is A-projective.

As a corollary we have that a subring of a left n-FGFP-ring is a left n-FGFP-ring, too. This result is a generalization of theorem 2.2 in [7].

In section 4 we shall prove that a right noetherian ring is a left n-FGFP-ring for all n. By means of this result and the main theorem in section 3 we shall derive some results concerning left and right semihereditary rings.

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Notation. In this note all rings considered are associative. All rings have a unit element, denoted by 1, homomorphisms preserve 1, subrings have the same 1 and modules are unital.

2. Flat modules and the dependence number.

We recall that a ring A is said to satisfy ${}_n$ACC (cf. Cohn [2]) in case any ascending chain of n-generated left ideals becomes stationary.

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For later purposes we need the following

LEMMA 2.1. *If the ring A satisfies ${}_n$ACC, then A is a left n-FGFP-ring.*

PROOF. It follows from the Morita-equivalence between A and A_n (the ring of $n \times n$-matrices) that it suffices to proves the lemma for $n=1$. In this case the lemma is well known (cf. Sahaev [10]).

PROPOSITION 2.2. *For any given $n \geqq 1$ there exists a ring A such that A is a left and right n-FGFP-ring, but neither a left $(n+1)$-FGFP-ring nor a right $(n+1)$-FGFP-ring.*

PROOF. For notation and definitions in the proof we refer to Cohn [2]. Let K be a commutative field. We take A to be the K-algebra on the generators x_{ij}^{ν}, $i,j=1,\ldots,n+1$, $\nu=1,2,\ldots$, and defining relations

$$\Sigma_j x_{ij}^{\nu} x_{jk}^{\nu'} = 0, \quad \nu \neq \nu' , \tag{2.1}$$

$$\Sigma_j x_{ij}^{\nu} x_{jk}^{\nu} = x_{ik}^{\nu} .$$

The matrix $E_\nu = (x_{ij}^{\nu})$ is clearly idempotent. Furthermore $E_\nu E_{\nu'} = E_{\nu'} E_\nu = 0$, $\nu \neq \nu'$. Thus A_{n+1} has an infinite set of non zero orthogonal idempotents. Hence A_{n+1} is not a left 1-FGFP-ring (cf. Sahaev [10]), and consequently A is not a left $(n+1)$-FGFP-ring.

It follows, as in [2, proposition 4.2], that theorem 3.1 in [2] can be applied, so we conclude that $\lambda(A)$ (the dependence number of A) $\geqq n+1$, Hence A satisfies ${}_n$ACC for $n \geqq 2$. We have now proved that A is a left n-FGFP-ring for $n \geqq 2$. If $n=1$, then A is an integral domain [2, proposition 2.2], and it is readily checked that an integral domain is a left 1-FGFP-ring.

Since all arguments are left and right symmetric, the proof of proposition 2.2 is complete.

PROPOSITION 2.3. *Any n-fir (cf. [2]) is a left n-FGFP-ring.*

PROOF. The proof requires only few modifications of the argument proving theorem 2.B in [6]. We will state the proof for the reader's convenience. It is based on the following result on n-firs (cf. [2, Introduction]).

If $a_1 b_1 + \ldots + a_m b_m = 0$, $m \leqq n$, (not all b_i equal to zero), then there exists a $m \times m$ unimodular matrix P such that

$$(a_1, \ldots, a_m) P = (a_1, \ldots, 0, \ldots, a_m)$$

with 0 on the ith place for a suitable i.

Let M be a flat left A-module generated by $f_1,\ldots,f_p$, $p \leqq n$. If there is no nontrivial relation between the f_i, then M is free. Assume

$$\sum_{i=1}^{r} a_i f_i = 0, \quad r \leqq p, \tag{2.2}$$

is a nontrivial relation between the f_i. Since M is flat, (2.2) is a consequence of a linear relation in A, i.e., there exists a finite set of elements $\hat{f}_j \in M$, $1 \leqq j \leqq k$, and $\hat{a}_{ij} \in A$, $1 \leqq i \leqq r$, $1 \leqq j \leqq k$, such that

$$f_i = \sum_j \hat{a}_{ij} \hat{f}_j \quad \text{for all } i, \qquad \sum_i a_i \hat{a}_{ij} = 0 \quad \text{for all } j. \tag{2.3}$$

