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IDEALS IN A C^*-ALGEBRA

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Introduction.

The aim of this work is to continue the investigations of non-closed ideals in C^*-algebras begun in [6]. In particular we shall, for a C^*-algebra A, study the minimal dense ideal K_A, introduced in [9], and the "inductive limit" topology of K_A given in [10]. Specializing in section 3 to C^*-algebras with continuous trace we explore the non-commutative Gelfand transformation $\hat{}: K_A \to K(\hat{A})$ given by $\hat{x}(\pi) = \mathrm{tr}\,\pi(x)$.

In [9, Theorem 1.5] the first author incorrectly stated that if A has continuous trace, then K_A consists of the elements $x \in A$ such that the dimension of $\pi(x)$ is finite and bounded for $\pi \in \hat{A}$, and $\pi(x) = 0$ for π outside some compact set in $\hat{A}$. Since K_A is minimal dense it is true that K_A is contained in the latter set, but we show by a counterexample that the inclusion may be proper, using homogeneous algebras whose corresponding fibre bundles have sufficiently many twists. Theorem 1.5 of [9] is cited in [12], but fortunately only the valid half of the theorem is needed for the conclusions.

It is a pleasure to thank H. Rischel, who provided and explained to us the building blocks of the above mentioned example.

We use the standard notation and terminology from [4]. Throughout the paper A will denote a C^*-algebra.

1. Order ideals.

In [6] E. G. Effros set up a bijective correspondence between closed order ideals of A^+ and closed left ideals of A. The extension of this correspondence to non-closed order ideals was considered in [9], and the distinction between invariant and strongly invariant order ideals was clarified in [12]. The following theorem gives the complete extension of the Effros correspondence.

THEOREM 1.1. *Let J be an order ideal of A^+ and define*

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$$I = I(J) = \{x \in A \mid x^*x \in J\}\,.$$

The map $J \to I(J)$ is a bijection between order ideals of A^+ and left ideals of A satisfying:

1) *If $x \in I$ and $\{u_n\}$ is a bounded sequence in A such that* $\lim u_n x \in A$, *then* $\lim u_n x \in I$.
2) *For any finite set $\{x_n\} \subset I$ there exists $x \in I$ such that $\sum x_n{}^* x_n = x^* x$.*

Furthermore:

J is invariant $\Leftrightarrow$ $I(J)$ is a two-sided ideal.

J is strongly invariant $\Leftrightarrow$ $I(J)$ is self-adjoint $\Leftrightarrow$ $I(J)$ is positively generated.

Proof. Let $\{u_n\}$ be a sequence in the unit ball of A and assume $x \in I(J)$, $\lim u_n x = y \in A$. Then

$$y^*y = \lim x^* u_n{}^* u_n x \leqq x^*x \in J\,,$$

hence $y \in I(J)$, so $I(J)$ satisfies 1). That $I(J)$ satisfies 2) as well is immediate from the definition.

Suppose now that I is a left ideal of A satisfying 1) and 2), and define

$$J = \{x^*x \in A^+ \mid x \in I\}\,.$$

Then J is a cone in A^+ since I is a linear space and satisfies 2). If $0 \leqq y \leqq x^*x \in J$, we put $u_n = y^{\frac{1}{2}}(n^{-1} + x^*x)^{-1}x^*$. We have $\|u_n\| \leqq 1$, and since $(n^{-1} + x^*x)^{-1}x^*x$ is an approximative unit for the hereditary (=order-related [9], =facial [2]) C^*-subalgebra generated by x^*x, we have $\lim u_n x = y^{\frac{1}{2}} \in A$, hence $y^{\frac{1}{2}} \in I$ and $y \in J$ by condition 1). This shows that J is an order ideal, hence the correspondence, which is clearly injective, is a surjection on left ideals satisfying 1) and 2).

That J is invariant iff $I(J)$ is a two-sided ideal follows from the expression

$$(xy)^*(xy) = y^*(x^*x)y\,.$$

The equivalence between J strongly invariant and $I(J)$ self-adjoint is seen by comparing

$$x \in I(J) \Leftrightarrow x^*x \in J\,,$$

$$x^* \in I(J) \Leftrightarrow xx^* \in J\,.$$

Clearly $I(J)$ is self-adjoint if it is positively generated. Conversely, if $I(J)$ *is* self-adjoint and $x = x^* \in I(J)$, we have

$$x = x_+ - x_- \quad \text{with} \quad x_+^2 + x_-^2 = x^2 \in J ,$$

hence $x_+, x_- \in I(J)$ and $I(J)$ is positively generated.

