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ON THE CONVERGENCE OF THE ADAMS SPECTRAL SEQUENCES

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In this note we will prove a theorem which at once solves the convergence problem of the Adams spectral sequence for cohomology theories satisfying certain conditions. The technique used here is very simple and is based on the fact that for certain extraordinary cohomology theories we have a natural transformation to ordinary cohomology. By using special properties of this natural transformation we obtain our results. These problems have also been discussed in [7] and [8].

In the following we will work in the stable homotopy category of J. M. Boardman, see [3]. We denote by $\mathscr{T}$ the class of highly connected spectra whose ordinary homology with Z-coefficients is finitely generated and torsion free in each dimension. Ordinary homology and cohomology will always in the following have Z-coefficients.

Let now M be a spectrum and M^* the cohomology theory defined by this spectrum, and let $\mathscr{A}$ denote the category in which M^* takes its values. Usually $\mathscr{A}$ will be an abelian category such as the category of modules over the ring $M^*(M)=A^M$ of operations or it can be the category of topologized modules over the topologized ring of operations (for details see [1], [2]), but all our following constructions go through without changes.

We are going to prove the following

THEOREM. *Let M be as above and assume that M satisfy the following conditions:*

a) i) $\pi_i(M)=0$ for $i<0$, $\qquad \pi_0(M)=\mathsf{Z}$,

ii) $H^1(M)=0$,

iii) $M \in \mathscr{T}$.

b) *There exists a natural transformation*

$$\mu:\quad M^n(\cdot) \to H^n(\cdot)$$

satisfying one of the following conditions:

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i) *μ is an epimorphism for elements in $\mathcal{T}$;*
ii) *if we have an exact triangle of elements in $\mathcal{T}$,*

$$X_1 \to X_0 \to M_0 \to X_1 ,$$

and μ is an epimorphism for X_0 and M_0, then it is an epimorphism for X_1. Assume also that μ is an epimorphism for M itself.

Let $X \in \mathcal{T}$ be of connectivity $n_0 - 1$; under condition b) ii) *we also assume $\mu(X)$ epi.*

Then there exists a diagram

$$\begin{tikzcd}[column sep=small] X = X_0 \arrow[dr] & X_1 \arrow[l] & \ldots \arrow[l] & X_i \arrow[l] \arrow[dr] & X_{i+1} \arrow[l] & \ldots \arrow[l] \\ & M_0 \arrow[u] & & & M_i \arrow[u] & \end{tikzcd} \tag{1}$$

such that:

A) *For every $i \geqq 0$,*

$$X_{i+1} \to X_i \to M_i \to X_{i+1}$$

is an exact triangle and

$$\deg(X_{i+1} \to X_i) = \deg(X_i \to M_i) = 0, \quad \deg(M_i \to X_{i+1}) = -1 .$$

The M_i's are of the form

$$\vee_{n_j} S^{n_j} M ,$$

where the n_j form an increasing sequence of natural numbers such that $n_j = n$ at most a finite number of times.

B) *If the connectivity of X_i is $n_i - 1$, then the n_j form a strictly increasing sequence,*

$$n_0 < n_1 < n_2 < \ldots < n_i < n_{i+1} < \ldots .$$

Before proving the theorem we make the following remarks:

The conditions are satisfied for $M^* = MU^*(= U^* = \Omega_u{}^*)$, see [4], [5], [6], and this case has been the model for our considerations.

A) gives that the diagram (1) forms a weakly injective M^*-complex over X in the terminology of [2].

The important thing in the theorem is B) which essentially solves the convergence problem for the Adams spectral sequence for M^*. For the groups $\{Y, X\}_*$ are being filtered by

$$F^n = \operatorname{im}[\{Y, X_n\}_* \to \{Y, X_0\}_*] ,$$

and if Y is finite-dimensional $F^n = 0$ for n big enough, and this implies

$$\bigcap_{n \geqq 0} F^n = 0 .$$

For further details, see [2], where the general machinery which is used in setting up the Adams spectral sequence is also discussed.

Proof of the theorem. According to the assumption, $X \in \mathscr{T}$. Hence from the Hurewicz Theorem and Univ. Coeff. Theorem it follows that its first non-vanishing ordinary cohomology group occurs in dimension n_0. Further for $n \geqq n_0$ the group $H^n(X)$ is free with a finite number d_n of generators,

$$H^n(X) = \mathsf{Z} \oplus \ldots \oplus \mathsf{Z} \quad (d_n \text{ summands}) ,$$

where $H^n(X) = 0$ if $d_n = 0$.

We write from now on $X = X_0$. From our assumptions we have an epimorphism

$$\mu(X_0)\colon \ M^n(X_0) \to H^n(X_0) .$$

Since $H^n(X)$ is free,

$$M^n(X_0) = H^n(X_0) \oplus \ker \mu(X_0) .$$

Let

$$f^{n,i_n}\colon \ X_0 \to S^n M, \quad i_n = 1, 2, \ldots, d_n ,$$

be morphisms representing the i_nth generator in $H^n(X_0)$. We construct the object

$$\bigvee_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} S^{n,i_n} M, \quad \text{where} \quad S^{n,i_n} M = S^n M .$$

Since the wedge is finite in each dimension it follows from (L 7) in [3] that

$$M_0 = \bigvee_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} S^{n,i_n} M \cong \prod_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} S^{n,i_n} M$$

(homotopy equivalence). Therefore by our morphisms f^{n,i_n} we get a morphism

$$f = (f^{n,i_n})\colon \quad X_0 \to M_0 ,$$

such that the following diagram commutes:

$$\begin{tikzcd}
\displaystyle\bigvee_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} S^{n,i_n} M \arrow[r, phantom, "\cong"] & \displaystyle\prod_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} S^{n,i_n} M \arrow[d, "\text{proj.}"] \arrow[r, phantom, "\cong"] & M_0 \\
X_0 \arrow[u, "f"] \arrow[r, "f^{n,i_n}"] & S^{n,i_n} M &
\end{tikzcd}$$

Observe that $M_0 \in \mathscr{T}$ since $\forall i: H^i(M_0) \cong \prod_{n,i} H^{i-n}(M) \cong \mathsf{Z} \oplus \ldots \oplus \mathsf{Z}$, a finite number of copies because

$$H^i(M) = 0 \quad \text{for } i < 0$$

(Hurewicz and Univ. Coeff. Theorems) and $M \in \mathscr{T}$.

