REMARKS ON DIOPHANTINE APPROXIMATION IN FUNCTION FIELDS

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Abstract

We study some problems in metric Diophantine approximation over local fields of positive characteristic.

1. Introduction

This short paper is motivated by a recent work of Kristensen, Pedersen and Weiss [14]. In it, the authors consider certain interesting Diophantine problems arising from K. R. Yu's generalisation [25] to higher dimensions, of Mahler's classification [18] of complex numbers according to their Diophantine properties. In this paper, which can be thought of as a sequel to our earlier work [8] and the paper [9] of the second named author, we consider function field analogues of their results. We recall Mahler's classification briefly for the reader, see [2] for details.

For $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ consider the Diophantine exponent

$$\omega_k(x) = \sup\{\omega \in \mathbb{R} : |P(x)| \le \mathrm{H}(P)^{-\omega} \text{ for infinitely many}$$
$$\text{irreducible } P \in Z[X], \ \deg(P) \le k\}.$$

Here $\mathrm{H}(P)$ denotes the naïve height of the polynomial P which is defined in §2. Let

$$\omega(x) = \varlimsup_{k \to \infty} \frac{\omega_k(x)}{k}.$$

Then, according to Mahler, there are four classes of real numbers:

- x is an *A-number* if $\omega(x) = 0$;
- x is an *S-number* if $0 < \omega(x) < \infty$;
- x is a *T-number* if $\omega(x) = \infty$, but $\omega_k(x) < \infty$ for all k;
- x is a *U-number* if $\omega(x) = \infty$ and $\omega_k(x) = \infty$ for all k large enough.

Mahler proved that almost all real numbers are *S*-numbers and conjectured [17] that in fact, $\omega_k(x) = k$ for almost every $x \in \mathbb{R}$. This conjecture was proven by Sprindhžuk [23], [24]. It is equivalent to showing that almost every point on the curve $(x, x^2, \ldots, x^n)$ is not very well approximable, i.e. for almost every $x \in \mathbb{R}$ and for every $\varepsilon > 0$ there are at most finitely many $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$ such that

$$|q_1 x + q_2 x^2 + \cdots + q_n x^n + p| \leq \|\mathbf{q}\|^{-n(1+\varepsilon)} \quad \text{for some } p \in \mathbb{Z},$$

where $\|\mathbf{q}\|$ denotes the supremum norm of $\mathbf{q}$. Sprindhžuk's proof of Mahler's conjecture initiated the subject of "metric Diophantine approximation with dependent quantities" which has seen dramatic progress since then and received an important boost with the proof of the Baker-Sprindhžuk conjectures by D. Kleinbock and G. A. Margulis [12] using methods from homogeneous dynamics.

The setting of this present paper involves Diophantine approximation over local fields of positive characteristic, i.e. Laurent series over a finite base field. The notation and pertinent definitions are introduced in the next section. Diophantine approximation in this setting has been extensively studied beginning with Mahler [19], [20]. Mahler's classification was undertaken in positive characteristic by Bundschuh [3]. We refer the reader also to [7] and to [22] for recent developments in analogues of Mahler's and Koksma's classification in positive characteristic. As regards Mahler's conjectures and their analogues, Sprindhžuk formulated and settled the function field analogue of Mahler's conjectures. The function field analogue of the Baker-Sprindhžuk conjectures were settled by the second named author in [9].

In [25], K. R. Yu developed a higher dimensional analogue of Mahler's classification. As far as we are aware, this higher dimensional classification has not been investigated in the setting of function fields although it seems a natural and interesting problem. In [14], Kristensen, Pedersen and Weiss use Yu's classification as a starting point to study various problems arising in metric Diophantine approximation with dependent quantities. For instance, they introduce notions of *k*-very well approximable and *k*-Dirichlet improvable vectors and show that they are contained in a set of zero Lebesgue measure. In fact their results, as well as ours, are more general; details are in subsequent sections. In this paper, we develop function field analogues of some of their results.