REMARK. If A is imbedded in a von Neumann algebra B, then condition 1) is equivalent with

1′) $$x \in I,\ u \in B,\ ux \in A \quad \Rightarrow \quad ux \in I .$$

An order ideal J of A^+ has roots if $x \in J$ implies $x^\alpha \in J$ for $\alpha = \frac{1}{2}$ and hence for all $\alpha \in \mathsf{R}$. An invariant order ideal with roots is strongly invariant. Since the property of having roots is equivalently expressed by $J = I(J)^+$, we have the following

COROLLARY 1.2. *An order ideal J is invariant and has roots iff* $I(J) = \operatorname{lin} J$.

An ideal generated by an invariant order ideal with roots is called *algebraic*. To give an idea of what an algebraic ideal can be like, consider the C^*-algebra $C_0(\mathsf{R})$ and the ideal consisting of those elements x such that $px \in C_0(\mathsf{R})$ for any polynomial p. As the next results show, the class of algebraic ideals has properties very similar to those of the class of closed ideals.

PROPOSITION 1.3. *The class of algebraic ideals is a distributive lattice under sum and intersection.*

PROOF. By [12, Theorem 1.6] the class of strongly invariant order ideals is a distributive lattice under sum and intersection. To show that the invariant order ideals with roots form a sublattice assume that J_1 and J_2 have roots. Clearly, then $J_1 \cap J_2$ has roots. If $x = y_1 + y_2$, $y_i \in J_i$, define

$$u_{in} = (n^{-1} + x^{\frac{1}{2}})^{-1} y_i^{\frac{1}{2}} .$$

Then

$$(n^{-1} + x^{\frac{1}{2}})^{-1}\, x\, (n^{-1} + x^{\frac{1}{2}})^{-1} = (u_{1n} y_1^{\frac{1}{2}})(u_{1n} y_1^{\frac{1}{2}})^* + (u_{2n} y_2^{\frac{1}{2}})(u_{2n} y_2^{\frac{1}{2}})^* .$$

We have $\|u_{in}\| \leq 1$, and since $(n^{-1} + x^{\frac{1}{2}})^{-1} x^{\frac{1}{2}}$ is an approximative unit for the hereditary C^*-subalgebra generated by x we have $u_{in} y_i^{\frac{1}{2}}$ convergent to $z_i \in A$. By condition 1) in theorem 1.1, $z_i \in I(J_i)$ hence

$$x^{\frac{1}{2}} = z_1 z_1^* + z_2 z_2^* \in J_1 + J_2$$

and we have shown that $J_1 + J_2$ has roots. The proposition now follows from the expressions

$$\operatorname{lin} J_1 + \operatorname{lin} J_2 = \operatorname{lin}(J_1+J_2)\,,$$
$$I(J_1) \cap I(J_2) = I(J_1 \cap J_2)\,.$$

PROPOSITION 1.4. *If I_1 and I_2 are algebraic ideals, then*

$$I_1 I_2 = I_1 \cap I_2\,.$$

PROOF. Put $J_i = I_i{}^+$. Then, since J_i has roots,

$$I_1 I_2 \subseteq I_1 \cap I_2 = \operatorname{lin}(J_1 \cap J_2) \subseteq \operatorname{lin}(J_1 J_2) \subseteq \operatorname{lin} J_1 \operatorname{lin} J_2 = I_1 I_2\,.$$

COROLLARY 1.5. *If I is algebraic, then $I = I^n$.*

COROLLARY 1.6. *If I_1 is an algebraic ideal of an algebraic ideal I_2 of A, then I_1 is an algebraic ideal of A.*

PROOF. The only non-trivial thing to check is that I_1 is an ideal of A. This follows from corollary 1.5 (cf. the implication 1.5.8 $\Rightarrow$ 1.8.5 in [4]).

2. On the minimal dense ideal.

Let K_A be the intersection of all dense, hereditary two-sided ideals of A. Then K_A is a dense algebraic ideal of A. (Cf. [9], [10], [11], [12].)

