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ON THE DIRICHLET PROBLEM FOR FUNCTIONS ON THE EXTREME BOUNDARY OF A COMPACT CONVEX SET

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1. Introduction.

The aim of this paper is to provide a different and hopefully simpler proof of necessary and sufficient conditions for solvability of the Dirichlet problem for bounded functions defined initially on the extreme boundary of a compact convex set. Originally, this problem was solved for the metrizable case by E. M. Alfsen in [2]. Working independently, A. J. Lazar [6], and E. Effros [4] recently removed the metrizability restraint for Choquet simplexes. Finally, E. M. Alfsen [3] gave a proof for the general case. However, since the proof in [3] pertains to a more intricate situation, it is somewhat cumbersome. It is hoped that the proof below removes some of this detail.

2. The theorem.

Throughout this section, K is an arbitrary compact convex subset of a locally convex Hausdorff topological linear space. The set of extreme points of K is denoted by eK. The terminology, notation, and basic notions concerning real affine functions, resultant of a measure (= inner and outer regular Borel), simplex, and maximal measure may be found in [7].

A closer analysis of the techniques used to develop the above notion of maximal measure leads one to suspect the possibility of extension to a slightly more general situation. A compact Hausdorff space, X, and a linear subspace of the space, $C(X)$, of continuous real valued functions on X form the setting in which the extension is to take place. Two methods produce the same results and are explained below. In subsequent paragraphs, E is a linear subspace of $C(X)$ which contains the constants and separates the points of X.

The first method replaces the set of continuous convex functions in the

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definition on p. 24 of [7] by the set $S(E)$ of all functions on X which are pointwise supremum of finite families of E and proceeds as in subsequent pages of [7]. This approach leads quite naturally via [2] to the concepts of upper and lower E-envelopes of a bounded function defined initially on a subset M of X. For such a function the respective envelopes are defined for $x \in X$ by

$$\bar{f}(x) = \inf\{g(x)\colon g \in E, g|M \geqq f\}$$

and

$$\underline{f}(x) = \sup\{g(x)\colon g \in E, g|M \leqq f\}.$$

Note that when $X = K$, $E = A(K)$, the space of continuous affine functions on K, and $M = K$ the above is formally equivalent to [7, p. 18].

The second approach takes cognizance of the fact that, under the assumptions above, E is an archimedean ordered normed space [5] and so $\bar{E}$ is linearly order isometric to $A(L)$ [5], where

$$L = \{R \in E^*\colon R(1) = 1 = \|R\|\}$$

has the weak-* topology, and $\bar{E}$ is the uniform closure of E in $C(X)$. Now $A(L)$ determines the maximal measures on L, and E determines the Choquet boundary of E in X [7, p. 38]. As E separates the points of X it is clear that, up to a homeomorphism, the closure of the Choquet boundary is $\overline{\mathrm{e}L}$. Since maximal measures on L are supported by $\overline{\mathrm{e}L}$, they may be identified in a canonical manner with certain measures on X which are supported by the closure of the Choquet boundary determined by E.

Either of these two methods produce the same set of maximal measures. Moreover, an E-maximal measure, u, may be shown to be maximal if and only if $u(f) = u(\bar{f})$ for each f in $C(X)$ [7, p. 64].

Before proceeding to the theorem, a *boundary measure* [3] is a measure u on the σ-ring, F_0, generated by $\mathrm{e}K$ and the Baire sets of K, and such that $|u|(K \setminus \mathrm{e}K) = 0$. It is shown in [7] that each maximal measure u on K can be associated with a boundary measure Tu, in such a way that u and Tu have the same resultant. Of course, it is also true that u and Tu agree on all continuous and hence on all Baire functions on X. These facts are used below without further reference.

THEOREM [3]. *A bounded real valued function f on $\mathrm{e}K$ has a continuous affine extension to all of K if and only if*

(a) *the upper and lower $A(K)$-envelopes are continuous on $\overline{\mathrm{e}K}$, and*

(b) *$Tu_1(f) = Tu_2(f)$ for any two maximal probability measures u_1 and u_2 with common resultant.*

PROOF. *Necessity*: Obvious.