We may assume not all the $\hat{a}_{ij}$ are zero, say $\hat{a}_{11} \neq 0$. Consider the relation

$$a_1 \hat{a}_{11} + \ldots + a_r \hat{a}_{r1} = 0$$

and let $P = (p_{ij})$ be a unimodular matrix such that $\sum_i a_i p_{ij} = 0$ for a suitable j. Let $P^{-1} = (\hat{p}_{ij})$ be the (twosided) inverse of P and define

$$f_t' = \sum_\nu p_{t\nu} f_\nu, \qquad 1 \leqq t \leqq r,$$

$$a_t' = \sum_\mu a_\mu p_{\mu t}. \quad 1 \leqq t \leqq r,$$

Then

$$\sum_t a_t' f_t' = \sum a_\mu p_{\mu t} \hat{p}_{t\nu} f_\nu = \sum_\nu a_\nu f_\nu = 0.$$

We have now obtained a nontrivial relation between fewer elements in M. Furthermore the module generated by $(f_1',\ldots,f_r',f_{r+1},\ldots,f_p)$ is equal to M. If we continue this argument, we get a set $(\tilde{f}_1,\ldots,\tilde{f}_p)$ of generators for M and a nontrivial relation $a\tilde{f}_j = 0$ (for a suitable $j \in \{1,\ldots,p\}$ and $a \in A$). Since M is flat and A an integral domain, we infer that $\tilde{f}_j = 0$. If we choose p such that M is not generated by less than p elements, then it follows that M is free with $(f_1,\ldots,f_p)$ as a base.

3. The main results.

We start with the key-result.

THEOREM 3.1. *Let M be a flat and finitely generated left A-module and A a subring of a ring B. If the left B-module $B \otimes_A M$ is B-projective, then M is A-projective.*

PROOF. We have a short exact sequence

$$0 \to K \to F \xrightarrow{\varphi} M \to 0, \tag{3.1}$$

where F is a free left A-module with a finite base and K denotes the kernel of φ. From (3.1) we derive the exact sequence

$$(3.2) \qquad \operatorname{Tor}_1^A(B,M)=0 \to B\otimes_A K \to B\otimes_A F \to B\otimes_A M \to 0$$

of left B-modules.

Since $B\otimes_A F$ is a finitely generated B-module and $B\otimes_A M$ is a finitely generated projective B-module, we have that $B\otimes_A K$ is finitely generated. We choose a generating set for $B\otimes_A K$ of the form $1\otimes k_i$, $1\leqq i\leqq m$. From Bourbaki [1, exercise 23, p. 65] it follows that there exists a homomorphism u from F to K such that $u(k_i)=k_i$, $1\leqq i\leqq m$.

Let k be an arbitrary element in K. We will prove that $u(k)=k$. If

$$1\otimes k = \textstyle\sum_i b_i(1\otimes k_i), \quad b_i\in B,\ 1\leqq i\leqq m ,$$

say, then

$$\begin{aligned} 1\otimes u(k) = (1_B\otimes u)(1\otimes k) &= (1_B\otimes u)(\textstyle\sum_i b_i(1\otimes k_i)) \\ &= \textstyle\sum_i (1_B\otimes u)(b_i(1\otimes k_i)) \\ &= \textstyle\sum_i b_i\otimes u(k_i) = \sum_i b_i(1\otimes k_i) = k \end{aligned}$$

and hence

$$1\otimes(u(k)-k) = 0 .$$

Since K is A-flat, we have an exact sequence of left A-modules

$$(3.3) \qquad 0\to A\otimes_A K \to B\otimes_A K .$$

The element $1\otimes(u(k)-k)$ is zero in $B\otimes_A K$, hence also zero in $A\otimes_A K$, and consequently $u(k)=k$. This proves that (3.1) is split exact, hence M is A-projective.

COROLLARY 3.2. *Let A be a subring of B. If B is a left n-*FGFP*-ring, then A is a left n-*FGFP*-ring, too.*

PROOF. If M is a n-generated flat left A-module, then $B\otimes_A M$ is a n-generated flat left B-module (cf. [1, chap. 1, § 2, no. 7, proposition 8, cor. 2]) and by theorem 3.1 M is A-projective.