PROPOSITION 2.1. *If B and C are C^*-subalgebras of A contained in K_A, then the hereditary C^*-subalgebra generated by B and C is also contained in K_A.*

PROOF. It suffices to prove that the closed order ideal J generated by B^+ and C^+ is contained in $K_A{}^+$. Since the order ideal generated by B^+ and C^+ is dense in J by [6, Theorem 2.5], there exists for any $x \in J$ a sequence $\{x_n\} \subset A$ converging to x such that

$$x_n \leqq \alpha_n y_n + \beta_n z_n\,,$$
$$\alpha_n, \beta_n \in \mathsf{R}^+, \quad y_n \in B^+, \quad z_n \in C^+, \quad \|y_n\| = \|z_n\| = 1\,.$$

Define $y = \sum 2^{-n} y_n$, $z = \sum 2^{-n} z_n$. Then $y \in B^+$, $z \in C^+$, hence $y+z \in K_A{}^+$ and x is contained in the closed order ideal generated by $y+z$. By [11, Proposition 4] this order ideal is contained in $K_A{}^+$.

PROPOSITION 2.2. *Let A and B be C^*-algebras and let $\Phi\colon K_A \to B$ be a morphism. Then Φ extends canonically to a morphism $\tilde{\Phi}\colon A \to B$, and if $C = \tilde{\Phi}(A)$, then $\Phi(K_A) = K_C$.*

PROOF. Since Φ is norm continuous on every C^*-subalgebra of K_A by [4, Proposition 1.3.7], we have $\|\Phi\| \leqq 1$ on K_A. Hence Φ extends to a morphism $\tilde{\Phi}$ of A. The last statement follows from [11, Corollary 6].

COROLLARY 2.3. *If K_A and K_B are isomorphic as involutive algebras, then A and B are isomorphic C^*-algebras.*

In [10, Theorem 2.1] a vector-space topology τ was defined on K_A, such that in the commutative case τ was the usual inductive limit topology on functions with compact supports. It was proved in [12, Theorem 2.4] that τ is the weakest locally convex topology on K_A for which all invariant convex functionals on $K_A{}^+$ are continuous.

LEMMA 2.4. *Let $\Phi: K_A \to K_B$ be a linear positive and surjective map such that*

1) $\forall x \in K_A \, \exists y \in K_B$:
$$\Phi(x^*x) = y^*y, \quad \Phi(xx^*) = yy^* ,$$
2) $\forall a \in K_A{}^+ \, \forall y \in K_B \, \exists x \in K_A$:
$$y^*y \leqq \Phi(a) \Rightarrow x^*x \leqq a, \quad \Phi(xx^*) = yy^* .$$

Then Φ is open and continuous in the respective τ-topologies.

PROOF. For any invariant convex functional ϱ on $K_B{}^+$ the composite map $\varrho \circ \Phi$ is an invariant convex functional on $K_A{}^+$ by 1), hence Φ is continuous. If ϱ is any invariant convex functional on $K_A{}^+$, define

$$\sigma(b) = \inf \{\varrho(a) \mid \Phi(a) = b\} .$$

Then clearly σ is a convex functional on $K_B{}^+$. If $y,z \in K_B$ and $y^*y \leqq z^*z$, then for any $\varepsilon > 0$ there exists $a \in K_A{}^+$ such that

$$\sigma(z^*z) + \varepsilon > \varrho(a), \quad \Phi(a) = z^*z .$$

There exists $x \in K_A$ satisfying 2), hence

$$\sigma(z^*z) + \varepsilon > \varrho(x^*x) = \varrho(xx^*) \geqq \sigma(yy^*) .$$

Since ε is arbitrary this proves that σ is invariant.

Finally we have by definition of σ that

$$\Phi\{a \in K_A{}^+ \mid \varrho(a) < 1\} = \{b \in K_B{}^+ \mid \sigma(b) < 1\} .$$

This proves that Φ is open.

THEOREM 2.5. *Let $\Phi: A \to B$ be a surjective morphism. Then the restriction of Φ to K_A is an open and continuous map onto K_B in the respective τ-topologies.*

PROOF. That Φ satisfies 1) is obvious, and 2) follows from [2, Lemme 4.1].

