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THE SPACE OF LEBESGUE MEASURABLE FUNCTIONS ON THE INTERVAL [0; 1] IS HOMEOMORPHIC TO THE COUNTABLE INFINITE PRODUCT OF LINES

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Let R^{N} denote the countable infinite product of lines. In the present paper we prove the following

THEOREM. *Each of the following metric spaces is homeomorphic to* R^{N}.

(a) *The space D of* (*equivalence classes of*) *those measurable real valued functions $x(\cdot)$ defined on* $[0; 1]$ *for which*

$$\operatorname*{ess\,sup}_{0 \leqq t \leqq 1} |x(t)| \leqq 1$$

under the metric inherited from the function Hilbert space $L_2 = L_2[0,1]$, that is,

$$\|x-y\| = \left(\int_0^1 |x(t)-y(t)|^2\,dt\right)^{\frac{1}{2}}.$$

(b) *The space S of* (*equivalence classes of*) *real valued measurable functions on the interval* $[0; 1]$ *under the metric*

$$\varrho(x,y) = \int_0^1 \frac{|x(t)-y(t)|}{1+|x(t)-y(t)|}\,dt\,,$$

that is, in the topology of convergence in measure (cf. [3, pp. 9–10]).

(c) *The space M of* (*equivalence classes of*) *measurable sets in* $[0,1]$ *under the metric defined as the Lebesgue measure of the symmetric difference of sets.*

The statements (a) and (b) of the Theorem are related to the following general conjectures,

CONJECTURE 1. Every non-locally-compact closed convex subset of the Hilbert space L_2 (or more generally every non-locally-compact closed

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convex subset of a locally convex separable complete linear metric space) is homeomorphic to the whole space.

CONJECTURE 2. All infinite dimensional separable complete linear metric spaces are homeomorphic.

The proof of the Theorem is based on some properties of certain specific G_δ subsets of convex compact metric spaces, and it involves the concept of a Z-set introduced by R. D. Anderson (cf. [1], [8]). Actually, in the present paper we use the dual concept of a T-set, that is, a set the complement of which is a Z-set.

1. The apparatus of T-sets.

Let X be a metric space, with metric $d(\cdot,\cdot)$. By a T_δ-set in X we mean a subset A of X of type G_δ which satisfies the following condition:

(T) For every $\varepsilon > 0$, every integer $m \geqq 0$ and every map $f\colon I^m \to X$, there is a map $g\colon I^m \to A$ such that

$$d(f(a), g(a)) < \varepsilon \quad \text{for } a \in I^m .$$

Here by I^m we denote the closed m-cube, and by a map we mean a continuous transformation.

It is easily seen that the class of T_δ-sets of a given metric space X is topologically invariant in the following sense.

1.1. If $h\colon X \to X_1$ (onto) is a homeomorphism, then an A is a T_δ-set in X if and only if $h(A)$ is a T_δ-set in X_1.

In particular the property of being a T_δ-set does not depend on a choice of the metric on X provided the metric induces the same topology. An open T_δ-set is called a T-set. From the classical Baire's Theorem it follows that in complete metrizable spaces T_δ-sets are exactly those which are intersections of countably many T-sets. A G_δ-subset (with respect to the relative topology) of a G_δ-set in a metric space is a G_δ-set with respect to the whole space. Hence

1.2. If A is a T_δ-set in a metric space X and B is a T_δ-set in A, then B is a T_δ-set in X.

Another concept essentially employed in our proofs is that of the radial interior (briefly "rint"). Suppose that W is a convex set in a real linear space E. Then we define

$$\operatorname{rint} W = \{w \in W : \text{ for each } x \in W \text{ there exists } \varepsilon > 0 \text{ such that } w + \varepsilon(w - x) \in W\} .$$

In other words, rint W consists of the internal points of W (cf. [5, p. 410]), i.e. of those points $w \in W$ which have the property that every line in the space E passing through w either has only the point w in common with W or intersects W along a segment having w as an interior point. A simple elementary argument gives

1.3. If $0 \in \operatorname{rint} W$, in particular if W is symmetric with respect to zero, then

$$\operatorname{rint} W = \{cw : w \in W \text{ and } 0 \leqq c < 1\}.$$

By Q we denote the countable infinite product of the intervals $[-1; 1]$. The pseudointerior of Q is the set

$$P = \{x = (x(n))_{n=1}^{\infty} : |x(n)| < 1 \text{ for } n = 1, 2, \ldots\}.$$

Clearly we have

1.4. R^{N} is homeomorphic to P.

The proof of Theorem 1 is based upon the following facts.

