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THE MAXIMAL IDEAL SPACE OF A BANACH ALGEBRA OF MULTIPLIERS

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1. Introduction.

J. L. Taylor [12] has characterized the maximal ideal space of the convolution algebra $M(G)$ of bounded regular Borel measures on a locally compact abelian topological group G as the set of semi-characters on a compact abelian topological semigroup called the *Taylor structure semigroup* of $M(G)$. A different construction of the Taylor structure semigroup has later been given by Ramirez [7] and Rennison [8], who exploit the natural C^*-algebra structure in the dual of $M(G)$ regarded as the bidual of $C_0(G)$, the C^*-algebra of continuous functions on G vanishing at infinity. They note that the strongly closed span Q of the set of multiplicative linear functionals on $M(G)$ is a sub-C^*-algebra of $M(G)'$, and indentify the Taylor structure semigroup of $M(G)$ with the maximal ideal space of Q.

For a commutative semi-simple Banach algebra A, denote by $\hat{A}$ the set of the Gelfand transforms of the elements of A and by A^m the set of functions on the spectrum of A that keep $\hat{A}$ invariant by pointwise multiplication. Each $f \in A^m$ determines a bounded linear operator on A and the set A^m of such operators, the *multiplier algebra* of A, is a Banach algebra under the uniform operator norm. If A is $L^1(G)$, the convolution algebra of Haar integrable functions on G, A^m may be identified with $M(G)$. In this paper we generalize the structure theory of $M(G)$ sketched above to the multiplier algebras of certain commutative semi-simple Banach algebras with the distinctive feature that the strongly closed span P of the set of multiplicative linear functionals has the structure of a commutative Banach algebra, too. Following Birtel [4], we embed A^m in P'. We then define an Arens quotient product in the dual of A^m originating from the product in P and show — under some natural additional assumptions — that the spectrum of A^m spans a commutative subalgebra Q of $(A^m)'$. The spectrum of Q generalizes the Taylor structure semigroup S of $M(G)$, as is indicated in section 5. The main theorem is theorem 4.7, in which it is assumed that P is a C^*-algebra with

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identity, though most of the auxiliary results are proved under weaker hypotheses. Applied to the case of the group algebra $L^1(G)$, theorem 4.7 shows that the natural embedding of the dual group Γ of G into the spectrum of $M(G)$ may be interpreted as the dual mapping from Γ into the set of semicharacters on S of a continuous homomorphism from S onto the Bohr compactification of G.

CONVENTION. All Banach algebras considered in this paper are complex. For any commutative Banach algebra A, $\Delta(A)$ denotes the spectrum of A, that is, the set of non-zero multiplicative linear functionals on A. If $D \subset A'$, $[D]$ denotes the subspace of A' generated by D, and $[D]^-$ its norm closure.

2. Arens products and quotient products.

2.1. R. Arens [1], [2] has extended the product of an arbitrary Banach algebra A to its bidual A'' by the following rule. If $m: A \times A \to A$ denotes the product in A, a jointly continuous bilinear map m^*: $A' \times A \to A'$ may be defined by setting $m^*(x',x)y = x'm(x,y)$ for $x' \in A'$, $x,y \in A$. Iterating this procedure one obtains

$$m^{**}: A'' \times A' \to A', \quad m^{**}(x'',x')y = x''m^*(x',y),$$

and finally

$$m^{***}: A'' \times A'' \to A'', \quad m^{***}(y'',x'')x' = y''m^{**}(x'',x').$$

For any Banach algebra product m we denote by m^t the product for which $m^t(x,y) = m(y,x)$. A is called *Arens regular*, if $m^{t***t} = m^{***}$. When no confusion can arise, any Banach algebra product will be denoted in the usual way by juxtaposition.

THEOREM 2.1. *Let A be a Banach algebra and E a subspace of A'. Denote by E° the annihilator of E in A''. Consider the following five statements:*

(1) *E° is a right ideal of A'' in the m^{t***}-product,*
(2) $m^{t**}(A'' \times E) \subset \bar{\bar{E}}$,
(3) $m^*(E \times A) \subset \bar{\bar{E}}$,
(4) $m^{**}(E^\circ \times \bar{\bar{E}}) = \{0\}$,
(5) *E° is a left ideal of A'' in the m^{***}-product.*

We have the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), *and if A is Arens regular, all statements are equivalent.*

PROOF. (1) implies (2): Suppose $m^{t**}(x'',x) \notin \bar{\bar{E}}$ for $x \in E$, $x'' \in A''$.

It is a consequence of the Hahn–Banach theorem that we can find $y'' \in E^\circ$ such that

$$m^{t***}(y'',x'')x' = y''m^{t**}(x'',x') \neq 0 .$$

Thus E° cannot be a right ideal for m^{t***}.

(2) implies (1): If $x'' \in E^\circ$, $y'' \in A''$ and $x' \in E$, we have, assuming (2),

$$m^{t***}(x'',y'')x' = x''m^{t**}(y'',x') = 0 ,$$

as $\bar{E}^\circ = E^\circ$.

(2) implies (3): If $x' \in E$ and $\dot{x} \in A''$ is the canonical image of $x \in A$, we have for $y \in A$

$$m^*(x',x)y = x'm(x,y) = x'm^t(y,x) = \dot{x}m^{t*}(x',y) = m^{t**}(\dot{x},x')y .$$

Hence $m^*(x',x) = m^{t**}(\dot{x},x') \in \bar{E}$.

