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HARMONIC ANALYSIS AND REAL GROUP ALGEBRAS

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1. Introduction.

The main objects of study in abstract harmonic analysis are the group algebras $L_{\mathsf{C}}^1(G)$ and $M_{\mathsf{C}}(G)$ consisting respectively of all complex Haar-integrable functions on the locally compact abelian group G and all bounded regular Borel measures on G. Much less attention has been paid to the real counterparts of these algebras: the real subalgebras $L_{\mathsf{R}}^1(G)$ and $M_{\mathsf{R}}(G)$ consisting of the real-valued functions in $L_{\mathsf{C}}^1(G)$ and the real-valued measures in $M_{\mathsf{C}}(G)$ respectively. These two real group algebras acquire the additional structure of an *ordered* algebra over R — via the concepts of a positive function and a positive measure. The building blocks of these ordered algebras will accordingly be the *convex* ideals since these are exactly the kernels of order-preserving homomorphisms. This gives rise to the following general question: What can be said about the convex ideals of $L_{\mathsf{R}}^1(G)$ and $M_{\mathsf{R}}(G)$ in comparison with well-known results from the ideal theory of $L_{\mathsf{C}}^1(G)$ and $M_{\mathsf{C}}(G)$?

The object of the present paper is essentially to give an answer to the above mentioned question in the case of the algebra $L_{\mathsf{R}}^1(G)$. We showed in [1] that there is only one convex ideal among the maximal ideals of $L_{\mathsf{R}}^1(G)$; namely, the kernel of the Haar-measure (consisting of all functions with zero integral). (Another proof of this along more general lines was given later by G. Maltese in [6].) We also showed in [1] that an intersection of regular maximal ideals in $L_{\mathsf{R}}^1(G)$ is convex if and only if it is contained in the kernel of the Haar measure. But since we cannot rely on the validity of spectral synthesis in L_{R}^1 this does not necessarily take care of all the closed ideals in $L_{\mathsf{R}}^1(G)$. The main purpose of the present paper is to prove in full generality that a proper closed ideal in $L_{\mathsf{R}}^1(G)$ is convex if and only if it is contained in the kernel of the Haar-measure. This shows that the "ordered" versions of spectral analysis (Wiener–Tauberian theorem) and spectral synthesis in $L_{\mathsf{R}}^1(G)$ are of a rather trivial nature because of the scarcity of convex ideals in this

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algebra. In the proof of the main theorem (Theorem 2) it turns out that a crucial role is played by the positivity of certain convolution products. We also offer an entirely different and elementary approach to this question in the case of compactly generated abelian groups. By means of Theorem 2 the determination of the w^*-closed convex translation invariant subspaces of L_{R}^{∞} is quite easy. We also treat the commutativity of a certain diagram which naturally arises in this connection. Finally we give some remarks which seem to indicate that there are no easy counterparts to theorems 1 and 2 in the case of $M_{\mathsf{R}}(G)$.

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2. Convex maximal ideals in $L_{\mathsf{R}}^1(G)$.

We shall first give the relevant definitions. By $L_{\mathsf{R}}^1(G)$ — or simply L_{R}^1 — we denote the ordered group algebra of all real-valued integrable functions on a locally compact abelian group G under the ordering $f \geqq g$ whenever $f(x) \geqq g(x)$ almost everywhere on G. By L_{C}^1 we denote the usual group algebra of all complex-valued integrable functions on G. We recall that in ideal $\mathfrak{A}$ in a commutative ring R is called *regular* whenever $R/\mathfrak{A}$ has an identity, in the following we shall always assume that $\mathfrak{A}$ is a proper ideal; hence L_{R}^1 will not be considered as an ideal in L_{R}^1. The ideal $\mathfrak{A} \subset L_{\mathsf{R}}^1$ is said to be *convex* if $f, g \in \mathfrak{A}$ and $f \geqq h \geqq g$ implies $h \in \mathfrak{A}$. When dealing with maximal ideals one should carefully distinguish between the following two statements.