1.5. Proposition. *Suppose that W is an infinite dimensional compact set in a linear metric locally convex space with* rint $W \neq \emptyset$. *Let A be a T_δ-set in W which is disjoint from* rint W. *Then there is a homeomorphism of W onto Q which carries A onto P. In particular A is homeomorphic to* R^{N}.

1.6. Proposition (cf. [2]). *If A is a T_δ set in* R^{N}, *then A is homeomorphic to* R^{N}.

In the particular case $W = Q$, Proposition 1.5 has been established by R. D. Anderson [2] (for an elegant proof see Toruńczyk [9]). The general statement follows from that particular one and from the Generalized Keller-Klee Theorem (cf. [4, Corollary 6.3]) which asserts that for every compact convex set W in a linear metric space such that rint $W \neq \emptyset$, there exists a homeomorphism of W onto Q which carries rint W onto rint Q.

Proposition 1.6 is an easy consequence of Proposition 1.5. Indeed, by 1.1 and 1.4, we may regard A as a T_δ-set in P. By Proposition 1.5 there exists a homeomorphism of Q onto itself, say F, such that $F(P) = Q \setminus \operatorname{rint} Q$. Clearly $F(A)$ is a T_δ-subset of $Q \setminus \operatorname{rint} Q$. Since $Q \setminus \operatorname{rint} Q$ is a T_δ-set in Q, the statement 1.2 implies that $F(A)$ is a T_δ-subset of Q. Thus, again by Proposition 1.5, there exists a homeomorphism of Q onto itself which carries $F(A)$ onto P. Thus A is homeomorphic to P. This completes the proof.

2. Projecting into the upper hemisphere and the Main Lemma.

Let R denote the real line. We shall consider the Hilbert space $\mathsf{R}\times L_2$ equipped with the inner product

$$\langle (u,x),\, (v,y)\rangle = uv + \int_0^1 x(t)\,y(t)\,dt$$

and with the norm $\|(u,x)\| = \langle (u,x),\, (u,x)\rangle^{\frac{1}{2}}$. We shall identify L_2 with the subspace $\{0\}\times L_2$ of the space $\mathsf{R}\times L_2$, and we shall use the same symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for denoting the inner product and the norm in L_2. By B we shall denote the closed unit ball of $\mathsf{R}\times L_2$, that is,

$$B = \{x \in \mathsf{R}\times L_2 :\ \|x\| \leqq 1\}\,.$$

We shall deal with two topologies on B. The first one induced by the norm of the space $\mathsf{R}\times L_2$ and the second one induced by the weak topology of $\mathsf{R}\times L_2$. Both topologies are metrizable on B. The metric for the weak topology on B is defined by

$$d^*(x,y) = \sum_{i=1}^{\infty} \frac{|\langle x-y, z_i^*\rangle|}{2^i(1+|\langle x-y, z_i^*\rangle|)} \quad \text{for } x \text{ and } y \in B\,,$$

where (z_i^*) is a dense sequence in B (cf. [5, p. 426]).

Therefore, for every subset Z of B we shall distinguish between the metric spaces $(Z, \|\cdot\|)$ and (Z, d^*) with the metric inherited by the norm and the d^* metric of B respectively.

Let

$$K = \{x \in L_2 :\ \|x\| \leqq 1\} \quad \text{and} \quad \partial K = \{x \in L_2 :\ \|x\| = 1\}$$

and define the map $G\colon K \to \mathsf{R}\times L_2$ by

$$G(x) = \big((1-\|x\|^2)^{\frac{1}{2}}, x\big) \quad \text{for } x \in K\,.$$

Clearly

$$G(K) = \{(u,x) \in \mathsf{R}\times L_2 :\ \|(u,x)\| = 1 \text{ and } u \geqq 0\}$$

is the "upper" unit hemisphere. Next we have

2.1. *G is a homeomorphism of the space $(K, \|\cdot\|)$ onto $(G(K), d^*)$.*

Proof. Clearly G is a homeomorphism between the spaces $(K, \|\cdot\|)$ and $(G(K), \|\cdot\|)$. To complete the proof it is enough to observe that on the unit sphere of a Hilbert space the weak and the norm topologies coincide (cf. [3, p. 139]).

