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CERTAIN EXTENDED MOCK THETA FUNCTIONS AND GENERALISED BASIC HYPERGEOMETRIC TRANSFORMATIONS

SNEH D. PRASAD

1. Introduction.

About thirty years back G. N. Watson [8], [9] studied certain q-series which were earlier called "mock" theta functions by Ramanujan. Recently, G. E. Andrews [4], [5] and R. P. Agarwal [1] studied these functions and some of their extensions as consequences of much more general transformations of basic hypergeometric series.

Watson [9] in his paper had defined certain bilateral series also which he called the "complete" mock theta functions. However, he defined the bilateral extensions of only four mock theta functions of the fifth order in the form

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-q;q)_n}, \quad \sum_{n=-\infty}^{\infty} \frac{q^{4n^2}}{(q^2;q^4)_n},$$

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}}{(-q;q)_n}, \quad \sum_{n=-\infty}^{\infty} \frac{q^{4n(n+1)}}{(q^2;q^4)_{n+1}},$$

which are the "complete" forms of Ramanujan's $f_0(q)$, $f_0(q^2)$, $f_1(q)$ and $f_1(q^2)$ respectively.

It seems rather surprising as to why Watson did not go a step ahead to define "complete" forms for other mock theta functions also.

The object of this paper is an attempt to answer this question to a fair extent. The "complete" forms that we have studied are extensions of the generalised third order mock theta functions given recently by Andrews [4]. Following Andrews we define the "complete" extended third order functions as follows:

$$\varphi_1(\alpha;q) = \sum_{m=-\infty}^{\infty} \frac{q^{m^2}}{(-\alpha q;q^2)_m}, \quad |\alpha|<1$$

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$$\psi_1(\alpha; q) = \sum_{m=-\infty}^{\infty} \frac{q^{m^2}}{(\alpha; q^2)_m}, \quad |\alpha| < |q|$$

$$v_1(\alpha; q) = \sum_{m=-\infty}^{\infty} \frac{q^{m(m+1)}}{(-\alpha^2 q^{-1}; q^2)_{m+1}}, \quad |\alpha| < |q|^{\frac{1}{2}}$$

$$f_1(\alpha; q) = \sum_{m=-\infty}^{\infty} \frac{q^{m(m-1)} \alpha^m}{(-q; q)_m (-\alpha; q)_m},$$

$$\omega_1(\alpha; q) = \sum_{m=-\infty}^{\infty} \frac{q^{2m^2} \alpha^{2m}}{(q; q^2)_{m+1} (\alpha^2 q^{-1}; q^2)_{m+1}}.$$

The unilateral forms of the above functions become mock theta functions for $\alpha = q^r$ as shown by Andrews [4].

The "complete" functions $\varphi_1(\alpha; q)$, $\psi_1(\alpha; q)$ and $v_1(\alpha; q)$ are easily seen to be expressible in terms of Lerch's Transcendant [6] which in its turn is a limiting case of a bilateral basic hypergeometric ${}_1\Psi_1$-series. In particular, it can be easily shown that

$$\varphi_1(\alpha; q) = f(\alpha^{\frac{1}{2}}, \alpha^{-\frac{1}{2}}; 1, q) = \prod_{n=0}^{\infty} \frac{(1+q^{2n+1})(1+q^{2n+1})(1-q^{2n+2})}{(1-\alpha q^{2n})(1+\alpha q^{2n+1})},$$

$$\psi_1(\alpha; q) = f(i\alpha^{\frac{1}{2}} q^{-\frac{1}{2}}, iq^{\frac{1}{2}} \alpha^{-\frac{1}{2}}; 1, q) = \prod_{n=0}^{\infty} \frac{(1+q^{2n+1})(1+q^{2n+1})(1-q^{2n+2})}{(1+q^{2n-1}\alpha)(1-\alpha q^{2n})},$$

$$v_1(\alpha; q) = \frac{1}{(1+\alpha^2 q^{-1})} f(\alpha q^{-\frac{1}{2}}, \alpha^{-1}; 1, q) = \prod_{n=0}^{\infty} \frac{(1+q^{2n+2})(1+q^{2n})(1-q^{2n+2})}{(1-\alpha^2 q^{2n-1})(1+\alpha^2 q^{2n+1})},$$

where Lerch's Transcendant is defined as

$$f(x, \xi; q, q_1) = \sum_{n=-\infty}^{\infty} \frac{(qq_1)^{n^2} (\xi^{-2n} x^{-2n})}{(-q_1 \xi^{-2}; q_1{}^2)_n}.$$

The next two "complete" extended mock theta function $f_1(\alpha; q)$ and $\omega_1(\alpha; q)$ are easily seen to be limiting cases of a basic bilateral ${}_2\Psi_2$-series.