3. For C^*-algebras with continuous trace.

Throughout this section we shall assume that A is a C^*-algebra with continuous trace. In contrast to CCR-algebras and algebras of type I, it need not be true that C^*-subalgebras of A have continuous trace. However the following result holds.

PROPOSITION 3.1. *If B is a hereditary C^*-subalgebra of A, then B has continuous trace.*

PROOF. By [10, Theorem 1.6] any irreducible representation of B is the restriction of some irreducible representation (π,H) of A to the subspace $\pi(B)H$, and this restriction map induces a homeomorphism between $\hat{B}$ and $\hat{A} \setminus \operatorname{hull} B$. It follows that an element $x \in B$ has bounded and continuous trace on $\hat{B}$ iff x as an element of A has bounded and continuous trace on $\hat{A}$.

PROPOSITION 3.2. *If I is a closed two-sided ideal of A, then I and A/I have continuous trace.*

PROOF. The first statement is a corollary of proposition 3.1. To prove the second let $\Phi: A \to A/I$ be the natural morphism. By [4, Proposition 3.2.1] any irreducible representation of A/I arises from an irreducible representation (π,H) of A for which $\pi(I)=0$, and since this induces a homeomorphism between $(A/I)^\wedge$ and hull I, we see that if $x \in A$ has bounded and continuous trace on $\hat{A}$, then $\Phi(x)$ has bounded and continuous trace on $(A/I)^\wedge$.

We define the map $\hat{}: K_A \to K(\hat{A})$ by

$$\hat{x}(\pi) = \operatorname{tr}\pi(x), \quad x \in K_A,\ \pi \in \hat{A} .$$

Since K_A is minimal dense, this is well defined and by [3, Lemme 23] it is a positive, linear and surjective map. When A is commutative, $\hat{}$ is the Gelfand transformation, and even though $\hat{}$ is not multiplicative in the non-commutative case, it seems to be the best substitute we can get. By the Dauns–Hofmann theorem [5, p. 379] there exists for any $x \in A$ and $f \in C^b(\hat{A})$ an element in A denoted $f \cdot x$ such that $f(\pi)x - f \cdot x \in \ker \pi$ for all $\pi \in \hat{A}$. It is immediate that we have

$$(f \cdot x)\hat{} = f\hat{x}, \quad f \in C^b(\hat{A}), \ x \in K_A .$$

With the correspondence in mind between invariant C^*-integrals on A and Radon measures on $\hat{A}$, proved in [3, Théorème 1], it is only natural that we have

THEOREM 3.3. *The map* $\hat{}: K_A \to K(\hat{A})$ *is open and continuous in the respective* τ*-topologies.*

PROOF. That $\hat{}$ satisfies condition 1) of lemma 2.4 is trivial. To prove that $\hat{}$ satisfies condition 2) as well, assume $a \in K_A{}^+$, $f \in K(\hat{A})^+$, $f \leqq \hat{a}$. Define $x_n = (n^{-1} + \hat{a})^{-1} f \cdot a$. Then $\{x_n\}$ converges to an element $x \leqq a$ such that $\hat{x} = f$. Hence lemma 2.4 applies and the theorem follows.

Clearly $\hat{}$ is not injective although $x \geqq 0$ and $\hat{x} = 0$ imply $x = 0$. Thus $\hat{}$ divides $K_A{}^+$ in equivalence classes, where $x, y \in K_A{}^+$ are equivalent when $\hat{x} = \hat{y}$. By the Riesz decomposition for C^*-algebras [12, Proposition 1.1] we may define another equivalence relation on A^+ by putting $x \sim y$ if there exists a finite set $\{z_n\} \subset A$ such that

$$x \in \sum z_n{}^* z_n, \quad y = \sum z_n z_n{}^* .$$

In terms of these definitions the following theorem can now be stated:

THEOREM 3.4. *For* $x, y \in K_A{}^+$ *the following conditions are equivalent*:

(1) $$\hat{x} = \hat{y} ,$$

(2) $$x \sim y .$$

PROOF. (2) $\Rightarrow$ (1) is obvious. Since $x + y \in K_A{}^+$ there exist by definition two finite sets $\{a_m\}$ and $\{b_m\}$ in A^+ such that

$$x + y \leqq \sum a_m, \quad [a_m] \leqq b_m .$$

Put $b = \sum b_m$ and let I be the closed two-sided ideal of A generated by b. Then I has continuous trace by proposition 3.2, and $x, y \in K_I{}^+$. Since each set

$$\mathscr{F}_n = \{\pi \in \hat{I} \mid \|\pi(b)\| \geqq n^{-1}\}$$

is compact by [4, Proposition 3.3.7] and since $\cup \mathscr{F}_n = \hat{I}$, the proof of (1) $\Rightarrow$ (2) is reduced to the case where $\hat{A}$ is σ-compact.