Next define for a subset X of K the set $W(X) \subset B$ by

$$W(X) = \{(u,x) \in \mathsf{R}\times L_2 :\ x \in X \text{ and } 0 \leqq u \leqq (1-\|x\|^2)^{\frac{1}{2}}\}\,.$$

2.2. The Main Lemma. *Let X be a non-one-point closed convex subset of $(K,\|\cdot\|)$ such that* $\operatorname{rint} X \neq \emptyset$ *and* $X \cap \partial K$ *satisfies the condition* (T) *in* (X,d^*). *Then*

(i) $G(X)$ *is a* T_δ*-set in* $(W(X),d^*)$,
(ii) $(X,\|\cdot\|)$ *is homeomorphic to* R^{N},
(iii) $(X\cap\partial K,\|\cdot\|)$ *is homeomorphic to* R^{N}.

Proof. (i) Since X is a closed subspace in the (complete metric) Hilbert space, X is an absolute G_δ (cf. [7, § 31, III]). Thus, by 2.1, we infer that $G(X)$ is a G_δ-subset of $(W(X),d^*)$. Evidently $\operatorname{rint} W(X)\neq\emptyset$. It remains to show that $G(X)$ satisfies condition (T) in $(W(X),d^*)$.

Let $\varepsilon>0$ and let $f\colon I^m \to W(X)$ be a map. Observe first that without loss of generality one can assume that $f(I^m)\subset \operatorname{rint} B$. (Otherwise take an $x_0\in X\cap \operatorname{rint} K$ and replace f by the function $f+\frac{1}{2}\varepsilon(x_0-f)$.) Next put $f_1=pf$, where $p\colon \mathsf{R}\times L_2\to L_2$ denotes the natural projection defined by $p((u,x))=x$ for $(u,x)\in\mathsf{R}\times L_2$. Since $X\cap\partial K$ satisfies the condition (T) in (X,d^*), there exists a map $g_1\colon I^m\to X\cap\partial K$ such that $d^*(f_1(a),g_1(a))<\varepsilon$ for $a\in I^m$. Let us set

$$s(a) = \inf\{0\leqq s\leqq 1 \mid \|f(a)+s(g_1(a)-f_1(a))\|=1\},$$
$$g(a) = f(a)+s(a)(g_1(a)-f_1(a)).$$

Since $\|f(a)\|=\|f(a)+0(g_1(a)-f_1(a))\|<1$ (because $f(a)\in\operatorname{rint}B$) and since

$$\|f(a)+1(g_1(a)-f_1(a))\| \geqq \|p(f(a)+1(g_1(a)-f_1(a)))\| = \|g_1(a)\| = 1,$$

we infer that the functions $s(\cdot)$ and g are well defined by the above formulas. Clearly $\|g(a)\|=1$ for $a\in I^m$. By the convexity of $W(X)$, we have

$$g(a) = (1-s(a))f(a)+s(a)g_1(a) \in W \quad \text{for } a\in I^m.$$

Since $0\leqq s(a)\leqq 1$, we also have

$$d^*(f(a),g(a)) = d^*(s(a)f_1(a),s(a)g_1(a)) \leqq d^*(f_1(a),g_1(a)).$$

Therefore, to complete the proof of (i) it is enough to check the continuity of g.

Let $(a_n)_{n=0}^\infty$ be a sequence in I^m and let $\lim_n a_n=a_0$. Since $0\leqq s(\cdot)\leqq 1$, every subsequence of (a_n) contains a subsequence, say (a_{k_n}), such that there exists a limit, $\lim_n s(a_{k_n})=c$. Clearly, we have

$$1 = \|f(a_0)+c(g_1(a_0)-f_1(a_0))\| = \lim_n\|f(a_{k_n})+s(a_{k_n})(g_1(a_{k_n})-f_1(a_{k_n}))\|.$$

On the other hand,

$$\|f(a_0)+s(a_0)(g_1(a_0)-f_1(a_0))\| = 1.$$

Since $\|f(a_0)\| < 1$, the half-line

$$\{x \in \mathsf{R} \times L_2 \;:\; x = f(a_0) + t(g_1(a_0) - f_1(a_0)),\ t \geqq 0\}$$

has exactly one point in common with the unit sphere of $\mathsf{R} \times L_2$. Therefore

$$f(a_0) + s(a_0)(g_1(a_0) - f_1(a_0)) \;=\; \lim_n \big(f(a_{k_n}) + s(a_{k_n})(g_1(a_{k_n}) - f_1(a_{k_n}))\big)\,.$$

This proves that g is continuous and completes the proof of (i).