Sufficiency: Let f be any bounded real function on eK which satisfies (a) and (b). Denote the upper and lower $A(K)$-envelopes of f by $\bar{f}$ and $\underline{f}$ respectively. Since $\bar{f}$ and $\underline{f}$ are continuous on $\overline{\mathrm{e}K}$, it follows [1, p. 4] that $\bar{f}(x)=\underline{f}(x)$ for all $x \in \overline{\mathrm{e}K}$. Thus f already has a continuous extension to $\overline{\mathrm{e}K}$. For simplicity of notation this first extension is again labeled f.

A simple application of the Krein-Milman theorem shows that $A(K)$ is isometrically isomorphic to

$$A = \{g|\overline{\mathrm{e}K}\colon\ g \in A(K)\}\,.$$

Since A-envelopes of functions defined on subsets of $\overline{\mathrm{e}K}$ are restrictions of the $A(K)$-envelopes of these functions, notational simplicity is again preserved by denoting these envelopes by the same symbols.

Now the remarks above make it clear that the theorem obtains provided A can be shown to be the set

$$B = \{f \in C(\overline{\mathrm{e}K})\colon\ f|\mathrm{e}K \text{ satisfies (a) and (b)}\}\,. \tag{1}$$

Using the results in [7, p. 19] and their duals, in conjunction with (a), it is easy to show that B is a linear subspace of $C(\overline{\mathrm{e}K})$ and obviously $B \supseteq A$. A straightforward combination of (a) with the definition of infimum yields

$$\inf\{g(x)\colon\ g \in B,\ g|M \geqq h\} = \inf\{g(x)\colon\ g \in A,\ g|M \geqq h\}$$

for any subset M of $\overline{\mathrm{e}K}$ and bounded real function h on M. A similar result holds for lower envelopes. Thus B-envelopes of functions may be denoted by the same symbols as used for A.

One implication of the above envelope agreement is coincidence of the Choquet boundaries determined by A and B. That is, the common Choquet boundary of A and B is eK. This follows from the second method of determining the E-maximal measures by way of the characterization of the extreme points of a compact convex set in [7, p. 27] and the definition of Choquet boundary. Of course, one of the principle ingredients above is the identification of $\bar{B}$ with $A(L)$ and A with $A(K)$. Here

$$L = \{R \in B^*\colon\ R(1) = 1 = \|R\|\}\,.$$

Now identify eL ($\overline{\mathrm{e}L}$) with eK ($\overline{\mathrm{e}K}$) via the preceding remarks. Let p be the restriction map of L onto K (K is identified with $\{R \in A^*\colon R(1) = 1 = \|R\|\}$). Observe that if p were known to be one-to-one, then the nature of the order structure (see [5]) of $(\bar{B})^*$ and A^* forces the restriction map between these spaces to be one-to-one and onto. Duality then implies that $\bar{B} = A$, or $B = A$, since $B \supseteq A$.

To show that p is one-to-one, observe that the remark following (1) (see also [7, p. 64]) together with the agreement of the A and B envelopes allows one to conclude that A-maximal measures are B-maximal and conversely. As each point of L (or K) is represented by at least one maximal probability measure, p will be one-to-one if it can be shown that two maximal probability measures u_1 and u_2 which have a common resultant in K also have this same resultant in L. To do this, recall that if u_1 and u_2 have a common resultant, then Tu_1 and Tu_2 also have this common resultant. By assumption (b) the boundary measures Tu_1 and Tu_2 must have a common resultant in L. But this can happen only when u_1 and u_2 have a common resultant in L. Thus p must be one-to-one. This completes the proof.

REFERENCES

1. E. M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand. 15 (1964), 97–110.
2. E. M. Alfsen, *Boundary values for homeomorphisms of compact convex sets*, Math. Scand. 19 (1966), 113–121.
3. E. M. Alfsen, *On the Dirichlet problem of the Choquet boundary*, Acta Math. 120 (1968), 149–159.
4. E. Effros, *Structure in simplexes* II, J. Functional Anal. 1 (1967), 361–391.
5. R. V. Kadison, *A representation theory for commutative topological algebras*, Mem. Amer. Math. Soc. 7 (1951).
6. A. J. Lazar, *Affine products of simplexes*, Math. Scand. 22 (1968), 165–175.
7. R. R. Phelps, *Lectures on Choquet's theorem* (Van Nostrand Mathematical Studies 7), D. Van Nostrand Co., Inc., Princeton, New Jersey, 1966.

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