For any $\pi_0 \in \hat{A}$ there exists by [4, Proposition 4.5.3] an element $e \in A^+$ such that $\pi(e)$ is a one-dimensional projection in a neighbourhood $\mathcal{O}$ of π_0. From the way e is constructed in [4, Lemme 4.4.2] we see that one may assume $e \in K_A{}^+$. Let J denote the strongly invariant order ideal in A^+ generated by e and let I denote the closed two-sided ideal of A generated by e. Then J consists of the elements $a \in A^+$ such that there exist $b \in A^+$ and $\alpha \geqq 0$ with

$$a \sim b, \quad b \leqq \alpha e ,$$

while I can be described as the elements $a \in A$ such that for any $\pi \in \hat{A}$

$$\pi(e) = 0 \Rightarrow \pi(a) = 0 .$$

We claim that for any $a \in K_A{}^+$ and any positive function $f \in C^b(\hat{A})$ with support in $\mathcal{O}$ we have

$$(*) \qquad f\cdot a \in J .$$

Since the elements in A^+ satisfying (*) constitute an order ideal it is enough to prove the formula for $a \in K_0{}^+$, hence we may assume $[a] \leqq b$ for some $b \in A^+$. Since $\hat{e}=1$ on the support of f, this gives

$$[f\cdot a] \leqq \hat{e}\cdot b .$$

Now $\hat{e}\cdot b \in I$ hence $f\cdot a \in K_I{}^+$ by definition, and since J is dense in I^+, we have $K_I{}^+ \subset J$ and the claim is established.

Since $\hat{A}$ is locally compact and σ-compact, it is paracompact and normal, hence the covering by sets of the form $\mathcal{O}$ has a locally finite refinement $\{\mathcal{U}_i\}$ and there exists a partition of unity $\{f_i\}$ subordinate to the covering $\{\mathcal{U}_i\}$ (see [8]). For each i we select $e_i \in K_A{}^+$ such that $\pi(e_i)$ is a one-dimensional projection for $\pi \in \mathcal{U}_i$.

Now consider the pair $x,y \in K_A{}^+$. By (*) there exist for each i elements $a_i, b_i \in A^+$ and constants α_i, β_i such that

$$f_i\cdot x \sim a_i, \quad f_i\cdot y \sim b_i ,$$
$$a_i \leqq \alpha_i e_i, \quad b_i \leqq \beta_i e_i .$$

Since $\hat{x}=\hat{y}$, we have $\hat{a}_i=\hat{b}_i$, but as $\dim \pi(e_i)=1$ for any $\pi \in \mathcal{U}_i$, this implies $a_i=b_i$, hence $f_i\cdot x \sim f_i\cdot y$. Since the covering is locally finite and $\hat{x}$ has compact support, only finitely many elements $f_i\cdot x$ and $f_i\cdot y$ are non-zero, hence

$$x = \sum f_i\cdot x \sim \sum f_i\cdot y = y .$$

It is interesting to compare the above theorem with the result of Dixmier which one can read out of [3, Lemme 21], namely that if $x,y \in A^+$

have bounded and continuous trace on $\hat{A}$, then $\hat{x}=\hat{y}$ iff there exists a sequence $\{z_i\}\subset A$ such that $x=\sum z_i^* z_i$, $y=\sum z_i z_i^*$. Clearly, it is the minimality condition on K_A which allows us to pass from a convergent sequence to a finite number of elements when $x,y\in K_A{}^+$.