(ii). By 2.1, the spaces $(X, \|\cdot\|)$ and $(G(X), d^*)$ are homeomorphic. By (i), $G(X)$ is a T_δ-subset of $(W(X), d^*)$. Obviously $\operatorname{rint} W(X)$ is non-empty and disjoint from $G(X)$. Since $(W(X), d^*)$ is a compact convex set which can be affinely and homeomorphically embedded in R^{N} (for instance putting $F(w) = (\langle w, z_i^*\rangle)_{i=1}^{\infty}$ for $w \in W(X)$, where $(z_i^*)_{i=1}^{\infty}$ is an orthonormal basis in $\mathsf{R} \times L_2$), Proposition 1.5 implies that $(G(X), d^*)$ is homeomorphic to R^{N}.

(iii) Since X is a norm closed convex subset of K, the space (X, d^*) is compact. As in (ii) we observe that the space (X, d^*) is affinely homeomorphic to a convex compact subset of R^{N}. Furthermore, the identity embedding

$$(X \cap \partial K, \|\cdot\|) \to (X \cap \partial K, d^*)$$

is a homeomorphism, because on ∂K the norm and the weak topologies coincide (cf. [3, p. 139]). Thus $(X \cap \partial K, d^*)$ is a G_δ subset of (X, d^*), because $(X \cap \partial K, \|\cdot\|)$ is a closed subset of L_2. Hence, by the assumption, $(X \cap \partial K, d^*)$ is a T_δ-subset of (X, d^*). Obviously the sets $\operatorname{rint} X$ and $X \cap \partial K$ are disjoint. Therefore, by Proposition 1.5, the spaces $(X \cap \partial K, d^*)$ and R^{N} are homeomorphic. Hence the space $(X \cap \partial K, \|\cdot\|)$ is homeomorphic to R^{N}. This completes the proof of the Main Lemma.

3. Proof of the Theorem.

Proof of (a). Clearly the space D is a closed convex subset of $(K, \|\cdot\|)$ and $\operatorname{rint} D \neq \emptyset$. Therefore, by the Main Lemma 2.2 (ii), it is enough to establish the following

3.1. Lemma. *The set $D \cap \partial K$ satisfies the condition* (T) *in* (D, d^*).

Proof. Let $x_{i,n}$ denote the characteristic function of the interval $[i2^{-n}; (i+1)2^{-n})$ for $i = 0, 1, \ldots, 2^n - 1$ and for $n = 0, 1, \ldots$. Let X_n be the 2^n-dimensional linear subspace spanned by the functions $x_{0,n}$, $x_{1,n}, \ldots, x_{2^n-1,n}$. Clearly the sets $\bigcup_{j=0}^{\infty} X_j$ and $\bigcup_{j=0}^{\infty} (D \cap X_j)$ are dense in the norm topology in L_2 and D respectively. Hence they are also dense in the weak topology. Thus for $\varepsilon > 0$ and for a map $f\colon I^m \to (D, d^*)$, using the convexity of the sets $D \cap X_j$, $j = 0, 1, \ldots$, one can construct a simplicial map $f': I^m \to (D,d^*)$ such that $d^*(f(a),f'(a)) < \frac{1}{2}\varepsilon$ for $a \in I^m$ and $f'(I^m) \subset X_n$ for some index n. Next for each $k > n$ we construct a function $h_k: D \cap X_n \to D$ such that

(1) $h^k(D \cap X_n) \subset D \cap \partial K$,

(2) $\langle h_k(x), x^* \rangle = \langle x, x^* \rangle$ for $x \in D \cap X_n$ and for $x^* \in X_k$,

(3) h_k is a continuous function in the norm topology .

Assume that we have done this. Since the linear space $\bigcup_{j=0}^{\infty} X_j$ is norm dense in L_2, the sets

$$\{z \in L_2 : \ |(z, x^*\rangle| < i^{-1} \text{ for } x^* \in X_k \cap K\}, \quad k > n, \ i = 1, 2, \ldots ,$$

form a base of neighbourhoods of zero for the weak topology of K. Thus the condition (2) implies that the sequence $(h_k f')(a)$ converges to $f'(a)$ at each point $a \in I^m$ uniformly in the metric $d^*(\cdot,\cdot)$. Therefore there exists an index k_0 such that for $g = h_{k_0} f'$ we have $d^*(f'(a), g(a)) < \frac{1}{2}\varepsilon$ for $a \in I^m$. Hence $d^*(f(a), g(a)) < \varepsilon$. It follows from (1) that $g(I^m) \subset D \cap \partial K$. Finally, since the set $D \cap X_n$ is compact, the norm continuity of h_k implies the weak continuity of h_k. Therefore g is continuous.