The proof of the next corollary requires only trivial modifications of the argument proving theorem 1.7 in [7].

COROLLARY 3.3. *A is a left n-*FGFP*-ring if and only if $A[[X]]$ is a left n-*FGFP*-ring.*

By combining corollary 3.2 and corollary 3.3 we get the following

COROLLARY 3.4. *A is a left n-FGFP-ring if and only if $A[X]$ is a left n-FGFP-ring.*

For further corollaries we refer to [7].

REMARK. Theorem 3.1 is known in special cases (cf. S. Endo [4, theorem 1] and F. Sandomierski [11, theorem 2.8]).

4. Flat embeddings of certain rings.

We start this section with some elementary results concerning rings A with $\mathrm{wgl\,dim}\, A \leqq 1$.

LEMMA 4.1. *Let A be a commutative ring. If $\mathrm{whd}_A(Aa+Ab)=0$ for all $a,b \in A$, then $\mathrm{wgl\,dim}\, A \leqq 1$.*

PROOF. It is enough to prove the lemma for A local (localization). Let $\mathfrak{a}$ be a finitely generated ideal and a,b two elements in $\mathfrak{a}$. $Aa+Ab$ is flat and finitely generated, hence $Aa+Ab$ is free [1, Chap. 1, exercise 23 p. 65] and consequently $Aa+Ab$ is generated by a single element. This proves that $\mathfrak{a}$ is generated by a single element. Since any finitely generated ideal is flat, we have that $\mathrm{wgl\,dim}\, A \leqq 1$.

We recall that a ring is left semi-hereditary if each finitely generated ideal is projective. If A is left semi-hereditary, then $\mathrm{wgl\,dim}\, A \leqq 1$.

PROPOSITION 4.2. *Let A be a commutative ring. If $\mathrm{wgl\,dim}\, A \leqq 1$ and $\mathrm{hd}_A(Aa)=0$ for all $a \in A$, then A is semi-hereditary.*

PROOF. Since $\mathrm{wgl\,dim}\, A \leqq 1$, we have that any ideal is flat. If $\mathfrak{a}$ is a finitely generated ideal, then we have to prove that the rank function $r\colon \mathrm{Spec}(A) \to Z$ is locally constant (cf. [1, chap. 2, § 5, theorem 1]). The ideal $\mathfrak{a}_\mathfrak{p}$ is a finitely generated flat ideal in $A_\mathfrak{p}$ (for any prime ideal $\mathfrak{p}$), hence $\mathfrak{a}_\mathfrak{p}$ is zero or free of rank one.

Define

$$U_0 = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid r_\mathfrak{p} = 0\}.$$

Then

$$U_0 = \mathrm{Spec}(A) \setminus \mathrm{Supp}(\mathfrak{a})$$

is an open subset of $\mathrm{Spec}(A)$ [1, chap. 2, § 4, prop. 17] and consequently we are done if we can prove that $\mathrm{Supp}(\mathfrak{a})$ is open. Let $\mathfrak{a}$ be generated by (a_i), $1 \leqq i \leqq n$. Since Aa_i is projective, we conclude that $\mathrm{Supp}(Aa_i)$ is open, hence $\mathrm{Supp}(\mathfrak{a}) = \bigcup_{i=1}^n \mathrm{Supp}(Aa_i)$ is open.

PROPOSITION 4.3. *Given a ring-homomorphism ϱ from A to B such that B is flat as a right A-module. Suppose that ϱ is a monomorphism. If* $\operatorname{wgl\,dim} A \leqq 1$ *and any flat and finitely generated left ideal in B is projective, then A is left semi-hereditary.*

PROOF. Let $\mathfrak{a}$ be a finitely generated left ideal in A. Then $B \otimes_A \mathfrak{a}$ is a flat left B-module. Since B is A-flat, $B \otimes_A \mathfrak{a}$ is an ideal in B. From theorem 3.1 we infer that $\mathfrak{a}$ is A-projective.

REMARK 1. Any finitely generated flat left ideal in A is projective if A satisfies one of the following conditions:

i) A is left coherent.

ii) A is a left n-FGFP-ring for all n.