(3) implies (4): If $x'' \in E^\circ$, $x' \in \bar{E}$ and $x \in A$, we have by (3),

$$m^{**}(x'',x')x = x''m^*(x',x) = 0 ,$$

as $\bar{E}^\circ = E^\circ$.

(4) implies (5): Take $x'' \in A''$, $y'' \in E^\circ$ and $x' \in E$. If (4) holds,

$$m^{***}(x'',y'')x' = x''m^{**}(y'',x') = x''(0) = 0 .$$

Thus $m^{***}(x'',y'') \in E^\circ$. Finally, if A is Arens regular, (5) implies (1), since a left ideal for $m^{***} = m^{t***t}$ is a right ideal for m^{t***}.

2.2. We assume in this subsection that A is a *commutative* Banach algebra and E a subspace of A' such that the condition (2) of theorem 2.1 holds, that is, $m^{**}(A'' \times E) \subset \bar{E}$. Then $m^*(\bar{E} \times A) \subset \bar{E}$ by theorem 2.1 and the continuity of m^*. Thus we may define $\mathbf{m}^{**}: E' \times \bar{E} \to A'$ by setting

$$\mathbf{m}^{**}(F,x')x = Fm^*(x',x) = x''m^*(x',x) = m^{**}(x'',x')x ,$$

where $F \in E'$ and x'' is its extension to A' such that $\|x''\| = \|F\|$. Then $\mathbf{m}^{**}$ is jointly continuous and $\mathbf{m}^{**}(E' \times \bar{E}) \subset \bar{E}$, so that we may define

$$\mathbf{m}^{***}: E' \times E' \to E' \quad \text{by} \quad \mathbf{m}^{***}(F,G)x' = F\mathbf{m}^{**}(G,x') .$$

By theorem 2.1., the ideal E° is a closed two-sided ideal in (A'',m^{***}). Let $p: A'' \to A''/E^\circ$ be the natural homomorphism. It is a well-known consequence of the Hahn–Banach theorem that $\varphi: A''/E^\circ \to E'$ defined by $\varphi \circ p(x'') = x''|E$ is a linear onto isometry.

THEOREM 2.2. *If A'' is given the product m^{***}, then φ transfers to $\mathbf{m}^{***}$ the natural product in A''/E°.*

PROOF. If x'', $y'' \in A''$ and $F = x''|E$, $G = y''|E$, we have for $x' \in E$,

$$\begin{aligned} m^{***}(\varphi \circ p(x''), \varphi \circ p(y''))x' &= Fm^{**}(G, x') = x''m^{**}(y'', x') \\ &= m^{***}(x'', y'')x' , \end{aligned}$$

and the theorem follows.

We call m^{***} the *Arens quotient product* in E', and denote usually $m^{***}(F, G) = F_*G$.

2.3. If A is a commutative Banach algebra and $D \subset \Delta(A)$,

$$m^{**}(x'', \textstyle\sum_{i=1}^n \lambda_i \alpha_i) = \sum_{i=1}^n \lambda_i x''(\alpha_i)\alpha_i$$

for $\lambda_i \in \mathbf{C}$, $\alpha_i \in D$, as is easily verified. Denote $E = [D]^-$. By continuity, $m^{**}(A'' \times E) \subset E$, and the Arens quotient product in E' is given by

$$F_*G(\textstyle\sum_{i=1}^n \lambda_i \alpha_i) = \sum_{i=1}^n \lambda_i F(\alpha_i) G(\alpha_i)$$

(cf. [4, p. 816]).

THEOREM 2.3. *Let A be a commutative Banach algebra and $E = [D]^-$, where $D \subset \Delta(A)$. If E is a commutative Banach algebra having D as a multiplicative subsemigroup, then $\Delta(E) \cup \{0\}$ with the weak* topology is a compact abelian topological semigroup under the Arens quotient product. If E has an identity $e \in D$, $\Delta(E)$ is a compact subsemigroup of $\Delta(E) \cup \{0\}$.*

PROOF. If $F, G \in \Delta(E)$, $\lambda_i, \mu_j \in \mathbf{C}$, $\alpha_i, \beta_j \in D$

for $i = 1, \ldots, n$, $j = 1, \ldots, p$, then we have

$$\begin{aligned} F_*G(\textstyle\sum_{i=1}^n \lambda_i \alpha_i)(\sum_{j=1}^p \mu_j \beta_j) &= F_*G(\textstyle\sum_{i=1}^n \sum_{j=1}^p \lambda_i \mu_j \alpha_i \beta_j) \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^p \lambda_i \mu_j F(\alpha_i \beta_j) G(\alpha_i \beta_j) \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^p \lambda_i F(\alpha_i) G(\alpha_i) \mu_j F(\beta_j) G(\beta_j) \\ &= F_*G(\textstyle\sum_{i=1}^n \lambda_i \alpha_i) F_*G(\sum_{j=1}^p \mu_j \beta_j) . \end{aligned}$$

By continuity, F_*G extends to a multiplicative linear functional on E. If E has an identity $e \in D$, F_*G is non-zero, as $F_*G(e) = F(e)G(e) = 1$. It is well known that $\Delta(E) \cup \{0\}$ is compact in the weak* topology [9, p. 113], and $\Delta(E)$ is compact if E has an identity. Since the Arens quotient product of E' can be defined by pointwise multiplication of the restrictions to D of the elements of E' and the elements of D suffice to define the relative weak* topology on $\Delta(E) \cup \{0\}$ [9, pp. 112–113] the multiplication in $\Delta(E) \cup \{0\}$ is jointly continuous.