A. $\mathfrak{A}$ is a maximal ideal having the property P.

B. $\mathfrak{A}$ is maximal among those ideals having the property P.

It is clear that $A \Rightarrow B$. By using Zorn's lemma it is trivial that $A \Leftrightarrow B$ in case P stands for "regular". It is a consequence of Theorem 2 that $A \Leftrightarrow B$ even if P stands for "closed and convex".

The solution to the problem of finding all maximal ideals in L_{R}^1 which are regular and convex is given by the following

THEOREM 1. *The only regular and convex maximal ideal in L_{R}^1 is the maximal ideal consisting of all functions in L_{R}^1 with zero integral. Otherwise expressed: If μ is an order preserving homomorphism of L_{R}^1 onto a partially ordered field F then F is isomorphic to the field of real numbers and μ is the Haar measure of G.*

For proofs of this theorem we refer to [1] and [6].

If $\mathfrak{M}_\alpha$ denotes the maximal ideal in L_{C}^1 which corresponds to the character $\alpha \in \hat{G}$ we have the following

COROLLARY. *The following statements are equivalent.*
(i) $\mathfrak{M}_\alpha$ *is the kernel of the Haar measure.*
(ii) $\mathfrak{M}_\alpha \cap L_{\mathsf{R}}^1$ *is convex.*
(iii) $\mathfrak{M}_\alpha$ *does not contain any strictly positive function.*

If G is connected (i) *is also equivalent to the following three statements:*
(iv) $\mathfrak{M}_\alpha \cap L_{\mathsf{R}}^1$ *is of real codimension one in* L_{R}^1.
(v) $\hat{f}(\alpha)$ *is real for all* $f \in L_{\mathsf{R}}^1$.
(vi) $\mathfrak{M}_\alpha \cap L_{\mathsf{R}}^1 = \mathfrak{M}_\beta \cap L_{\mathsf{R}}^1 \Rightarrow \alpha = \beta$.

PROOF. The equivalence of (i) and (ii) is the content of Theorem 1. The equivalence of (i) and (iii) is obvious from the proof of Theorem 1 in [1]. From the Gelfand–Mazur theorem it follows that $L_{\mathsf{R}}^1/\mathfrak{M}_\alpha \cap L_{\mathsf{R}}^1$ is isomorphic to either R or C. This quotient algebra is isomorphic to R if and only if α is a real-valued character which means that α is the identity character if G is connected. From this, together with

$$(\mathfrak{M}_\alpha \cap L_{\mathsf{R}}^1 = \mathfrak{M}_\beta \cap L_{\mathsf{R}}^1) \Leftrightarrow (\alpha = \pm \beta)\,, \tag{2.1}$$

we easily deduce the latter part of the corollary. The equivalence (2.1) expresses that a maximal ideal in L_{R}^1 can be extended to a maximal ideal in L_{C}^1 in at most two ways and that the extension is unique if and only if the given maximal ideal in L_{R}^1 corresponds to a real-valued character (which means that $\alpha = -\alpha$). For more general information about the relationship between real and complex Banach algebras we can refer the reader to [4] and [5].

3. Convex closed ideals in $L_{\mathsf{R}}^1(G)$.

We shall now prove that any closed convex ideal in L_{R}^1 is contained in

$$\mathfrak{M}_0{}^{\mathsf{R}} = \left\{ f \mid f \in L_{\mathsf{R}}^1 \ \& \ \int f(x)\,dx = 0 \right\}.$$

The essential step in the proof is the following lemma which may have some independent interest (see also the remarks at the end of this section).