We now briefly introduce the setting of Diophantine approximation in function fields. The subsequent sections are devoted, respectively, to the notion of *k*-Dirichlet improvable vectors and *k*-very well approximable vectors. Dirichlet improvable vectors and very well approximable vectors have been studied

in the function field setting in a previous paper of the authors [8] and in a paper [9] of the second named author respectively. Following the ideas in [14], we explain how one can use these previously established results to study the refined notion of k-Dirichlet improvable and k-very well approximable vectors. We also use the occasion to record a result which characterises Dirichlet improvable Laurent series, which shows a striking contrast with the case of real numbers.

Finally, we should mention that [14] also has a nice section devoted to badly approximable vectors which is based on the beautiful work of Beresnevich [1] on badly approximable vectors on manifolds. Since there are as yet no function field analogues of these results, we have nothing to say about it.

1.1. The set-up

Let p be a prime and $q := p^r$, where $r \in \mathbb{N}$, and consider the function field $\mathbb{F}_q(T)$. We define a function $|\cdot|: \mathbb{F}_q(T) \to \mathbb{R}_{\geq 0}$ as follows:

$$|0| := 0 \quad \text{and} \quad \left| \frac{P}{Q} \right| := e^{\deg P - \deg Q} \quad \text{for all non-zero } P, Q \in \mathbb{F}_q[T].$$

Clearly $|\cdot|$ is a non-trivial, non-archimedian and discrete absolute value in $\mathbb{F}_q(T)$. This absolute value gives rise to a metric on $\mathbb{F}_q(T)$.

The completion field of $\mathbb{F}_q(T)$ is $\mathbb{F}_q((T^{-1}))$, i.e. the field of Laurent series over $\mathbb{F}_q$. The absolute value of $\mathbb{F}_q((T^{-1}))$, which we again denote by $|\cdot|$, is given as follows. Let $a \in \mathbb{F}_q((T^{-1}))$. For $a = 0$, define $|a| = 0$. If $a \neq 0$, then we can write

$$a = \sum_{k \leq k_0} a_k T^k, \quad \text{where } k_0 \in \mathbb{Z}, \ a_k \in \mathbb{F}_q \ \text{and} \ a_{k_0} \neq 0.$$

We define k_0 as the *degree* of a, which will be denoted by $\deg a$, and $|a| := e^{\deg a}$. This clearly extends the absolute value $|\cdot|$ of $\mathbb{F}_q(T)$ to $\mathbb{F}_q((T^{-1}))$ and moreover, the extension remains non-archimedian and discrete. Let Λ and F denote $\mathbb{F}_q[T]$ and $\mathbb{F}_q((T^{-1}))$ respectively from now on. It is obvious that Λ is discrete in F. For any $d \in \mathbb{N}$, F^d is throughout assumed to be equipped with the supremum norm which is defined as follows

$$\|\mathbf{x}\| := \max_{1 \leq i \leq n} |x_i| \quad \text{for all } \mathbf{x} = (x_1, x_2, \ldots, x_d) \in F^d,$$

and with the topology induced by this norm. Clearly Λ^n is discrete in F^n. Since the topology on F^n considered here is the usual product topology on F^n, it follows that F^n is locally compact as F is locally compact. Let λ be the Haar measure on F^d which takes the value 1 on the closed unit ball $\|\mathbf{x}\| = 1$.

Diophantine approximation in the positive characteristic setting consists of approximating elements in F by 'rational' elements, i.e. those from $\mathbb{F}_q(T)$. As we have mentioned before, this subject has been extensively studied, beginning with Mahler who developed Minkowski's geometry of numbers in function fields and continuing with Sprindžuk who, in addition to proving the analogue of Mahler's conjectures, also proved some transference principles in the function field setting. The subject has also received considerable attention in recent times; we refer the reader to [21], [15], [16] and other works.

2. k-Dirichlet improvability

Throughout we let $N = \binom{k+d}{d} - 1$ be the number of non-constant monomials in $\Lambda[X_1, \ldots, X_d]$ of total degree $\leq k$, where $k \in \mathbb{N}$. For $P \in \Lambda[X_1, \ldots, X_d]$, let $\mathrm{H}(P)$ and $\widetilde{\mathrm{H}}(P)$ be the maximum absolute value among all the coefficients of P and the maximum absolute value of the coefficients of the non-constant terms of P respectively.