EXAMPLE. The following example shows the role of the requirement that the identity of E belong to D. Take $A = \mathbf{C}^2$ with componentwise operations and norm $\|(z_1,z_2)\| = \sup\{|z_1|,|z_2|\}$. Then $A' = \mathbf{C}^2$ with the norm $\|(z_1,z_2)\| = |z_1| + |z_2|$, and A' is generated by $\Delta(A) = \{(1,0),(0,1)\}$. If A' is considered as a Banach algebra under componentwise operations, A' has the identity $(1,1)$, but

$$\Delta(A') = \{(1,0),(0,1)\} \subset A'' = A$$

is not closed under (Arens) multiplication.

2.4. The next result is essentially indicated in [7, pp. 1395–1396], but we give it an integration free proof. A *semicharacter* on a topological semigroup is a non-zero continuous homomorphism into the multiplicative semigroup of complex numbers z with $|z| \leqq 1$.

THEOREM 2.4. *Let A be a commutative semi-simple Banach algebra. Suppose that $P = [\Delta(A)]^-$ is a commutative C^*-algebra having $\Delta(A)$ as a multiplicative subsemigroup with identity. Then the semicharacters of $\Delta(P)$ are precisely the Gelfand transforms of the elements of $\Delta(A)$.*

PROOF. Since

$$F_* G(\textstyle\sum_{i=1}^n \lambda_i \alpha_i) = Fm^{**}(G, \sum_{i=1}^n \lambda_i \alpha_i) = F(\sum_{i=1}^n \lambda_i G(\alpha_i)\alpha_i)$$

for $F,G \in P'$, $\lambda_i \in \mathbf{C}$ and $\alpha_i \in \Delta(A)$, it follows by continuity that the adjoint of the operator $F \mapsto F_* G$ maps P into itself. Hence the Arens quotient product in P' is separately $\sigma(P',P)$-continuous [11, p. 128]. As P is semi-simple, $H = [\Delta(P)]^-$ is a separating subspace of P'. Hence H is $\sigma(P',P)$-dense in P' [11, p. 125].

Let $f\colon \Delta(P) \to \mathbf{C}$ be a semicharacter. As $f \neq 0$ is continuous, it is the Gelfand transform of some non-zero $x' \in P$. It follows from theorem 2.3 that H is a subalgebra of P', and since f is multiplicative on $\Delta(P)$, it is clear that the continuous linear functional defined by x' on P' is multiplicative on H. Since the $\sigma(P',P)$-continuous functions $F \mapsto F(x')G(x')$ and $F \to F_* G(x')$ for fixed $G \in H$ coincide on H, we have $F_* G(x') = F(x')G(x')$ for all $F \in P'$. Keeping next $F \in P'$ fixed and letting G vary, we conclude that the functional defined by x' is multiplicative on the whole of P', and as A is homomorphically embedded in P', $x' \in \Delta(A)$. As

$$F_* G(\alpha) = F(\alpha)G(\alpha) \quad \text{and} \quad |F(\alpha)| \leqq \|\alpha\| \, \|F\| \leqq 1$$

for $F,G \in \Delta(P)$, $\alpha \in \Delta(A)$, it is, conversely, clear that the Gelfand transform of any $\alpha \in \Delta(A)$ is a semicharacter of $\Delta(P)$.

3. Preliminaries on multipliers.

3.1. In this section A is a *commutative semi-simple* Banach algebra and $P=[\Delta(A)]^-$. We denote by A^m the set of functions $f\colon \Delta(A)\to \mathbf{C}$ such that $f\hat{x}\in\hat{A}$ for all $x\in A$. Each $f\in A^m$ defines a unique bounded linear operator $T_f\colon A\to A$ such that $(T_f x)\hat{}=f\hat{x}$. The set $A^m=\{T_f\mid f\in A^m\}$ with the operations induced by pointwise addition and multiplication in A^m is a commutative semi-simple Banach algebra under the uniform operator norm. It is called the *algebra of multipliers* of A. For the basic theory of multiplier algebras we refer to [3]. Since the elements of $\Delta(A)$ are linearly independent, each $f\in A^m$ extends uniquely to a linear functional on $[\Delta(A)]$. If A has a weak bounded approximate identity [4, p. 817] this extension is continuous and determines therefore a unique element $\mathbf{f}$ of P'. Since A is semi-simple, the mapping $x\mapsto\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ denotes the continuous linear functional on P defined by x, is injective. It is clearly norm decreasing. If it is a homeomorphism, A is said to be *topologically embeddable* in P'.

THEOREM 3.1. *If A has a weak bounded approximate identity and is topologically embeddable in P', the mapping $T_f\to\mathbf{f}$ is a topological isomorphism from A^m into P'.*

PROOF. As noted in [4, p. 817], the mapping $T_f\to\mathbf{f}$ is a continuous homomorphism, and obviously injective. Let $C>0$ be such that $\|\hat{\mathbf{x}}\|>C\|x\|$ for all $x\in A$. Then we have

$$\|\mathbf{f}\| \geqq \sup_{x\in A,\|\hat{\mathbf{x}}\|\leqq 1}\|\mathbf{f}_*\hat{\mathbf{x}}\| \geqq \sup_{\|x\|\leqq 1}\|\mathbf{f}_*\hat{\mathbf{x}}\| \geqq C\sup_{\|x\|\leqq 1}\|T_f x\| = C\|T_f\|,$$

and the assertion follows.