LEMMA 1. *Let $f \in L_{\mathsf{R}}^1$ with $\int f(x)dx \neq 0$. Then there exists a $g \in L_{\mathsf{R}}^1$ such that $f * g > 0$.*

PROOF. Since $\int f(x)dx \neq 0$ we have $\hat{f}(\alpha) \neq 0$ for all α in a certain compact neighborhood K of the identity element in $\hat{G}$. Let $\hat{p}$ be a non-zero positive definite function on $\hat{G}$ with support contained in K. The function p defined by

$$p(x) = \int_{\hat{G}} \hat{p}(\alpha)\,(x,\alpha)\,d\alpha$$

will (by Bochner's theorem) be a non-zero positive function in $L_R{}^1$. By an extension of a well-known theorem of Wiener (see Godement [2, Théorème A]) we can further determine a function $F \in L_C{}^1$ such that

$$\hat{F}(\alpha) = \hat{f}(\alpha)^{-1} \tag{3.1}$$

for all $\alpha \in K$. We now put $F*p = g+ih$ and get

$$[f*(g+ih)]\hat{} = \hat{f}\hat{F}\hat{p}\,. \tag{3.2}$$

Inserting (3.1) in (3.2) we obtain $\hat{f}\hat{F}\hat{p}=\hat{p}$ on K and since $\hat{p}(\alpha)=0$ for $\alpha \notin K$ this shows that $\hat{f}\hat{F}\hat{p}=\hat{p}$ holds for all $\alpha \in \hat{G}$. By Fourier inversion we thus have

$$f*(g+ih) = f*g + i(f*h) = p\,. \tag{3.3}$$

Since f, g, h and p all belong to $L_R{}^1$ this gives the desired result

$$f*g = p > 0\,.$$

We are now ready to prove

THEOREM 2. *Any proper closed convex ideal in $L_R{}^1(G)$ is contained in the kernel of the Haar measure.*

PROOF. The proof is a repetition of the last part of the proof of Theorem 1 in [1]: Assume that $\mathfrak{A}$ is a proper closed convex ideal which is not contained in the kernel of the Haar measure. The ideal $\mathfrak{A}$ must then contain a function f such that $\int f(x)dx \neq 0$. By Lemma 1, $\mathfrak{A}$ must therefore also contain a non-zero positive function p. Being translation invariant, $\mathfrak{A}$ will further contain a positive function h which is $>\varepsilon>0$ on a neighborhood of the zero-element o of G. Now, nh also belongs to $\mathfrak{A}$ for any positive integer n and we can choose for any sufficiently small neighborhood U of o a function h_U such that

$$0 < h_U < nh \quad \text{on } U$$

for a suitable n and such that the h_U's constitute an approximate identity for $L_R{}^1$, that is,

$$\lim_U (h_U * f) = f \quad \text{for any } f \in L_{\mathsf{R}}^1 .$$

Since $\mathfrak{A}$ is supposed to be convex we have $h_U \in \mathfrak{A}$ and since $\mathfrak{A}$ is closed

$$f = \lim_U (h_U * f) \in \mathfrak{A} \quad \text{for all } f \in L_{\mathsf{R}}^1$$

contradicting that $\mathfrak{A}$ is proper.

For later reference we give the following

COROLLARY 1. *A closed ideal $\mathfrak{A}$ is convex if and only if $\mathfrak{A}$ does not contain a strictly positive function.*

If we formulate Theorem 2 in terms of homomorphisms we get the following

COROLLARY 2. *Any order-preserving ring-homomorphism of L_{R}^1 onto a partially ordered ring T is a factor in the canonical order-preserving homomorphism of L_{R}^1 onto* R.

COROLLARY 3. *Spectral analysis holds for closed convex ideals in L_{R}^1 while spectral synthesis does not hold.*

Corollary 3 is valid since the converse of Theorem 2 is obviously true: Any ideal contained in $\mathfrak{M}_0^{\mathsf{R}}$ is convex.

