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# DENSE $p$-SUBSPACES OF PROXIMITY SPACES

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## Introduction.

Let $(X,\delta_1)$ be a proximity space, where $X$ is a dense (topological) subspace of a completely regular Hausdorff space $T$. In this paper we obtain conditions equivalent to the following property: $T$ admits a compatible proximity relation $\delta$ for which $(X,\delta_1)$ is a $p$-subspace of $(T,\delta)$. In particular, this property is characterized by means of maximal round filters on $(X,\delta_1)$. Examples are provided which concern the results.

## 1. Characterization of dense $p$-subspaces.

Let $P^*(X)$ denote the algebra of bounded real-valued proximity functions on a proximity space $(X,\delta_1)$. Then $P^*(X)$ induces an admissible, totally bounded uniform structure $\mathscr{P}^*$ on $X$. For definitions and results concerning round filters, see [2] or [7]. Further notation will follow that of [6].

In [6] it is shown that (for $X$ dense in $T$) the following conditions are equivalent:

(A) Every point $x$ in $T$ is a cluster point of a unique maximal round filter $\mathscr{F}_x$ on $(X,\delta_1)$.
(B) Every member of $P^*(X)$ has an extension to a member of $C^*(T)$.

Part of the motivation for the theorem that follows arises from (A), for suppose that each $\mathscr{F}_x$ in (A) is also required to converge to $x$. As the theorem shows, this is equivalent to the condition that $T$ admit a compatible proximity relation $\delta$ for which $(X,\delta_1)$ is a $p$-subspace of $(T,\delta)$. The latter condition is equivalent to the condition that $T$ admit a compatible proximity relation $\delta$ such that $P^*(T)\,|\,X = P^*(X)$, where $P^*(T)\,|\,X$ is the collection of restrictions to $X$ of members of $P^*(T)$. (See Theorem 7 of [5].)

Example 2 shows that $(X,\delta_1)$ and $T$ can occur such that each point $x$ in $T$ is a limit point of a unique maximal round filter $\mathscr{F}_x$ on $(X,\delta_1)$, and there can exist $x \in (T-X)$ and a (non-convergent) maximal round filter $\mathscr{F} \neq \mathscr{F}_x$ on $(X,\delta_1)$ which clusters at $x$. Thus, in the present theo-

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rem, (i) cannot be replaced by the weaker condition that each point $x$ in $T$ is a limit point of a unique maximal round filter $\mathscr{F}_x$ on $(X,\delta_1)$.

THEOREM. *Let $(X,\delta_1)$ be a proximity space, where $X$ is a dense (topological) subspace of a completely regular Hausdorff space $T$. Then the following are equivalent:*

(i) *Every point $x$ in $T$ is a limit point of a unique maximal round filter $\mathscr{F}_x$ on $(X,\delta_1)$, and $\mathscr{F}_x$ is the unique maximal round filter on $(X,\delta_1)$ which clusters at $x$.*
(ii) *Every gauge $\sigma \in \mathscr{P}^*$ has a unique extension to a continuous pseudometric $\bar{\sigma}$ on $T$, and the collection $\mathscr{D} = \{\bar{\sigma}: \sigma \in \mathscr{P}^*\}$ is an admissible uniform structure for $T$.*
(iii) *The canonical injection of $(X,\delta_1)$ into its Smirnov compactification $\delta_1 X$ has an extension to a homeomorphism $\tau$ of $T$ into $\delta_1 X$.*
(iv) *There is a proximity relation $\delta$ on $T$ for which $(X,\delta_1)$ is a $p$-subspace of $(T,\delta)$.*

PROOF. (i) implies (ii). By condition (ii) of the extension theorem of [6], every gauge $\sigma \in \mathscr{P}^*$ has a unique extension to a continuous pseudometric $\bar{\sigma}$ on $T$. Thus every basic $\bar{\sigma}$-neighborhood of a point $x$ in $T$ is a $T$-neighborhood of $x$.