3.2. A commutative Banach algebra A is called *regular*, if for any closed set $F\subset\Delta(A)$ and $\alpha\notin F$ there is $x\in A$ such that $\hat{x}\mid F\equiv 0$ and $x(\alpha)=1$. Let $j_A(\infty)$ denote the set of elements of A with Gelfand transforms of compact support. A function $f\colon\Delta(A)\to\mathbf{C}$ *belongs locally* to $\hat{A}$ at $\alpha\in\Delta(A)$, if there is a neighborhood U of α and $x\in A$ such that $f\mid U=\hat{x}\mid U$. Augmented with theorem 3.1. above, the theorem in [4, p. 819], may be rephrased as follows:

THEOREM 3.2. *Suppose that A is regular, topologically embeddable in P', and has an approximate identity $\{u_\delta\}\subset j_A(\infty)$. Then $f\colon\Delta(A)\to\mathbf{C}$ belongs to A^m if and only if it can be extended to a continuous linear functional $\mathbf{f}$ on P and belongs locally to A at each point of $\Delta(A)$. The correspondence $T_f\to\mathbf{f}$ is a topological isomorphism from A^m into P'.*

4. The spectrum of the multiplier algebra.

4.1. Our main concern here is roughly to define a semigroup structure in $\Delta(A^m)$ if one is given in $\Delta(A)$. In view of theorem 3.2 it is reasonable to have first a closer look at the algebra of functions belonging locally to $\hat{A}$. — In this section we assume throughout that

a) *A is a commutative semi-simple Banach algebra,* and

b) *$P=[\Delta(A)]^-$ is a commutative Banach algebra under a product m such that*

c) $m(\Delta(A)\times\Delta(A))\subset\Delta(A)$.

For a function $f: \Delta(A)\to \mathbf{C}$ and $\alpha,\beta\in\Delta(A)$ we define $f^\alpha(\beta)=f(\alpha\beta)$. A subset $\mathscr{F}$ of $\mathbf{C}^{\Delta(A)}$ is *translation invariant*, if $f^\alpha\in\mathscr{F}$ for all $f\in\mathscr{F}$, $\alpha\in\Delta(A)$.

THEOREM 4.1. *The mapping $\xi\mapsto m(\alpha,\xi)$ on P is weak*-continuous for all $\alpha\in\Delta(A)$ if and only if $\hat{A}$ is translation invariant.*

PROOF. When A is regarded (algebraically) as a subspace of P', the adjoint $m^*(\cdot,\alpha)$ of the norm continuous linear operator $\xi\mapsto m(\alpha,\xi)$ maps A into P'. Since $m^*(\hat{x},\alpha)\beta=\hat{x}^\alpha(\beta)$ for all $\beta\in\Delta(A)$, it follows therefore by linearity and continuity that $m^*(\cdot,\alpha)$ maps A into A if and only if $\hat{x}^\alpha\in\hat{A}$ for all $x\in A$. But the condition $m^*(A,\alpha)\subset A$ is equivalent to the $\sigma(P,A)$-continuity of the operator $\xi\mapsto m(\alpha,\xi)$ [11, p. 128], and the theorem is proved.

We assume henceforth that

d) *$\hat{A}$ is translation invariant.*

NOTATION. We denote by B the subspace of P' consisting of those functionals whose restrictions to $\Delta(A)$ belong locally to $\hat{A}$ at each point of $\Delta(A)$. Let B_0 denote the set of restrictions $\xi'\,|\,\Delta(A)$, where $\xi'\in B$.

THEOREM 4.2. *B_0 is translation invariant.*

PROOF. Let $f\in B_0$ and $\alpha\in\Delta(A)$. Since the multiplication in P is norm continuous, it is clear that f^α is the restriction to $\Delta(A)$ of a continuous linear form on P. We show that f^α belongs locally to $\hat{A}$ at $\beta\in\Delta(A)$. There is a neighborhood U of $\alpha\beta$ and $x\in A$ such that $f\,|\,U=\hat{x}\,|\,U$. As the multiplication in $\Delta(A)$ is separately continuous for the Gelfand topology by theorem 4.1, there is a neighborhood V of β such that $\alpha V\subset U$. Then we have for any $\gamma\in V$,

$$f^\alpha(\gamma)=f(\alpha\gamma)=\hat{x}(\alpha\gamma)=\hat{x}^\alpha(\gamma)\,,$$

and the conclusion follows by the translation invariance of $\hat{A}$.

THEOREM 4.3. *If P is Arens regular, so that (P'', m^{***}) is commutative, the annihilator B° of B is a closed ideal of P'' and $\bar{B}' = B' = P''/B^\circ$ is a commutative Banach algebra under the Arens quotient product $\boldsymbol{m}^{***}$ which, moreover, is separately $\sigma(\bar{B}', \bar{B})$-continuous.*

PROOF. By theorem 4.2, $m^*(B, \alpha) \subset B$ for each $\alpha \in \Delta(A)$, and by linearity and continuity, $m^*(B \times P) \subset \bar{B}$. It follows from theorem 2.1 that B° is an ideal and the Arens quotient product, obviously commutative, is defined. Since

$$(F_* G)\xi' = F\boldsymbol{m}^{**}(G, \xi') \quad \text{for } F, G \in \bar{B}',\ \xi' \in \bar{B}\,,$$
$$\boldsymbol{m}^{**}(G, \xi') = m^{**}(\xi'', \xi') \in \bar{B}$$

(theorem 2.1), where $G = \xi'' | P$, the adjoint of the operator $F \mapsto F_* G$ maps $\bar{B}$ into itself, whence the separate $\sigma(\bar{B}', \bar{B})$-continuity of $\boldsymbol{m}^{***}$ [11, p. 128].