We first prove the following analogue of the Dirichlet's theorem in this set up.

THEOREM 2.1. *For given* $\mathbf{x} = (x_1, \ldots, x_d) \in F^d$, *there exists* $c(\mathbf{x})$ *such that for all* $m \in \mathbb{N}$, *one has* $P \in \Lambda[X_1, \ldots, X_d]$ *of total degree* $\leq k$ *and* $\mathrm{H}(P) \leq e^m$ *satisfying*

$$|P(\mathbf{x})| < \frac{c(\mathbf{x})}{e^{mN}}.$$

We can choose $c(\mathbf{x}) = 1$ *if the condition* $\mathrm{H}(P) \leq e^m$ *is replaced by* $\widetilde{\mathrm{H}}(P) \leq e^m$.

The above theorem indeed almost follows from Theorem 2.1 of [8]. In fact, it is just a restatement of [8, Theorem 2.1] if we consider $\widetilde{\mathrm{H}}(P)$. To prove the theorem for $\mathrm{H}(P)$, we shall actually prove the the following which yields Theorem 2.1 at once.

THEOREM 2.2. *Assume* $n \in \mathbb{N}$, $\mathbf{y} := (y_1, y_2, \ldots, y_n) \in F^n$ *and* $k := \max\{0, \deg y_1, \ldots, \deg y_n\}$. *For any* $m \in \mathbb{N}$ *there exist* $(q_1, \ldots, q_n, p) \in \Lambda^{n+1} \setminus \{0\}$ *such that*

$$|y_1q_1 + y_2q_2 + \cdots + y_nq_n - p| < \frac{e^{nk}}{e^{nm}} \quad and \quad |p|, \max_{1 \leq j \leq n} |q_j| \leq e^m. \quad (2.1)$$

PROOF. If $k = 0$ then the conclusion follows immediately from [8, Theorem 2.1]. Suppose now $k > 0$. Consider $m > k$. By [8, Theorem 2.1], we have $(q_1, \ldots, q_n) \in \Lambda^n \setminus \{0\}$ and $p \in \Lambda$ such that

$$|y_1q_1 + y_2q_2 + \cdots + y_nq_n - p| < \frac{1}{e^{n(m-k)}} \quad and \quad \max_{1 \leq j \leq n} |q_j| \leq e^{m-k}.$$

It is easy to see that $|p| \le e^m$ as $\deg(y_1 q_1 + y_2 q_2 + \cdots + y_n q_n) \le (m-k)+k = m$. So (2.1) holds in this case. When $m = k$, take $q_1 = 1$, $q_2 = \cdots = q_n = 0$ and p to be the polynomial part of $q_1 y_1 = y_1$ which has degree $\le k = m$. Thus (2.1) is obvious. We are now left with the case $m < k$. Choose $q_1 = q_2 = \cdots = q_n = 0$ and $p = 1$ so that (2.1) holds again.

COROLLARY 2.3. *For given* $\mathbf{x} = (x_1, \ldots, x_d) \in F^d$*, there exists* $c(\mathbf{x})$ *such that*

$$|P(\mathbf{x})| < \frac{c(\mathbf{x})}{\mathrm{H}(P)^N}$$

holds for infinitely many $P \in \Lambda[X_1, \ldots, X_d]$ *of total degree* $\le k$.

A similar statement holds for $\widetilde{\mathrm{H}}(P)$ and it is easily checked that $c(\mathbf{x})$ can be taken to be 1 in that case.

The notion of *singular* vectors was introduced by Khintchine [10], [4] and the more general notion of *Dirichlet improvable* numbers and vectors was introduced by Davenport and Schmidt [6], [5]. Kristensen, Pedersen and Weiss [14] introduced natural extensions of these notions, calling them (k, ε)-Dirichlet improvable vectors and k-singular vectors. We now introduce the function field analogues.