Now let $x \in T$ and let $N_x$ be any $T$-neighborhood of $x$. Choose a $T$-neighborhood $N_x{}^*$ of $x$ for which $\mathrm{cl}_T N_x{}^* \subseteq N_x$. Then there exists $F \in \mathscr{F}_x$ for which $F \subseteq N_x{}^*$. Choose $F_1 \in \mathscr{F}_x$ such that $F_1 \ll F$. Then there are $\sigma \in \mathscr{P}^*$ and $\varepsilon > 0$ for which $\sigma(F_1, X-F) \geqq \varepsilon$. If $y \in T$ and $\bar{\sigma}(x,y) < \varepsilon$, then $\bar{\sigma}(y, X-F) > 0$, so that $y \notin \mathrm{cl}_T(X-F)$. Hence, if $N(\bar{\sigma},\varepsilon)$ is the $\varepsilon$-ball about $x$ determined by $\bar{\sigma}$, we have $N(\bar{\sigma},\varepsilon) \subseteq T - \mathrm{cl}_T(X-F)$.

Since $\mathrm{cl}_T(X-F) \cup \mathrm{cl}_T F = T$ and $\mathrm{cl}_T F \subseteq N_x$, evidently $T - \mathrm{cl}_T(X-F) \subseteq N_x$. Thus $N(\bar{\sigma},\varepsilon) \subseteq N_x$, and $\mathscr{D}$ is admissible.

(ii) implies (iii). Let $(T^*, \mathscr{D}^*)$ be the completion of the separated, totally bounded uniform space $(T,\mathscr{D})$. There is a uniform isomorphism $\tau$ (see [7]) of $(T,\mathscr{D})$ into $(T^*,\mathscr{D}^*)$ ,and $\tau[X]$ is dense in $T^*$. Since every gauge $\bar{\sigma}$ in $\mathscr{D}$ agrees with $\sigma$ on $X$, $T^*$ is the Smirnov compactification of $(X,\delta_1)$, and (iii) now follows from the uniqueness of the Smirnov compactification.

(iii) implies (iv). For $A,B \subseteq T$, define $A \delta B$ if and only if $\tau[A]$ is close to $\tau[B]$ in $\delta_1 X$. It is readily verified that $\delta$ is a proximity relation for $T$ which is compatible with the topology on $T$. Clearly, $\delta$ agrees with $\delta_1$ on $X$, so that $(X,\delta_1)$ is a $p$-subspace of $(T,\delta)$.

(iv) implies (i). Let $x \in T$ and let $\mathscr{F}_x$ be the trace in $(X,\delta_1)$ of the filter of $T$-neighborhoods of $x$. Then $\mathscr{F}_x$ is a round filter on $(X,\delta_1)$ which converges to $x$. Since $\mathscr{P}^*$ is generated by $P^*(X)=P^*(T)|X$, the filter $\mathscr{F}_x$ is Cauchy relative to $\mathscr{P}^*$. Thus, by Theorem 1 of [2], $\mathscr{F}_x$ is maximal. Now by the extension theorem of [6], $\mathscr{F}_x$ is also the unique maximal round filter on $(X,\delta_1)$ which clusters at $x$.

This completes the proof.

## 2. Examples.

EXAMPLE 1. Let $T$ be the subset $\{(x,y): y \geqq 0\}$ of the plane. The topology on $T$ is generated by the usual neighborhoods of points in $T$ together with the following neighborhoods of the points $(x,0)$:

$$N_\varepsilon(x,0) = \{(x,0)\} \cup \{(u,v) \in T: (u-x)^2+(v-\varepsilon)^2<\varepsilon^2\},$$

where $\varepsilon>0$. Then $T$ is a completely regular, Hausdorff space. (See Example 3.K of [3].)

Let $X$ be the subspace $\{(x,y): y>0\}$ of $T$ and let $\delta_1$ be the proximity relation on $X$ generated by the usual metric in the plane. Now $X$ is a dense subspace of $T$, and every point of $T$ is a cluster point of a unique maximal round filter on $(X,\delta_1)$, but for points $(x,0)$ of $T-X$, no maximal round filter on $(X,\delta_1)$ converges to $(x,0)$. Now $(X,\delta_1)$ has the extension property of the corollary in [6], so that every member of $P^*(X)$ has an extension to a member of $C^*(T)$, but by the theorem of the present paper, there is no compatible $\delta$ for which $(X,\delta_1)$ is a $p$-subspace of $(T,\delta)$.

EXAMPLE 2. Let $X$ be the positive integers with the discrete topology. Take $f(x)=x^{-1}$ and $g(x)=1$, if $x$ is even, and $g(x)=0$, if $x$ is odd. Then the pseudometrics $\psi_f,\psi_g$ on $X$ determined by $f$ and $g$, respectively, generate an admissible uniform structure $\mathscr{D}$ for $X$. Let $\delta_1$ be the proximity relation for $X$ generated by $\mathscr{D}$.