4.2. As in theorem 4.3, we generally identify B' and $\bar{B}'$ as Banach spaces (note, however, that the weak* topologies on B' and $\bar{B}'$ are distinct unless B is closed). As $\bar{B}$ is a commutative Banach algebra in the Arens quotient product of P', $\Delta(\bar{B})$ is then identified with the set of *continuous* multiplicative linear functionals on B.

THEOREM 4.4. *If P is Arens regular, then $\Delta(\bar{B}) \cup \{0\}$ is a multiplicative subsemigroup of B'. If the function $e(\alpha) \equiv 1$ belongs to B_0, then $\Delta(\bar{B})$ is a subsemigroup of $\Delta(\bar{B}) \cup \{0\}$.*

PROOF. Let $F, G \in \Delta(\bar{B}) \cup \{0\}$. We show that $\boldsymbol{m}^{***}(F, G) = F_* G$ is multiplicative on B. For $\boldsymbol{f}, \boldsymbol{g} \in B$ and $\alpha, \beta \in \Delta(A)$ we have

$$m^*(\boldsymbol{fg}, \alpha)\beta = \boldsymbol{fg}\, m(\alpha\beta) = \boldsymbol{f} m(\alpha\beta)\, \boldsymbol{g} m(\alpha\beta) = m^*(\boldsymbol{f}, \alpha)\beta\, m^*(\boldsymbol{g}, \alpha)\beta\,.$$

Thus

$$m^*(\boldsymbol{fg}, \alpha) = m^*(\boldsymbol{f}, \alpha) m^*(\boldsymbol{g}, \alpha)\,,$$

where $m^*(\boldsymbol{f}, \alpha)$ and $m^*(\boldsymbol{g}, \alpha)$ belong to B by theorem 4.2. Therefore,

$$\begin{aligned} \boldsymbol{m}^{**}(G, \boldsymbol{fg})\alpha &= Gm^*(\boldsymbol{fg}, \alpha) = Gm^*(\boldsymbol{f}, \alpha)\, Gm^*(\boldsymbol{g}, \alpha) \\ &= \boldsymbol{m}^{**}(G, \boldsymbol{f})\alpha\, \boldsymbol{m}^{**}(G, \boldsymbol{g})\alpha \end{aligned}$$

for all $\alpha \in \Delta(A)$, so that

$$\boldsymbol{m}^{**}(G, \boldsymbol{fg}) = \boldsymbol{m}^{**}(G, \boldsymbol{f})\, \boldsymbol{m}^{**}(G, \boldsymbol{g})\,,$$

where $\boldsymbol{m}^{**}(G, \boldsymbol{f})$ and $\boldsymbol{m}^{**}(G, \boldsymbol{g})$ belong to $\bar{B}$ by theorems 4.2 and 2.1.

Hence we have

$$F_* G(fg) = Fm^{**}(G,fg) = Fm^{**}(G,f)Fm^{**}(G,g) = F_* G(f)F_* G(g) .$$

Finally, if F and G are non-zero and $e \in B_0$, where $e(\alpha) \equiv 1$, then

$$m^{**}(G,e)\alpha = Gm^*(e,\alpha) = G(e) = 1$$

for all $\alpha \in \Delta(A)$. It follows that

$$F_* G(e) = Fm^{**}(G,e) = F(e) = 1 .$$

4.3. In important applications, for example when A is the group algebra of a locally compact abelian group, P has the structure of a C^*-algebra. The next result is concerned with this situation.

THEOREM 4.5. *If, in addition to the assumptions* a)–d), *an involution* $\xi \mapsto \xi^*$ *is defined on* P *such that* P *becomes a* C^**-algebra, then* B' *is a commutative* C^**-algebra with identity* u. *If* $\xi' \in \bar{B}$, *then* $\bar{\xi}' \in \bar{B}$ *where* $\bar{\xi}'(\xi) = \overline{\xi'(\xi^*)}$, *and the involution* $F \mapsto F^*$ *in* $B' = \bar{B}'$ *is given by*

$$F^*(\xi') = \overline{F(\bar{\xi}')} .$$

The involution in B' *is* $\sigma(B',\bar{B})$*-continuous. The subspace* $Q = [\Delta(\bar{B})]$ *is a sub-*C^**-algebra of* B'. *If* P *has an identity, its canonical image in* B' *is* u *and* $u \in Q$.