Let $0 < \varepsilon \le 1/e$. A point $\mathbf{x} \in F^d$ is said to be (k, ε)-*Dirichlet improvable* if there exists $m_0 \in \mathbb{N}$ such that for every $m \ge m_0$, the following system of inequalities admits a non-zero solution $P \in \Lambda[X_1, \ldots, X_d]$ with total degree $\le k$

$$\begin{cases} |P(\mathbf{x})| < \dfrac{\varepsilon}{e^{mN}}, \\ \widetilde{\mathrm{H}}(P) < \varepsilon e^m. \end{cases}$$

When $k = 1$, this coincides with the usual Dirichlet improvable vectors in F^d, studied in [8]. Furthermore, one has the following observation in the case $d = 1$.

THEOREM 2.4. *For any* $\alpha \in F$*,* α *is Dirichlet* ε-*improvable for some* $0 < \varepsilon \le 1/e$ *if and only if it is a rational function.*

PROOF. Let $\alpha \in F$. Assume that the Dirichlet's theorem can be ε-improved for α, where $0 < \varepsilon \le 1/e$. So there exists $N_0 \in \mathbb{N}$ such that for any $n \ge N_0$, one has $p_n, q_n \in \Lambda$ with

$$\begin{cases} |q_n \alpha - p_n| < \dfrac{\varepsilon}{e^n}, \\ 1 \le |q_n| < \varepsilon e^n. \end{cases}$$

As a consequence, we have

$$\forall n \in \mathbb{N} \text{ with } n \geq N_0, \quad \begin{cases} |q_n\alpha - p_n| < \dfrac{1}{e^{n+1}}, \\ 1 \leq |q_n| < e^{n-1}, \end{cases} \quad (2.2)$$

as $\varepsilon \leq 1/e$. Without any loss in generality, we assume always that the pairs p_n, q_n are relatively prime and q_n is monic. Now observe that, for each such n,

$$q_n p_{n+1} - q_{n+1} p_n = q_{n+1}(q_n\alpha - p_n) - q_n(q_{n+1}\alpha - p_{n+1}).$$

Taking the absolute value of the above and using the ultrametric property,

$$|q_n p_{n+1} - q_{n+1} p_n| \leq \max\{|q_{n+1}||q_n\alpha - p_n|, |q_n||q_{n+1}\alpha - p_{n+1}|\}. \quad (2.3)$$

In view of (2.2), it is obvious that, $\forall n \in \mathbb{N}$ with $n \geq N_0$,

$$|q_{n+1}||q_n\alpha - p_n| < \frac{e^n}{e^{n+1}} = \frac{1}{e} < 1$$

and

$$|q_n||q_{n+1}\alpha - p_{n+1}| < \frac{e^{n-1}}{e^{n+2}} = \frac{1}{e^3} < 1.$$

This shows that the right-hand side of (2.3) is < 1 and hence

$$q_n p_{n+1} - q_{n+1} p_n = 0 \quad \text{i.e.} \quad \frac{p_n}{q_n} = \frac{p_{n+1}}{q_{n+1}} \quad \text{for all } n \geq N_0.$$

Since all q_n are monic and the pairs pairs p_n, q_n are relatively prime, it follows that $p_n = p_{N_0}$ and $q_n = q_{N_0}$ for all $n \geq N_0$. Therefore, from (2.2), one obtains

$$|q_{N_0}\alpha - p_{N_0}| < \frac{1}{e^{n+1}} \quad \text{for all } n \geq N_0,$$

and this makes $\alpha = p_{N_0}/q_{N_0} \in \mathbb{F}_q(T)$. Thus if at all any improvement of Dirichlet's theorem is possible for α then it has to be a rational function. On the other hand, any rational function is trivially Dirichlet ε-improvable for any $\varepsilon \in (0, 1/e]$.

We note that in the case of real numbers, the situation is quite different, namely there do exist irrational real numbers which are Dirichlet improvable. In fact, it was shown by Davenport and Schmidt [6] that an irrational number is Dirichlet improvable if and only if it is badly approximable. But for function fields with positive characteristic, the sets consisting of badly approximable and Dirichlet Laurent series respectively are disjoint.