REMARK. The result of Lemma 1 may be considered as a contribution to the following general type of problem which has been treated in several special instances: If A, B and C are three classes of functions in L_{C}^1, what can then be said about the "size" of the set $(A * B) \cap C$? (Here $A * B$ denotes the set of all convolution products $f * g$ where $f \in A$ and $g \in B$.) If we put $A = \complement \mathfrak{M}_0^{\mathsf{R}}$ (the complement of $\mathfrak{M}_0^{\mathsf{R}}$ in L_{R}^1), $B = L_{\mathsf{R}}^1$, and

$$C = (L_{\mathsf{R}}^1)^+ = \{f \mid f \in L_{\mathsf{R}}^1 \text{ and } f > 0\}$$

Lemma 1 says that $(A * B) \cap C \neq \emptyset$.

4. The positivity of certain convolution products on compactly generated abelian groups.

The scarcity of closed convex ideals in L_{R}^1 was shown to be mainly due to the existence of certain positive convolution products on G. Though the proof of Lemma 1 was not difficult, it used a couple of fairly deep-lying results of harmonic analysis. We shall in this section show that in certain cases we can establish the existence of the pertinent positive convolution products in a quite elementary way. In fact if we restrict the given function f in Lemma 1 to have compact support we can obtain an everywhere strictly positive integrable function by convoluting f with a function which is "almost constant" — in a sense which will be made precise below. This, however, raises the question as to which groups G possess such almost constant integrable functions as well as which closed convex ideals possess functions with compact support and non-vanishing integral.

DEFINITION. When ε is a strictly positive real number we shall say that a nowhere vanishing function f on G is *almost constant of type* (ε, K) if

$$1-\varepsilon < \frac{f(x_1)}{f(x_2)} < 1+\varepsilon$$

whenever $(x_1 - x_2) \in K$, where K is a compact subset of G. We shall say that G *possesses integrable almost constant functions* if for any given ε and K there exists a (positive) integrable almost constant function of type (ε, K) on G.

LEMMA 2. *Any compactly generated abelian group G possesses integrable almost constant functions.*

PROOF. By the structure theorem for compactly generated abelian groups (see [3, p. 90]) any such group may be written as $G = \mathsf{R}^n \times \mathsf{Z}^m \times F$ where R denotes the reals, Z the integers, F a compact group, and m and n are non-negative integers. This essentially reduces the question to proving the lemma for the three groups R, Z and F, which indeed is quite easy. For a compact F we can just take any strictly positive constant function. If K is a compact subset of R and $\varepsilon > 0$, the function f defined on R by

$$f(x) = e^{-\delta |x|/a} \tag{4.1}$$

is an integrable almost constant function of type (ε, K) if the positive real numbers a and δ are chosen such that $K \subset [-a, a]$ and $e^{\delta} < 1+\varepsilon$. In fact if $|x_2 - x_1| \leqq a$ then also $||x_2| - |x_1|| \leqq a$ and

$$1-\varepsilon < \frac{1}{1+\varepsilon} < e^{-\delta} \leqq \frac{f(x_1)}{f(x_2)} \leqq e^{\delta} < 1+\varepsilon .$$

The restriction of the function (4.1) to Z will give an integrable almost constant function on Z which is of type (ε, K) if a is again chosen such that $K \subset [-a,a]$. Such a choice is obviously possible since K is finite, being a compact subset of a discrete group.

Roughly speaking we therefore only have to show how two integrable almost constant functions on the groups G and H, respectively, can be used to define an integrable almost constant function on $G \times H$. Assume therefore that g and h are two integrable almost constant functions on G and H and are of type (ε_1, K_1) and (ε_2, K_2) respectively. Without loss of generality we can assume that $\varepsilon_1, \varepsilon_2 < 1$. Then define f as a function on $G \times H$ by putting

$$f(x,y) = g(x)\,h(y)\,.$$

It is clear that f is an integrable, everywhere positive function on $G \times H$ such that

$$(1-\varepsilon_1)(1-\varepsilon_2) < \frac{f(x_1,y_1)}{f(x_2,y_2)} < (1+\varepsilon_1)(1+\varepsilon_2)$$

whenever $(x_1-x_2, y_1-y_2) \in K_1 \times K_2$. In order to produce an integrable almost constant function f on $G \times H$ of type (ε, K) it is therefore enough to choose ε_1, ε_2, K_1 and K_2 such that $K \subset K_1 \times K_2$ and $(1+\varepsilon_1)(1+\varepsilon_2) < 1+\varepsilon$. This completes the proof of Lemma 2.