PROOF. It is shown by Civin and Yood [5, p. 869] that P'' is Arens regular and is in fact isometrically isomorphic to the von Neumann algebra $\mathscr{A}$ enveloping P, i.e. the von Neumann algebra generated by the image of P in the universal representation (cf. [6, p. 236]). In particular, P'' is a commutative C^*-algebra with identity having B° as a closed ideal (theorem 4.3). Hence B° is also self-adjoint, and $P''/B^\circ = B'$ is a commutative C^*-algebra with identity [6, proposition 1.8.2., p. 17]. It is shown by Ramirez [7, p. 1392] that the involution $\xi'' \mapsto (\xi'')^*$ induced by $\mathscr{A}$ on P'' can be defined by

$$(\xi'')^*\xi' = \overline{\xi''(\bar{\xi}')}, \quad \text{where} \quad \bar{\xi}'(\xi) = \overline{\xi'(\xi^*)} .$$

Another way of seeing this is to note that the above involution is weak* continuous and coincides with the original involution of P on the weak* dense subspace P of P'', whereas the weak* topology of P'' corresponds to the weak operator topology of $\mathscr{A}$ [6, p. 237], for which the involution in $\mathscr{A}$ is continuous. To the quotient algebra the involution is transferred as follows: If $F \in B'$, $F^* = (\xi'')^* | B$, where $\xi'' \in P''$ is any extension of F. If we now had $\xi' \notin \bar{B}$ for some $\xi' \notin \bar{B}$, we could find $\xi'' \in B^\circ$ such that $\xi''(\overline{\xi'}) \neq 0$. Then $\xi'' \mid \bar{B} = 0$, and $\overline{\xi''(\overline{\xi'})} = (\xi'' \mid \bar{B})^* \xi' = 0$, which is a contradiction. It follows that the involution can be defined as stated in the theorem. It is then immediate that the involution is $\sigma(B', \bar{B})$-continuous. It follows from theorem 4.4 that Q is a subalgebra of B'. Now let $F \in \Delta(\bar{B})$ and $\xi', \eta' \in \bar{B}$. Then

$$\widetilde{\xi' \eta'}(\alpha) = \overline{\xi' \eta'(\alpha^*)} = \overline{\xi'(\alpha^*)}\,\overline{\eta'(\alpha^*)} = \tilde{\xi}'(\alpha)\tilde{\eta}'(\alpha)$$

for $\alpha \in \Delta(A)$, so that $\widetilde{\xi' \eta'} = \tilde{\xi}' \tilde{\eta}'$. Since

$$F^*(\xi' \eta') = \overline{F(\widetilde{\xi' \eta'})} = \overline{F(\tilde{\xi}' \tilde{\eta}')} = \overline{F(\tilde{\xi}')}\,\overline{F(\tilde{\eta}')} = F^*(\xi')F^*(\eta')\,,$$

$F^* \in \Delta(\bar{B})$, and by the continuity of the involution Q is a self-adjoint subalgebra of B'. Finally, if P has an identity e, a straightforward verification shows that its canonical image $\dot{e}$ in P'' is an identity for P''. Clearly, the natural homomorphism maps $\dot{e}$ onto the uniquely determined identity u of $P''/B^\circ = B'$, and since each $\alpha \in \Delta(A)$ may be regarded as an element of $\Delta(\bar{B})$, $u \in Q$.

4.4. For $\xi \in P = [\Delta(A)]^-$ and $f \in B$ we define

$$\Phi(\xi)f = f(\xi)\,.$$

Then Φ is a norm decreasing injection from P into $Q = [\Delta(\bar{B})]^-$, since $\Phi\alpha \in \Delta(\bar{B})$ for all $\alpha \in \Delta(A)$. The range of Φ is a subalgebra of B' and Φ is a homomorphism onto its range, as is readily seen by the definition of the Arens quotient product, noting that the Arens product in P'' extends the product of P.

THEOREM 4.6. *The adjoint $\Phi^*\colon Q' \to P'$ of Φ is a homomorphism for the Arens quotient products of Q' and P'.*

PROOF. Let $F, G \in Q'$ and $\alpha \in \Delta(A)$. Then $\Phi\alpha \in \Delta(\bar{B})$ and

$$\begin{aligned}\Phi^*(F_* G)\alpha &= (F_* G)\Phi\alpha = F(\Phi\alpha)G(\Phi\alpha)\\ &= \Phi^*(F)\alpha\,\Phi^*(G)\alpha = (\Phi^* F_* \Phi^* G)\alpha\,.\end{aligned}$$

Thus $\Phi^*(F_* G) = \Phi^* F_* \Phi^* G$.

4.5. If the assumptions of theorem 3.2 are fulfilled, then $B_0 = A^m$ and $B = \bar{B}$, which simplifies some notations. We conclude this section with a theorem in which we collect the most important results of the above theory in a specialized situation.

THEOREM 4.7. *Let A be a commutative, regular, semi-simple Banach algebra with an approximate identity $\{u_\delta\} \subset j_A(\infty)$. Suppose that $P = [\Delta(A)]^-$ is a commutative C^*-algebra having $\Delta(A)$ as a multiplicative subsemigroup with identity, such that $\hat{A}$ is translation invariant. Suppose, furthermore, that A is topologically embeddable in P'. Then $Q = [\Delta(A^m)]$ is a commutative C^*-algebra with identity having $\Delta(A^m)$ as a multiplicative subsemigroup. With their respective Gelfand topologies and Arens quotient products, $\Delta(P)$ and $\Delta(Q)$ are compact abelian topological semigroups, and $\Delta(A)$ (resp. $\Delta(A^m)$) may be identified, via the Gelfand transform, with the set of the semicharacters on $\Delta(P)$ (resp. on $\Delta(Q)$). There is a continuous homomorphism Ψ from $\Delta(Q)$ onto $\Delta(P)$ such that the mapping $\Psi' : \Delta(A) \to \Delta(A^m)$ defined by*

$$(\Psi'(\alpha))^\wedge = \hat{\alpha} \circ \Psi$$

is a topological isomorphism onto an open subsemigroup of $\Delta(A^m)$.