We shall finally give an example of a C^*-algebra with continuous trace and homogeneous of degree two. By [7, Theorem 3.2] such an algebra is isomorphic with the family of continuous cross-sections of a suitable fibre bundle which we shall first describe. Let P^n denote the complex n-dimensional projective space, i.e. the set of one-dimensional subspaces $\pi\subset\mathsf{C}^{n+1}$. The total space of the bundle is the set

$$E = \{(a,b,\boldsymbol{c},\boldsymbol{d},\pi)\in \mathsf{C}\times\mathsf{C}\times\mathsf{C}^{n+1}\times\mathsf{C}^{n+1}\times\mathsf{P}^n \mid \boldsymbol{c}\in\pi,\ \boldsymbol{d}^*\in\pi\}\,,$$

where $\boldsymbol{d}^*$ means complex conjugation at each coordinate. The base space is P^n and the projection $p\colon E\to\mathsf{P}^n$ is projection on the last coordinate. Clearly, then for any $\pi\in\mathsf{P}^n$ the fibre $p^{-1}(\pi)$ is homeomorphic with C^4, hence with M_2, and if $\mathscr{V}_j$ is the open subset of points of P^n whose jth homogeneous coordinates are non-zero, then $p^{-1}(\mathscr{V}_j)$ is homeomorphic with $\mathsf{M}_2\times\mathscr{V}_j$. It follows that the system $\mathscr{B}_n=(E,\mathsf{P}^n,\mathsf{M}_2,p)$ is a fibre bundle, where the bundle group is the subgroup of automorphisms of M_2 induced by inner transformations by the unitaries of the form

$$u_\Theta = \begin{pmatrix}\exp i\Theta & 0\\ 0 & \exp -i\Theta\end{pmatrix}.$$

We shall write the elements of E in the form

$$e = \begin{pmatrix} a & \boldsymbol{c}\\ \boldsymbol{d} & b\end{pmatrix},\ \pi$$

and we can then restore the matrix operations in the fibre over each π by the definitions

$$ee' = \begin{pmatrix} aa'+\boldsymbol{c}\cdot\boldsymbol{d}' & a\boldsymbol{c}'+\boldsymbol{c}b'\\ \boldsymbol{d}a'+b\boldsymbol{d}' & \boldsymbol{d}\cdot\boldsymbol{c}'+bb'\end{pmatrix},\ \pi;\qquad e^* = \begin{pmatrix} a^* & \boldsymbol{d}^*\\ \boldsymbol{c}^* & b^*\end{pmatrix},\ \pi\,.$$

If we choose a unit vector $\boldsymbol{v}$ in π, then e is represented by the matrix $\boldsymbol{e}\in\mathsf{M}_2$, where

$$\boldsymbol{e} = \begin{pmatrix} a & c\\ d & b\end{pmatrix};\qquad e = \begin{pmatrix} a & c\boldsymbol{v}\\ d\boldsymbol{v}^* & b\end{pmatrix},\ \pi\,.$$

Since any other representative of e is of the form $u_\Theta{}^*\boldsymbol{e}u_\Theta$, the norm of $\boldsymbol{e}$ depends only on e and is denoted $\|e\|$.

Now let A_n be the set of continuous cross-sections of $\mathscr{B}_n$, that is, the set of continuous maps $x: \mathsf{P}^n \to E$ such that $p(x(\pi))=\pi$ for all $\pi \in \mathsf{P}^n$. With the definitions

$$xy(\pi) = x(\pi)y(\pi), \quad x^*(\pi) = x(\pi)^*, \quad \|x\| = \sup \|x(\pi)\|,$$

A_n becomes a C^*-algebra. By [7, Theorem 3.2] A_n is homogeneous of degree two and $\hat{A}_n = \mathsf{P}^n$.

We define elements $x_n, y_n \in A_n^+$ by

$$x_n(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \pi; \quad y_n(\pi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \pi,$$

and claim the following

Lemma 3.5. *If* $z_i \in A_n$, $i=1,2,\ldots,k$, *and* $\sum z_i z_i^* = x_n$, $\sum z_i^* z_i = y_n$, *then* $k > n$.