Since an almost constant function is nowhere equal to zero it is clear that a group must in any case be σ-compact in order to possess integrable almost constant functions. We do not know, however, whether the existence of integrable almost constant function characterizes the class of compactly generated groups — or may be the class of σ-compact groups. In any case we have the following

Lemma 1B. *Let G be any locally compact abelian group which possesses integrable almost constant functions. If f is a function in $L_{\mathbf{R}}^1(G)$ with compact support and non-vanishing integral, then there exists a function $g \in L_{\mathbf{R}}^1(G)$ such that $f * g > 0$.*

Proof. In contradistinction to the more general result of Lemma 1 we are here able to give an entirely elementary proof. Let K denote the support of f. We can suppose without loss of generality that

$$\int_G f(x)\,dx = \int_K f(x)\,dx = 1 \quad \text{and} \quad \int_K |f(x)|\,dx = \alpha\,.$$

Let g be an integrable almost constant function of type (ε, K). Then

$$f * g(x) = \int_K g(x-y)\,f(y)\,dy = g(x) - \int_K (g(x)-g(x-y))\,f(y)\,dy$$

which gives

$$f*g(x) \geqq g(x) - \int_K |g(x)-g(x-y)|\,|f(y)|\,dy$$

or

$$f*g(x) \geqq g(x) - \int_K g(x)\left|1-\frac{g(x-y)}{g(x)}\right|\,|f(y)|\,dy$$
$$\geqq g(x) - g(x)\varepsilon\alpha\,.$$

By choosing g of type (ε, K) with $\varepsilon < 1/\alpha$ we therefore get the desired inequality $f*g>0$.

Lemma 1B is more restrictive than Lemma 1 in two ways: We have imposed conditions both on G and on f. The condition on f would be inessential if it turned out that any closed ideal which is not contained in the kernel of the Haar measure contains a function with compact support and non-vanishing integral. (It can be shown that this is not true in the case $G=\mathsf{R}$.) Then the above Lemma 1B gives a new proof for Theorem 2 in the case of groups which possess integrable almost constant functions. Whether a closed ideal $\mathfrak{M}_0{}^{\mathsf{R}}$ contains a function with compact support and non-vanishing integral is again a question concerning the "size" of $(A*B)\cap C$. If we can prove that

$$(A*B)\cap C \neq \emptyset$$

with

$$A = \mathfrak{A}-\mathfrak{M}_0{}^{\mathsf{R}}, \quad B = L_{\mathsf{R}}{}^1, \quad C = L_{00}^1-\mathfrak{M}_0{}^{\mathsf{R}}$$

this would prove that $\mathfrak{A}$ contains a function of the desired type.

5. Convex translation invariant subspaces of $L_{\mathsf{R}}{}^{\infty}$.

We denote by $L_{\mathsf{R}}{}^{\infty}$ the real dual of $L_{\mathsf{R}}{}^1$, consisting of all bounded measurable real-valued functions on G. Thus $L_{\mathsf{R}}{}^{\infty}$ is nothing else than the family of all the real-valued functions in the usual complex L^{∞}-space which we shall here denote by $L_{\mathsf{C}}{}^{\infty}$. It is a well-known fact that there is a one-to-one correspondence between the closed ideals in $L_{\mathsf{C}}{}^1$ and the w^*-closed translation invariant subspaces of $L_{\mathsf{C}}{}^{\infty}$ (see [7, p. 184]). The same correspondence persists between the real spaces $L_{\mathsf{R}}{}^1$ and $L_{\mathsf{R}}{}^{\infty}$. By means of this correspondence we shall easily describe the convex w^*-closed translation invariant subspaces of $L_{\mathsf{R}}{}^{\infty}$.