COROLLARY 4.4. *For a commutative ring A with* $\operatorname{wgl\,dim} A \leqq 1$ *the following conditions are equivalent:*

i) *A is semi-hereditary.*

ii) *Q_{cl} (the classical quotient ring of A) is von Neumann regular.*

iii) *The complete direct product B of the quotient fields of $A/\mathfrak{p}$, where $\mathfrak{p}$ ranges over all minimal prime ideals in A, is A-flat.*

PROOF. ii) implies i) by proposition 4.3.

A is a subring of B (cf. [5]). So iii) implies i), since B is von Neumann regular.

It follows from S. Endo [3, § 4, proposition 1] that the quotient field of $A/\mathfrak{p}$ is isomorphic to $Q_{cl}/\mathfrak{p}Q_{cl}$. Hence B is a module over Q_{cl}. If A is semi-hereditary, then Q_{cl} is a von Neumann regular ring [3, § 4, proposition 1] so B is flat as a module over Q_{cl}. Since Q_{cl} is A-flat, B is A-flat. We have now proved that i) implies ii) and iii).

That implication i) implies iii) is due to C. U. Jensen.

REMARK 2. It is easy to give examples of commutative non semi-hereditary rings A with $\operatorname{wgl\,dim} A \leqq 1$ (see for instance C. U. Jensen [6]). Since any ring A with $\operatorname{wgl\,dim} A \leqq 1$ can be embedded in a von Neumann regular ring [5], we see that the flatness of B is essential for the validity of proposition 4.3.

COROLLARY 4.5. (Sandomierski [11, theorem 2.10]) *Let A be a ring with* $\operatorname{wgl\,dim} A \leqq 1$ *and zero left singular ideal. If Q, the complete ring of left quotients of A, is flat as a right A-module, then A is left semi-hereditary.*

PROOF. It follows [8, § 4.5, proposition 2] that Q is a von Neumann regular ring, and hence any finitely generated flat left ideal in Q is projective.

We conclude this section by the following

THEOREM 4.6. *If the ring A is right noetherian, then A is a left n-FGFP-ring for all n.*

PROOF. Let $\mathrm{rad}(A)$ denote the prime radical of A (cf. Lambek [8]). $B = A/\mathrm{rad}(A)$ is a semiprime (in Lambek's notation), right noetherian ring. Hence the complete ring of right quotients of B is completely reducible [8, § 4.5, proposition 3, corollary]. Moreover B is a left n-FGFP-ring for all n [corollary 3.2].

Let M be a finitely generated flat left A-module. We have a short exact sequence of left A-modules

$$0 \to K \to F \to M \to 0 , \tag{4.1}$$

where F is free and finitely generated. Since $M/\mathrm{rad}(A)M$ is B-flat and finitely generated, $M/\mathrm{rad}(A)M$ is B-projective.

From (4.1) we derive the short exact sequence of left B-modules

$$0 \to K/\mathrm{rad}(A)K \to F/\mathrm{rad}(A)F \to M/\mathrm{rad}(A)M \to 0 . \tag{4.2}$$

Since $M/\mathrm{rad}(A)M$ is projective, (4.2) is split exact and hence $K/\mathrm{rad}(A)K$ is finitely generated as a B-module. From [8, § 3.5, proposition 4] it follows that $\mathrm{rad}(A)$ is nilpotent and consequently K is finitely generated [1, Chap. 2, § 3, no. 2]. So we conclude that M is A-projective.

COROLLARY 4.7. *If the ring A can be embedded in a right noetherian ring, then A is a left n-FGFP-ring for all n.*

COROLLARY 4.8. *Suppose A can be embedded in a right noetherian ring. If $\mathrm{wgl\,dim}\, A \leqq 1$, then A is left and right semi-hereditary.*

PROOF. It follows that any flat and finitely generated left or right A-module is projective. Since any finitely generated left or right ideal in A is flat, the proof of the corollary is complete.

A similar result has been obtained by Small (cf. [13]).

ADDED IN PROOF. After writing this paper I have become aware that proposition 4.2 has also been proved by W. V. Vasconcelos [*On finitely generated flat modules*, Trans. Amer. Math. Soc. 138 (1969), 505–512] and that corollary 3.2 has been proved by I. I. Sahaev [*On rings over which any finitely generated module is projective*, Izv. Vysš. Učebn. Zaved. Matematika 9 (1969), 65–73. (In Russian.)].

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