The set of all (k, ε)-Dirichlet improvable vectors will be denoted by $DI(k, \varepsilon)$. We call a vector $\mathbf{x} \in F^d$ k-*singular* if it is (k, ε)-Dirichlet improvable for every $0 < \varepsilon \leq 1/e$. As an obvious corollary of [8, Theorem 3.7], we have $\lambda(DI(1, \varepsilon)) = 0$ for $\varepsilon \ll 1$.

In this section, we aim to establish that for any $k \in \mathbb{N}$, one has $\varepsilon_0 = \varepsilon_0(d, k) > 0$ such that $\lambda(DI(k, \varepsilon)) = 0$ whenever $\varepsilon < \varepsilon_0$. In fact, we will be proving a much more general result. For proceeding towards that, we need a few definitions.

Let μ be a Radon measure on F^d and $U \subseteq F^d$ be open with $\mu(U) > 0$. The measure μ is said to be *Federer* on U if there exists $D > 0$ such that for every ball B with center in supp(μ) such that $3B \subseteq U$, one has

$$\frac{\mu(3B)}{\mu(B)} \leq D.$$

Consider a continuous function $f : U \to F$. For any $B \subseteq U$, we set

$$\|f\|_{\mu, B} := \sup_{\mathbf{x} \in B \cap \text{supp}(\mu)} |f(\mathbf{x})|.$$

DEFINITION 2.5. For $C, \alpha > 0$, the function f is said to be (C, α)-*good* on U with respect to μ if for every ball $B \subseteq U$ with center in supp(μ), one has

$$\mu(\{\mathbf{x} \in B : |f(\mathbf{x})| < \varepsilon\}) \leq C \left(\frac{\varepsilon}{\|f\|_{\mu, B}} \right)^\alpha \mu(B).$$

We say a map $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ from U to F^n, where $n \in \mathbb{N}$, is (C, α)-*good* on U with respect to μ, or simply $(\mathbf{f}, \mu)$ is (C, α)-*good* on U, if every F-linear combination of $1, f_1, \ldots, f_n$ is (C, α)-good on U with respect to μ.

DEFINITION 2.6. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ be a map from U to F^n, where $n \in \mathbb{N}$. We say that $\mathbf{f}$ is *non-planar* with respect to μ or $(\mathbf{f}, \mu)$ is *non-planar* if for any ball $B \subseteq U$ with center in supp(μ), the restrictions of the functions $1, f_1, \ldots, f_n$ on $B \cap \text{supp}(\mu)$ are linearly independent.

The notion of non-planar maps was introduced in [13] and generalizes the notion of non-degenerate maps introduced in [12].

For $m \in \mathbb{N}$ and a ball $B = B(\mathbf{x}; r) \subseteq F^d$, where $\mathbf{x} \in F^d$ and $r > 0$, we shall use the notation $3^m B$ to denote the ball $B(\mathbf{x}; 3^m r)$.

DEFINITION 2.7. We call μ k-*friendly* on U if it is Federer on U and the function

$$\mathbf{f} : U \to F^N \text{ given by } (x_1, x_2, \ldots, x_d) \mapsto (x_1, x_2, \ldots, x_{d-1} x_d^{k-1}, x_d^k),$$
$$(2.4)$$

i.e. $\mathbf{f}$ maps $(x_1, x_2, \ldots, x_d)$ to N-distinct monomials in d-variables of total degree $\leq k$, is (C, α)-good for some $C, \alpha > 0$ and non-planar with respect to μ.

Our definition of k-friendly measures is slightly weaker than that in [14] leading to potentially stronger results. We refer the reader to §10.5 of [13] where these definitions are compared. The notion of friendly measures was originally introduced in [11].

EXAMPLE 2.8. It is not difficult to see that the prefixed Haar measure λ on F^d is Federer on any given open $U \subseteq F^d$ with $D = e^2$ and non-planar. By [13, Lemma 2.4], $\mathbf{f}$ is $(C, 1/dk)$-good, where C depends on d and k only. Thus λ is k-friendly on any open $U \subseteq F^d$.