If $\mathfrak{A}$ is a closed ideal in $L_{\mathsf{C}}{}^1$ we put

$$\mathfrak{A}^{\perp\mathsf{C}} = \{g \mid g\in L_{\mathsf{C}}{}^{\infty} \text{ and } g*f=0 \text{ for all } f\in\mathfrak{A}\}\,.$$

Similarly if $\mathfrak{A}$ is a closed ideal in L_R^1 we put

$$\mathfrak{A}^{\perp\mathrm{R}} = \{g \mid g \in L_\mathrm{R}^\infty \text{ and } g * f = 0 \text{ for all } f \in \mathfrak{A}\}\,.$$

The correspondence between closed ideals in L_R^1 and w^*-closed translation invariant subspaces of L_R^∞ is then given by the mapping $\mathfrak{A} \to \mathfrak{A}^{\perp\mathrm{R}}$.

Lemma 2. *$\mathfrak{A}$ contains a strictly positive function if and only if $\mathfrak{A}^{\perp\mathrm{R}}$ does not contain a strictly positive function.*

Proof. If $\mathfrak{A}$ contains a strictly positive function it is clear that $\mathfrak{A}^{\perp\mathrm{R}}$ can not contain such a function since $\mathfrak{A} * \mathfrak{A}^{\perp\mathrm{R}} = \{0\}$. Conversely if $\mathfrak{A}$ does not contain a strictly positive function we know from Corollary 1 of Theorem 2 that $\mathfrak{A} \subset \mathfrak{M}_0{}^\mathrm{R}$ and thus $1 \in \mathfrak{A}^{\perp\mathrm{R}}$ proving that $\mathfrak{A}^{\perp\mathrm{R}}$ contains a strictly positive function.

Proposition 1. *$\mathfrak{A}^{\perp\mathrm{R}}$ is convex if and only if $\mathfrak{A}$ is not convex.* ($\mathfrak{A} \neq 0$.)

Proof. If $\mathfrak{A}$ is convex then $\mathfrak{A} \subset \mathfrak{M}_0$ and $1 \in \mathfrak{A}^{\perp\mathrm{R}}$ showing that $\mathfrak{A}^{\perp\mathrm{R}}$ is not convex. Conversely if $\mathfrak{A}$ is not convex then $\mathfrak{A}$ contains a strictly positive function and hence by Lemma 2, $\mathfrak{A}^{\perp\mathrm{R}}$ does not contain a strictly positive function. This means that two functions in $\mathfrak{A}^{\perp\mathrm{R}}$ cannot be comparable without being identical and hence $\mathfrak{A}^{\perp\mathrm{R}}$ is convex.

Corollary. *A w^*-closed translation invariant subspace of L_R^∞ is convex if and only if it annihilates a strictly positive function.*

Let $\mathscr{I}_\mathrm{C}$ and $\mathscr{I}_\mathrm{R}$ denote the families of closed ideals in L_C^1 and L_R^1 respectively and let $\mathscr{S}_\mathrm{C}$ and $\mathscr{S}_\mathrm{R}$ denote the families of w^*-closed translation invariant subspaces of L_C^∞ and L_R^∞ respectively. We define the mappings

$$\varphi\colon \mathscr{I}_\mathrm{C} \to \mathscr{I}_\mathrm{R} \quad \text{and} \quad \psi\colon \mathscr{S}_\mathrm{C} \to \mathscr{S}_\mathrm{R}$$

by

$$\varphi(\mathfrak{A}) = \mathfrak{A} \cap L_\mathrm{R}^1 \quad \text{and} \quad \psi(\mathfrak{A}^{\perp\mathrm{C}}) = \mathfrak{A}^{\perp\mathrm{C}} \cap L_\mathrm{R}^\infty\,.$$