Proof. Clearly we have

$$z_i(\pi) = \begin{pmatrix} 0 & c_i(\pi) \\ 0 & 0 \end{pmatrix}, \pi,$$

where each c_i is continuous and $\sum |c_i|^2 = 1$. If S^{2n+1} is identified with the set

$$\{a = (a_0, a_1, \ldots, a_n) \in \mathsf{C}^{n+1} \mid \sum |a_j|^2 = 1\}$$

and $\Psi: \mathsf{S}^{2n+1} \to \mathsf{P}^n$ is given by $\Psi(a) = \mathsf{C}a$, then each composite map $c_i \circ \Psi: \mathsf{S}^{2n+1} \to \mathsf{C}^{n+1}$ satisfies $c_i(\Psi(a)) \in \Psi(a)$. It follows that there exist continuous functions $\Phi_i: \mathsf{S}^{2n+1} \to \mathsf{C}$ such that $c_i(\Psi(a)) = \Phi_i(a)a$. Since $\sum |\Phi_i(a)|^2 = 1$, this defines a map $\Phi: \mathsf{S}^{2n+1} \to \mathsf{S}^{2k-1}$. However $\Psi(a) = \Psi(-a)$ and hence $\Phi(a) = -\Phi(-a)$. By the Borsuk–Ulam theorem [13, Corollary 5.8.8.] this implies $2k-1 \geqq 2n+1$.

Now define $A_0 = \sum_0^{\oplus} A_n$ (see [4, 1.9.14]). Then A_0 is homogeneous of degree two and $\hat{A}_0 = \bigcup \mathsf{P}^n$. Since A_0 is an ideal in the full direct sum of the A_n, and since $q = \sum x_n$ is a projection in this sum, we infer from [4, Corollaire 1.8.4] that the set $A = A_0 + \mathsf{C}q$ is a C^*-algebra. It is immediately verified that $\hat{A} = \bigcup \mathsf{P}^n \cup \{\pi_\infty\}$, where $\pi_\infty(A) = A/A_0 = \mathsf{C}$ and $\hat{A}$ is a one-point compactification of $\hat{A}_0$. Clearly A has continuous trace.

Proposition 3.6. *There exists a C^*-algebra A with continuous trace for which K_A is properly contained in the set of elements $x \in A$ such that*

$$\sup \dim \pi(x) < \infty \quad \textit{and} \quad \hat{x} \in K(\hat{A}).$$

PROOF. Take A as above and consider the set J consisting of elements $y \in A^+$ such that there exist a finite set $\{z_i\} \subset A$ and $\alpha \in \mathsf{R}^+$ satisfying

$$y = \sum z_i^* z_i, \quad \sum z_i z_i^* \leqq \alpha q .$$

By definition J is the strongly invariant order ideal generated by q. Since q as a projection belongs to K_A, we have $J \subset K_A{}^+$. However, q is not contained in any closed two-sided ideal of A, since $\pi(q) \neq 0$ for each $\pi \in \hat{A}$. Hence J is dense in A^+ and so $J = K_A{}^+$.

Now consider $y = \sum n^{-1} y_n$. Then $\dim \pi(y) \leqq 1$ for any $\pi \in \hat{A}$, and $\hat{y}$ is clearly continuous, hence $\hat{y} \in K(\hat{A})$ since $\hat{A}$ is compact. However, by lemma 3.5. there cannot exist any finite set $\{z_i\} \subset A$ such that $y = \sum z_i^* z_i$, $\sum z_i z_i^* \leqq \alpha q$. Hence $y \notin K_A$ and the proposition follows.

Apart from disproving [9, Theorem 1.5] (cf. the introduction) the above proposition also provides a negative answer to the problem raised in [4, 4.7.24].

If $\tilde{A}_0$ denotes the C^*-algebra obtained by adjoining an identity to A_0, then trivially $K_{\tilde{A}_0} = \tilde{A}_0$. This gives an example of a C^*-algebra which is CCR and has a Hausdorff spectrum, but for which theorem 3.4 is false.

Furthermore if A'' denotes the enveloping von Neumann algebra of A and I is the smallest ideal of A'' containing K_A, introduced in [1, Section 3], our example answers to the negative the question whether one always has $I \cap A = K_A$. To see this we observe that the projections $x_n, y_n \in A_n$ are equivalent in A_n'' since they are abelian and have central support 1. Hence there exists $v_n \in A_n''$ such that $x_n = v_n v_n^*$, $y_n = v_n^* v_n$. Define $v = \sum n^{-\frac{1}{2}} v_n \in A''$. Then $v^* v = y$ and $v v^* \leqq q$, hence $y \in I \cap A \setminus K_A$.

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