From the naturality of μ we get the following commutative diagram:

$$\begin{array}{ccc} M^n(M_0) & \xrightarrow{M^n(f)} & M^n(X_0) \\ {\scriptstyle \mu(M_0)} \downarrow & & \downarrow {\scriptstyle \mu(X_0)} \\ H^n(M_0) & \xrightarrow[H^n(f)]{} & H^n(X_0) \end{array}$$

It follows from the construction of f that for $n \geqq n_0$ and $n \neq n_0 + 1$ we have

$$M^n(X_0) \supset \operatorname{im} M^n(f) \supset H^n(X_0) .$$

Since $\mu(X_0)$ is epi, this inclusion-chain gives, that the composition $\mu(X_0) \circ M^n(f)$ is epi. But this implies $H^n(f)$ epi for $n \geqq n_0$ and $n \neq n_0 + 1$. (Remark that we did not use here that $\mu(M_0)$ is epi!.)

Let us now look closer at the case $n = n_0$. We have

$$\begin{aligned} H^{n_0}(M_0) &= H^{n_0}\Bigg(\bigvee_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} S^{n, i_n} M \Bigg) \\ &\cong \prod_{\substack{n \geqq n_0,\, n \neq n_0+1 \\ i_n \in \{1, \ldots, d_n\}}} H^{n_0 - n}(M) \\ &= \prod_{i_{n_0} \in \{1, \ldots, d_{n_0}\}} H^0(M) = \mathsf{Z} \oplus \ldots \oplus \mathsf{Z} \cong H^{n_0}(X_0), \end{aligned}$$

since M_0 is just constructed such that the number of Z's is exactly the number of generators in $H^{n_0}(X_0)$. Here we used that $H^0(M) = \mathsf{Z}$ which follows from the fact that $M \in \mathscr{T}$ and $\pi_0(M) = \mathsf{Z}$ by using the Hurewicz and Univ. Coeff. Theorems. And $H^i(M) = 0$ for $i < 0$ as mentioned before.

The map

$$H^{n_0}(f): \ H^{n_0}(M_0) \to H^{n_0}(X_0)$$

is epi, but the two groups are isomorphic, hence $H^{n_0}(f)$ is an isomorphism.

We now extend the morphism $f: X_0 \to M_0$ to an exact triangle

$$X_1 \to X_0 \to M_0 .$$

This follows from one of the axioms of a stable category, and we may assume $\deg f = 0 = \deg(X_1 \to X_0)$.

Since X_0 and M_0 are (n_0-1)-connected it is clear from the exact homotopy sequence of this triangle

$$\dots \to \pi_{n_0}(X_1) \to \pi_{n_0}(X_0) \to \pi_{n_0}(M_0) \to \pi_{n_0-1}(X_1) \to 0 \to 0 \dots$$

that X_1 is at least (n_0-2)-connected.

Consider the long exact sequence of this triangle in ordinary cohomology:

$$\begin{aligned}
\dots \leftarrow H^{n_0+2}(X_1) &\xleftarrow{0} H^{n_0+2}(X_0) \xleftarrow{\text{epi}} H^{n_0+2}(M_0) \leftarrow \\
\leftarrow H^{n_0+1}(X_1) &\longleftarrow H^{n_0+1}(X_0) \longleftarrow H^{n_0+1}(M_0) \leftarrow \\
\leftarrow H^{n_0}(X_1) &\longleftarrow H^{n_0}(X_0) \xleftarrow{\cong} H^{n_0}(M_0) \longleftarrow \\
\leftarrow H^{n_0-1}(X_1) &\longleftarrow 0 \longleftarrow 0 \longleftarrow \dots .
\end{aligned}$$

But

$$H^{n_0+1}(M_0) = H^{n_0+1}\left(\bigvee_{\substack{n\geqq n_0,\, n\neq n_0+1\\ i_n\in\{1,\dots,d_n\}}} S^{n,i_n}M\right) \cong \prod_{\substack{n\geqq n_0,\, n\neq n_0+1\\ i_n\in\{1,\dots,d_n\}}} H^{n_0+1-n}(M) = 0$$

since we have assumed

$$H^1(M) = 0 \quad \text{and} \quad n \neq n_0+1 .$$

Hence

$$H^{n_0}(X_1) = H^{n_0-1}(X_1) = 0 .$$

And again using Univ. Coeff. and Hurewicz Theorems we get that X_1 is at least n_0 connected and of course $n_0 > n_0-1$. The long exact cohomology sequence in this case also implies $X_1 \in \mathscr{T}$, since $H^{n_0+1}(M_0)=0$ and $H^n(f)$ is epi for $n \geqq n_0+2$.

So by our assumptions (b) i) and ii)) the construction goes on by induction and we get the whole diagram in the theorem, since it has been shown that X_1 satisfies the conditions which were required by X_0. This completes the proof of the theorem.

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