The question arises whether the following diagram is commutative or not:

$$D: \quad \begin{array}{ccc} \mathscr{I}_\mathrm{C} & \xrightarrow{\perp\mathrm{C}} & \mathscr{S}_\mathrm{C} \\ \varphi\downarrow & & \downarrow\psi \\ \mathscr{I}_\mathrm{R} & \xrightarrow{\perp\mathrm{R}} & \mathscr{S}_\mathrm{R} \end{array}$$

If $\psi(\mathfrak{A}^{\perp_C}) = (\varphi(\mathfrak{A}))^{\perp_R}$ we shall say that *D is commutative for* $\mathfrak{A}$. It is clear that φ is a surjection since

$$\varphi(\mathfrak{B}) = \mathfrak{A} \quad \text{when} \quad \mathfrak{A} \subset L_R^1$$

and

$$\mathfrak{B} = \mathfrak{A} + i\mathfrak{A} = \{f_1 + if_2 \mid f_1, f_2 \in \mathfrak{A}\}.$$

This ideal $\mathfrak{B}$ is the unique minimal closed ideal in L_C^1 such that $\varphi(\mathfrak{B}) = \mathfrak{A}$. These ideals $\mathfrak{A} + i\mathfrak{A}$ which are in one-to-one correspondence with the closed ideals in L_R^1 will be called *symmetric ideals* in L_C^1.

PROPOSITION 2. *The diagram D is commutative for $\mathfrak{A}$ if and only if $\mathfrak{A}$ is a symmetric ideal.*

PROOF. If $\mathfrak{B} = \mathfrak{A} + i\mathfrak{A}$ with $\mathfrak{A} \in \mathscr{I}_R$, then

$$(\varphi(\mathfrak{B}))^{\perp_R} = \{g \mid g \in L_R^\infty \text{ and } g * f = 0 \text{ for all } f \in \mathfrak{A}\}$$

and this set is evidently the same as $\psi(\mathfrak{B}^{\perp_C})$. Assume conversely that $\mathfrak{B}$ is not symmetric, i.e., that $\mathfrak{B}$ *properly* contains $\varphi(\mathfrak{B}) + i\varphi(\mathfrak{B})$. Due to the fact that $\perp_C$ is one-to-one there exists a $g \in L_C^\infty$ such that $g * f = 0$ for all $f \in \varphi(\mathfrak{B}) + i\varphi(\mathfrak{B})$ but not for all $f \in \mathfrak{B}$. Writing $g = g_1 + ig_2$ we get

$$(g_1 + ig_2) * f = g_1 * f + i(g_2 * f) = 0 \quad \text{for all } f \in \varphi(\mathfrak{B}).$$

Thus $g_1 * f = g_2 * f = 0$ for all $f \in \varphi(\mathfrak{B})$. On the other hand both g_1 and g_2 cannot annihilate $\mathfrak{B}$ since $g = g_1 + ig_2$ would then do the same. This proves that either g_1 or g_2 will annihilate $\varphi(\mathfrak{B})$ without annihilating $\mathfrak{B}$. This implies that the diagram D is not commutative for $\mathfrak{B}$.

In order to determine more specifically the maximal ideals for which D is commutative it is convenient to have the following

LEMMA 3. *The ideal $\mathfrak{A}_\alpha = \mathfrak{M}_\alpha \cap \mathfrak{M}_{-\alpha}$ consists of all functions $f = f_1 + if_2$ such that $\hat{f}_1(\alpha) = \hat{f}_2(\alpha) = 0$. Otherwise expressed:*

$$\mathfrak{A}_\alpha = \varphi(\mathfrak{M}_\alpha) + i\varphi(\mathfrak{M}_\alpha) \quad (= \varphi(\mathfrak{M}_{-\alpha}) + i\varphi(\mathfrak{M}_{-\alpha})).$$