With these definitions, we now discuss the main theorem of this section.

THEOREM 2.9. *Suppose μ is a k-friendly measure on $U \subseteq F^d$. Then there exist $\varepsilon_0 = \varepsilon_0(d, \mu)$ such that $\mu(DI(k, \varepsilon)) = 0$ whenever $\varepsilon < \varepsilon_0$. Thus in particular, the set of all k-singular vectors is μ-null.*

PROOF. The key observation is that if $\mathbf{x} \in U$ is (k, ε)-Dirichlet improvable, then $\mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is defined in (2.4), is a Dirichlet ε-improvable vector in F^N in the sense of [8]. To see this, let us write the N monomials in the variables $X_1, \ldots, X_d$ having total degree $\leq k$ as

$$M_1(X_1, \ldots, X_d), \ldots, M_N(X_1, \ldots, X_d)$$

so that a polynomial $P \in \Lambda[X_1, \ldots, X_d]$ will look like

$$a_0 + a_1 M_1(X_1, \ldots, X_d) + \cdots + a_N M_N(X_1, \ldots, X_d),$$

where $a_0, \ldots, a_N \in \Lambda$. Clearly $\mathbf{f}(\mathbf{x}) = (M(\mathbf{x}), \ldots, M_N(\mathbf{x})) \ \forall \mathbf{x} \in F^d$. From this, one can see that, if $\mathbf{x} \in U$ is (k, ε)-Dirichlet improvable, where $\varepsilon \in (0, 1/e]$, then for all $m \gg 1$, there exist $a_0, \ldots, a_N \in \Lambda$ such that

$$\begin{cases} |a_0 + (a_1, \ldots, a_N) \cdot \mathbf{f}(\mathbf{x})| = |a_0 + a_1 M_1(\mathbf{x}) + \cdots + a_N M_N(\mathbf{x})| < \dfrac{\varepsilon}{e^{mN}}, \\[2mm] \hspace{5cm} \|(a_1, \ldots, a_N)\| < \varepsilon e^m. \end{cases}$$

This establishes the claim. We therefore apply [8, Theorem 3.7] with $\mathbf{f}$ given by (2.4). From the definition of k-friendly measure, the hypothesis of [8, Theorem 3.7] is already satisfied, whence considering

$$\mathscr{T} := \{(mN, m, m, \ldots, m) : m \in \mathbb{N}\},$$

Theorem 2.9 follows.

3. *k*-very well approximable points

A point $\mathbf{x} \in F^d$ will be called *k-very well approximable* (*k*-VWA) if for some $\varepsilon > 0$, one has

$$|P(\mathbf{x})| < \frac{1}{\mathrm{H}(P)^{N+\varepsilon}} \quad \text{for infinitely many } P \in \Lambda[X_1, \ldots, X_d].$$

We aim to prove that this property is exceptional in the sense that the set of all such points in F^d is λ-null. In fact, we prove this for any *k*-friendly measure.

THEOREM 3.1. *For any k-friendly measure μ on a given $U \subseteq F^d$, The set of all k-VWA points in U is null with respect to μ.*

The main observation to prove this theorem is the following. For given $\mathbf{x} \in U$, denote $\mathbf{y} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is given by (2.4). Now for any polynomial P having total degree $\leq k$, we have $P(\mathbf{x}) = \mathbf{q} \cdot \mathbf{y} + p$ where $\mathbf{q}, p \in \Lambda$, so that $\mathrm{H}(P) \geq \|\mathbf{q}\|$. Hence, if $\mathbf{x} \in F^d$ is a *k*-VWA point then $\mathbf{f}(\mathbf{x})$ is a VWA vector in the sense of [9, Definition 1.1]. Thus the above theorem follows immediately from [9, Theorem 1.6] when $\mu = \lambda$. For the general case, namely to consider more general measures than Haar measure, one has to adopt the modifications considered in [13].

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