Take $\alpha \notin X$ and set $T=X\cup\{\alpha\}$. Let the basic neighborhoods of $\alpha$ be defined as follows:

$$N_\alpha = \{\alpha\} \cup \{2n: n \geqq m\} \cup \{4n+1: n \geqq k\},$$

where $m,k \in X$. (Thus, in $T$, each point $x \neq \alpha$ is isolated, and the neighborhoods of $\alpha$ are determined as above.) Then $X$ is dense in $T$, and it is easily verified that $T$ is a completely regular, Hausdorff space.

If $A$ and $B$ are the sets of even and odd integers, respectively, then $A\,\delta_1 B$, but $\mathrm{cl}_T A \cap \mathrm{cl}_T B \neq \emptyset$. Thus $(X,\delta_1)$ cannot be a $p$-subspace of $(T,\delta)$ for any compatible proximity relation $\delta$ for $T$.

Let $\mathscr{F}_\alpha$ be the round hull of the filter generated by the sets $F_m=\{2n: n \geqq m\}$, where $m \in X$. Then $\mathscr{F}_\alpha$ is a round filter, and each $F_m \in \mathscr{F}_\alpha$.

Since $\psi[F_m] \leqq 1/(2m)$, where $\psi = \psi_f \vee \psi_g$, $\mathscr{F}_\alpha$ is a maximal round filter on $(X, \delta_1)$.

Let $\mathscr{F}^*$ be the round hull of the filter generated by the sets

$$F_k^* = \{2n+1 : n \geqq k\}, \quad k \in X .$$

Then $\mathscr{F}^*$ is also a maximal round filter on $(X, \delta_1)$. Evidently, $\mathscr{F}_\alpha$ converges to $\alpha$, and $\mathscr{F}^*$ clusters at $\alpha$ but does not converge. We note that $\mathscr{F}_\alpha$ and $\mathscr{F}^*$ are the only free maximal round filters on $(X, \delta_1)$.

REMARK. If $T$ admits a compatible proximity $\delta$ such that $(X, \delta_1)$ is a dense $p$-subspace of $(T, \delta)$, then each point $x$ in $T$ is a cluster point of a unique cluster $\pi_x$ from $(X, \delta_1)$. (See Theorem 3 of [5].) Example 2 shows that the converse of this statement is false. Now $\mathscr{F}_\alpha$ contains small sets relative to $\mathscr{D}$. Thus, by Theorem 8 of [4] (which remains true for funnels with $\mathscr{D}$-small sets), $\mathscr{F}_\alpha$ is a subclass of a unique cluster $\pi_\alpha$ from $(X, \delta_1)$. Evidently, $\alpha$ is a cluster point of $\pi_\alpha$. Similarly, $\mathscr{F}^*$ is a subclass of a unique cluster $\pi^*$ from $(X, \delta_1)$, and $\pi_\alpha \neq \pi^*$. It is easily seen that $\pi_\alpha$ and $\pi^*$ are the only clusters from $(X, \delta_1)$ which do not contain a point. Now $\alpha$ is not a cluster point of $\pi^*$, so that each point $x$ in $T$ is a cluster point of a unique cluster $\pi_x$ from $(X, \delta_1)$, but there is no compatible proximity $\delta$ for $T$ such that $(X, \delta_1)$ is a $p$-subspace of $(T, \delta)$.

## REFERENCES


1. E. M. Alfsen and J. E. Fenstad, *On the equivalence between proximity structures and totally bounded uniform structures*, Math. Scand. 7 (1959), 353–360.
2. E. M. Alfsen and J. E. Fenstad, *A note on completion and compactification*, Math. Scand. 8 (1960), 97–104.
3. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton · Toronto · London · New York, 1960.
4. S. Leader, *On completion of proximity spaces by local clusters*, Fund. Math. 48 (1960), 201–216.
5. S. Leader, *On clusters in proximity spaces*, Fund. Math. 47 (1959), 205–213.
6. D. A. Mattson, *Extensions of proximity functions*, Proc. Amer. Math. Soc. 26 (1970), 347–351.
7. W. Thron, *Topological structures*, Holt, Rinehart, and Winston, New York, 1966.


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