To complete the proof we have to construct the functions h_k. In the following summations, i runs through $0, \ldots, 2^n - 1$. For

$$x = \textstyle\sum_i c_i x_{i,n} \in D \cap X_n$$

we put

$$h_k(x) = 2\chi_{B(x)} - 1 .$$

Here $\chi_{B(x)}$ denotes the characteristic function of the set

$$B(x) = \textstyle\bigcup_i A_i(c_i) ,$$

where

$$A_i(c_i) = \bigcup_{j=i2^{k-n}}^{(i+1)2^{k-n}-1} \left[2^{-k}j;\, 2^{-k}(j + 2^{-1}(1+c_i))\right]$$

for $|c_i| \leqq 1$ and for $i = 0, 1, \ldots, 2^n - 1$. Clearly $h_k(x) \in D \cap \partial K$. Furthermore, for each $j = 0, 1, \ldots, 2^k - 1$, we have

$$\langle h_k(x), x_{j,k} \rangle = 2^{-k}\big(2^{-1}(1+c_i) - (1 - 2^{-1}(1+c_i))\big) = 2^{-k}c_i = \langle x, x_{j,k} \rangle ,$$

where $i = i(j)$ is chosen so that $2^{k-n} i \leqq j < 2^{k-n}(i+1)$. Thus, by linearity of the inner product with respect to the second variable, we get (2). Finally, (3) is an obvious consequence of the inequality

$$\|h_k(x) - h_k(x')\| = (2^{-n+1} \textstyle\sum_i |c_i - c_i'|)^{\frac{1}{2}} \leqq (2^{n-1}\|x - x'\|)^{\frac{1}{2}}$$

for

$$x = \textstyle\sum_i c_i x_{i,n} \quad \text{and} \quad x' = \textstyle\sum_i c_i' x_{i,n} .$$

This completes the proof of the Lemma and of (a).

Proof of (b). By Proposition 1.6 and by (a), it is enough to show that the space S is isomorphic to a T_δ-subset of the space $(D, \|\cdot\|)$. To this end define $H\colon S \to D$ by

$$H(z) = z(1+|z|)^{-1} \quad \text{for } z \in S .$$

Clearly H is one to one and

$$H(S) = \{x \in D:\ \operatorname{mes}\{t \in [0;\,1]\colon |x(t)| = 1\} = 0\} .$$

The inverse of H is defined by

$$H^{-1}(x) = x(1-|x|)^{-1} \quad \text{for } x \in H(S) .$$

Obviously, if both spaces S and D are equipped with the topology of convergence in measure, then H is a homeomorphism. Since the identical embedding $(D,\varrho) \to (D, \|\ \|)$ is a homeomorphism (cf. [6, p. 110, Theorem D]), we infer that the map $H\colon (S,\varrho) \to (H(S), \|\ \|)$ is a homeomorphism. Since S is a complete metric space, $H(S)$ is a G_δ-subset of $(D, \|\ \|)$. Finally observe that $H(S)$ satisfies the condition (T) in $(D, \|\cdot\|)$. Indeed, for a map $f\colon I^m \to (D, \|\cdot\|)$ and for $1 > \varepsilon > 0$ we put $g = (1-\varepsilon)f$. Thus $H(S)$ is a T_δ-subset of $(D, \|\ \|)$. This completes the proof of (b).

Proof of (c). Define $F\colon M \to D$ by

$$F(A) = 2\chi_A - 1 \quad \text{for } A \in M ,$$

where χ_A denotes the characteristic function of the set A. Clearly we have

$$\operatorname{mes}(A_1 \dot{-} A_2) = \int_0^1 |\chi_{A_1} - \chi_{A_2}|\,dt \leqq \|\chi_{A_1} - \chi_{A_2}\| = 2^{-1}\|F(A_1) - F(A_2)\|$$

and

$$\|F(A_1) - F(A_2)\| = 2\|\chi_{A_1} - \chi_{A_2}\| = 2(\operatorname{mes}(A_1 \dot{-} A_2))^{\frac{1}{2}} ,$$

where $\operatorname{mes}(A_1 \dot{-} A_2)$ denotes the Lebesgue measure of the symmetric difference of the sets A_1 and A_2. Hence F is a homeomorphic embedding of M into $(D, \|\ \|)$. Moreover $F(M)$ is precisely the set $D \cap \partial K$. Thus, the desired conclusion follows from the Main Lemma 2.2 (iii) and from Lemma 3.1. This completes the proof of the Theorem.

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