PROOF. By virtue of theorem 3.2, A^m may be identified with B (cf. subsection 4.1), which is closed, as A^m is a Banach algebra. By theorem 4.5, Q is a commutative C^*-algebra having $\Delta(A^m)$ as a multiplicative subsemigroup with identity by theorem 4.4 (the Arens regularity of P was noted in the proof of theorem 4.5). Theorem 2.3 then shows that $\Delta(P)$ and $\Delta(Q)$ are compact abelian topological semigroups and by theorem 2.4, $(\Delta(A))^\wedge$ (resp. $(\Delta(A^m))^\wedge$) is precisely the set of semicharacters on $\Delta(P)$ (resp. on $\Delta(Q)$). Consider the mapping Φ introduced in section 4.4. As Φ is a homomorphism and maps the identity of P onto that of Q, $F \circ \Phi$ is a non-zero multiplicative linear functional on P for any $F \in \Delta(Q)$. Thus the adjoint Φ^* of Φ maps $\Delta(Q)$ into $\Delta(P)$. We define $\Psi = \Phi^* | \Delta(Q)$. By theorem 4.6 Ψ is a homomorphism. It is continuous for the weak* topologies of $\Delta(Q)$ and $\Delta(P)$ since Φ^* is so [11, p. 128]. Since Φ is injective, $\Psi(\Delta(Q)) = \Delta(P)$, as may be seen by an argument used in [6, p. 17], in the proof of proposition 1.8.1. If $\alpha \in \Delta(A)$, then $\hat{\alpha} \circ \Psi$ is a continuous function on $\Delta(Q)$, and therefore the Gelfand transform of an element $\Psi'(\alpha)$ of Q. We show that $\Psi' | \Delta(A) = \Phi | \Delta(A)$, whence in particular $\Psi'(\Delta(A)) \subset \Delta(A^m)$ (this inclusion also follows from the fact that $\hat{\alpha} \circ \Phi$ is a semicharacter on $\Delta(Q)$). If $F \in \Delta(Q)$, then

$$F(\Psi'\alpha) = (\hat{\alpha} \circ \Psi)F = \hat{\alpha}(\Psi F) = (\Psi F)\alpha = F(\Phi\alpha) ,$$

and the semi-simplicity of Q implies that Ψ' coincides with Φ on $\Delta(A)$. But it follows from the proof of theorem 1 in [3, p. 204], that $\Phi | \Delta(A)$ is a homeomorphism onto an open subset of $\Delta(A^m)$.

5. Application to harmonic analysis.

5.1. Let G be a locally compact abelian topological group with a fixed Haar measure, denoted by dx, and dual group Γ. If $A = L^1(G)$, $\Delta(A)$ may be identified with Γ and $P = [\Delta(A)]^-$ with AP, the C^*-algebra of the almost periodic functions on G. The Gelfand transform on G is then the Fourier transform:

$$f(\gamma) = \int_G f(x)\overline{(x,\gamma)}\,dx \quad \text{for} \quad f \in L^1(G),\ \gamma \in \Gamma .$$

It is a consequence of Eberlein's theorem [10, p. 32] that $L^1(G)$ is topologically, even isometrically, embeddable in AP'. Also the other assumptions of theorem 4.7 are well-known basic results of harmonic analysis. Since the multiplier algebra of $L^1(G)$ is isometrically isomorphic to the convolution measure algebra $M(G)$ [10, p. 74], theorem 4.7 yields a connection between Γ and $\Delta(M(G))$. In this case $\Delta(P) = \Delta(AP)$ is the Bohr compactification of G.

5.2. Ramirez [7] and Rennison [8] have studied $\Delta(M(G))$ by considering the Arens product in $M(G)' = C_0(G)''$, which makes $M(G)'$ isomorphic to the enveloping von Neumann algebra of $C_0(G)$. In the next theorem we show that the C^*-algebra structure thereby introduced in $M(G)'$ is the same as that which is obtained by interpreting $M(G)$ as the multiplier algebra of $L^1(G)$ and using the techniques of section 4. For $\mu \in M(G)$, denote by $\hat{\mu}$ its Fourier–Stieltjes transform, i.e.

$$\hat{\mu}(\gamma) = \int_G \overline{(x,\gamma)}\,d\mu(x) \quad \text{for} \quad \gamma \in \Gamma ,$$

and by $\hat{\boldsymbol{\mu}}$ the continuous linear functional defined by $\hat{\mu}$ on AP. Then the mapping $\mu \mapsto \hat{\boldsymbol{\mu}}$ is a linear isometry into AP' [10, p. 32]. We denote by B the image of $M(G)$ in AP'.

THEOREM 5.1. *When $M(G)' = C_0(G)''$ is regarded as the enveloping von Neumann algebra of $C_0(G)$ and B' is given the C^*-algebra structure introduced in theorem 4.5, then the adjoint of the operator $\mu \mapsto \hat{\boldsymbol{\mu}}$ is a C^*-algebra isomorphism from B' onto $M(G)'$.*

PROOF. Let φ be the adjoint of the inverse isometry $\hat{\boldsymbol{\mu}} \mapsto \mu$, and denote $E = \varphi(C_0)$, where $C_0 = C_0(G)$ is canonically regarded as a subalgebra of $M(G)'$, whose C^*-algebra structure is defined analogously with that of P'' in the proof of theorem 4.5. Clearly, φ is an isometric vector space isomorphism onto B'. For $f \in C_0$ denote $\dot{f} = \varphi(f)$. We first show that $\varphi | C_0$ is an algebra isomorphism onto E. Denote the pointwise product in AP by m. Then we have for $\alpha, \beta \in \Gamma$ and $\mu \in M = M(G)$,

$$m^*(\boldsymbol{\mu}, \alpha)\beta = \hat{\mu}(\alpha\beta) = \int_G \overline{(x,\alpha)}\, \overline{(x,\beta)}\, d\mu(x) .$$