PROOF. If $\hat{f}_1(\alpha) = \hat{f}_2(\alpha) = 0$ then also $\hat{f}(\alpha) = 0$ and $f \in \mathfrak{M}_\alpha$. Since $f_1, f_2 \in \varphi(\mathfrak{M}_\alpha) = \varphi(\mathfrak{M}_{-\alpha})$ we also have $\hat{f}_1(-\alpha) = \hat{f}_2(-\alpha) = 0$ and hence $\hat{f}(-\alpha) = 0$. Thus $f \in \mathfrak{M}_\alpha \cap \mathfrak{M}_{-\alpha}$. Assume conversely that $h = h_1 + ih_2 \in \mathfrak{A}_\alpha$, that is,

$$\int h(x)\,\overline{(x,\alpha)}\,dx = \int h(x)\,(x,\alpha)\,dx = 0. \tag{5.1}$$

By adding and subtracting the two left hand terms of (5.1) we get

$$(5.2) \qquad \int h(x)\,(\overline{(x,\alpha)}+(x,\alpha))\,dx = \int h(x)\,(\overline{(x,\alpha)}-(x,\alpha))\,dx = 0\,.$$

Since $\overline{(x,\alpha)}+(x,\alpha)$ and $\overline{(x,\alpha)}-(x,\alpha)$ are real and purely imaginary respectively, (5.2) is also valid when substituting h_1 or h_2 for h. Adding up the two expressions on the left hand side of (5.2) with h_1 instead of h we get $\hat{h}_1(\alpha)=0$. Similarly $\hat{h}_2(\alpha)=0$.

PROPOSITION 3. *The diagram D is commutative for a maximal ideal* $\mathfrak{M}_\alpha$ *if and only if* α *is a real-valued character.*

PROOF. According to Proposition 2 and Lemma 3 D is commutative for $\mathfrak{M}_\alpha$ if and only if $\mathfrak{A}_\alpha=\mathfrak{M}_\alpha$ or equivalently if and only if $\mathfrak{M}_\alpha=\mathfrak{M}_{-\alpha}$. But $\alpha=-\alpha$ means that α is real-valued (i.e., α assumes only the values ± 1).

Since the identity character is the only continuous real character in case G is connected we get the following

COROLLARY. *If G is connected the diagram D is commutative for* $\mathfrak{M}_\alpha$ *if and only if* $\mathfrak{M}_\alpha$ *is the kernel of the Haar measure.*

6. Convex ideals in $M_{\mathsf{R}}(G)$.

It is well known that the ideal theory of $M_{\mathsf{C}}(G)$ is quite a bit of a mystery. Even the maximal ideals of $M_{\mathsf{C}}(G)$ have not been described in a satisfactory way. It seems that one has a similar increasing complexity when passing from $L_{\mathsf{R}}^1(G)$ to $M_{\mathsf{R}}(G)$. What corresponds to the kernel of the Haar measure in case of $M_{\mathsf{R}}(G)$ is the convex ideal $\mathfrak{M}$ consisting of all real measures μ with total mass equal to zero:

$$\mathfrak{M} = \{\mu \mid \mu(G)=\textstyle\int_G d\mu=0\}\,.$$

But $\mathfrak{M}$ is not the only convex maximal ideal in $\mathfrak{M}_{\mathsf{R}}(G)$. In fact, using the Lebesgue decomposition

$$\mu = \mu_d+\mu_s+\mu_a$$

where μ_d is discrete, μ_s is singular and μ_a is absolutely continuous, the set $\{\mu \mid \mu_d(G)=0\}$ is a convex maximal ideal in $M_{\mathsf{R}}(G)$ which in general is different from $\mathfrak{M}$. We therefore have no immediate counterpart to

Theorem 1 in the case of $M_{\mathsf{R}}(G)$. It is also easy on the basis of the Lebesgue decomposition to exhibit several closed non-maximal ideals of $M_{\mathsf{R}}(G)$ which are convex and not contained in $\mathfrak{M}$.

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