By the uniqueness of the Fourier-Stieltjes transform [10, p. 29],

$$m^*(\boldsymbol{\mu}, \alpha) = \widehat{\bar{\alpha}\boldsymbol{\mu}} ,$$

where $\bar{\alpha}\mu$ is the measure $f \mapsto \mu(\bar{\alpha} f)$. Therefore, if $g \in C_0$,

$$\boldsymbol{m}^{**}(\dot{g}, \hat{\boldsymbol{\mu}})\alpha = \dot{g} m^*(\hat{\boldsymbol{\mu}}, \alpha) = \bar{\alpha}\mu(g) = \int_G \overline{(x,\alpha)}\, g(x)\, d\mu(x) ,$$

and by the same uniqueness theorem $\boldsymbol{m}^{**}(\dot{g}, \hat{\boldsymbol{\mu}}) = \boldsymbol{g}\hat{\boldsymbol{\mu}}$, where $g\mu$ is the measure $f \mapsto \mu(gf)$. It follows that

$$(\dot{f}_* \dot{g})\hat{\boldsymbol{\mu}} = \dot{f} \boldsymbol{m}^{**}(\dot{g}, \hat{\boldsymbol{\mu}}) = \dot{f}(\widehat{\boldsymbol{g\mu}}) = (g\mu)f = \mu(fg) ,$$

i.e.

(1) $$\varphi(fg) = \varphi f_* \varphi g .$$

As B is closed, the involution in B' is weak* continuous (theorem 4.5), and multiplication separately weak* continuous (theorem 4.3). The corresponding statements are valid for M' (see the proof of theorem 4.5). Since φ is continuous with respect to the weak* topologies of M' and B' [11, p. 128], the mappings

$$S \mapsto \varphi(fS) \quad \text{and} \quad S \mapsto \varphi(f)_* \varphi(S)$$

on M' for fixed $f \in C_0$ are so, too. As they coincide by (1) on the weak* dense subspace C_0 of M',

$$\varphi(fS) = \varphi f_* \varphi S \quad \text{for all} \quad S \in M(G)' .$$

Fixing S and replacing f by a variable $T \in M'$, it is seen by the same argument that φ is an algebra homomorphism. To show that φ transfers the involution $S \mapsto S^*$ in $M(G)'$ to the involution $F \mapsto F^*$ defined in theorem 4.5, we first note that $C = C(G)$, the Banach space of bounded continuous functions on G may be regarded as a subspace of the dual of $M(G)$, and since C_0 is $\sigma(C, M)$-dense in C [11, p. 125], there is, in particular, for each $\gamma \in \Gamma$ a net $\{f_\delta\}$ of elements of C_0 converging to γ with respect to $\sigma(C, M)$. Using the Jordan decomposition of a measure into a linear combination of positive measures and the decomposition of the integrand into its real and imaginary parts for each of the positive measures separately, we infer that the net $\{\bar{f}_\delta\}$ converges to $\bar{\gamma}$ for $\sigma(C,M)$. Therefore,

$$\hat{\tilde{\mu}}(\gamma) = \int_G \overline{(x,\gamma)}\, d\tilde{\mu}(x) = \lim_\delta \int_G \overline{f_\delta(x)}\, d\tilde{\mu}(x)$$

$$= \lim_\delta \overline{\int_G f_\delta(x)\, d\mu(x)} = \overline{\int_G (x,\gamma)\, d\mu(x)} = \overline{\hat{\mu}(\bar{\gamma})}\,,$$

and we have for $\sum \lambda_i \gamma_i \in AP$, $i = 1,2,\ldots,n$,

$$\tilde{\hat{\boldsymbol{\mu}}}(\sum \lambda_i \gamma_i) = \overline{\hat{\boldsymbol{\mu}}(\sum \bar{\lambda}_i \bar{\gamma}_i)} = \sum \lambda_i \overline{\hat{\mu}(\bar{\gamma}_i)} = \sum \lambda_i \hat{\tilde{\mu}}(\gamma_i) = \hat{\tilde{\boldsymbol{\mu}}}(\sum \lambda_i \gamma_i)\,.$$

By continuity, $\tilde{\hat{\boldsymbol{\mu}}} = \hat{\tilde{\boldsymbol{\mu}}}$, so that

$$(\varphi S^*)\hat{\boldsymbol{\mu}} = S^*\mu = \overline{S(\tilde{\mu})} = \overline{\varphi S(\hat{\tilde{\boldsymbol{\mu}}})} = \overline{\varphi S(\tilde{\hat{\boldsymbol{\mu}}})} = (\varphi S)^*\hat{\boldsymbol{\mu}}\,,$$

that is

$$\varphi S^* = (\varphi S)^*\,.$$

We have proved that φ is a C^*-algebra isomorphism, and so is its inverse as stated in the theorem.

COROLLARY. *If $A = L^1(G)$ in theorem 4.7, then $\Delta(Q)$ is (topologically isomorphic to) the Taylor structure semigroup of $M(G)$.*

PROOF. The corollary follows from the above theorem and theorem 6.5 in [8].

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