The above "complete" functions are more conveniently expressible in the form of a bilateral ${}_A\Psi_B$-series.

Thus far a study has only been made of the type of series ${}_A\Psi_A$. In this paper we have deduced a general transformation between series of the type ${}_A\Psi_B$ in the hope that such transformations should throw some light by way of unification on the various mock theta functions. We give in Section 3 a proof of a general transformation between ${}_A\Psi_B$-

series by means of a transformation of ${}_A\Psi_A$-series given earlier by L. J. Slater [7; 7.2.5] and also give a direct proof of it, by integrals.

Later in the paper we derive some interesting special cases, of our general transformation, which give relations between the functions $f_1(\alpha; q)$ and $\omega_1(\alpha; q)$.

Finally, we have deduced a general transformation between "bibasic" series of the type ${}_A\Psi_B$.

2. Notations.

Let

$$[q^a; q]_n = (1-q^a)(1-q^{a+1})\dots(1-q^{a+n-1}), \quad [q^a; q]_0 = 1, \quad |q| < 1 .$$

(a_N) stands for a sequence of N parameters $a_1, a_2, \dots, a_N$; when $N = A$, it will be dropped out.

By

$$\prod^{P}\begin{bmatrix} x^{(\alpha)}; x \\ x^{(\beta)} \end{bmatrix} \quad \text{or simply} \quad \prod^{P}\begin{bmatrix} (\alpha); x \\ (\beta) \end{bmatrix}$$

we denote the product

$$\prod_{u=0}^{P-1}\left[\frac{(1-x^{(\alpha)+u})}{(1-x^{(\beta)+u})}\right].$$

For P infinite it will be dropped from the product symbol.

Bilateral ${}_A\Psi_B$-series and "bibasic" bilateral hypergeometric series of the type ${}_A\Psi_B$ are defined as follows:

$${}_A\Psi_B\begin{bmatrix} q^{(a)}; x; \lambda \\ q^{(b)} \end{bmatrix} \equiv {}_A\Psi_B\begin{bmatrix} (a); x; \lambda \\ (b) \end{bmatrix} = \sum_{n=-\infty}^{\infty} \frac{[q^{(a)}; q]_n x^n q^{\frac{1}{2}\lambda n(n+1)}}{[q^{(b)}; q]_n}, \quad \lambda \geqq 0 ,$$

when λ is zero it shall be omitted from the Ψ-symbol. For $\lambda = 0$ this series converges provided $A > B$ and $|x| < 1$ or $A = B$ and $|q^{\Sigma(b)-\Sigma(a)}| < |x| < 1$ and for λ positive the series converges provided $\lambda + A - B > 0$ or $\lambda + A - B = 0$ and $|q^{\Sigma(b)-\Sigma(a)+A-B}| < |x|$.

$${}_{A+D}\Psi_{B+C}\begin{bmatrix} q^{(a)}: q_1^{(d)}; x \\ q^{(b)}: q_1^{(c)}; q^{\lambda}q_1^{\lambda_1} \end{bmatrix} \equiv {}_{A+D}\Psi_{B+C}\begin{bmatrix} (a): (d); x \\ (b): (c); q^{\lambda}q_1^{\lambda_1} \end{bmatrix}$$

$$= \sum_{n=-\infty}^{\infty} \frac{[q^{(a)}; q]_n [q_1^{(d)}; q_1]_n x^n q^{\frac{1}{2}n\lambda(n+1)} q_1^{\frac{1}{2}\lambda_1 n(n+1)}}{[q^{(b)}; q]_n [q_1^{(c)}; q_1]_n}; \quad |q| < 1, |q_1| < 1 ,$$

and we take $|q| \leqq |q_1|$ for definiteness.

This series certainly converges when $\lambda=0$, $\lambda_1=0$, provided $A>B$, $D>C$ and $|x|<1$, or $A=B$, $D=C$ and

$$|q^{\Sigma(b)-\Sigma(a)}\, q_1^{\Sigma(c)-\Sigma(d)}| < |x| < 1;$$

when $\lambda>0$, $\lambda_1>0$ provided $\lambda+A-B>0$, $\lambda_1+D-C>0$ or $\lambda+A-B=0$, $\lambda_1+D-C=0$ and

$$|q^{\Sigma(b)-\Sigma(a)+A-B}\, q_1^{\Sigma(c)-\Sigma(d)+D-C}| < |x| .$$

A term of the type $(h)'-h_r$ stands for the sequence

$$h_1-h_r, h_2-h_r, \ldots, h_{r-1}-h_r, h_{r+1}-h_r, \ldots, h_n-h_r .$$

The parameter $-[\alpha]^*$ in the product denotes a term of the type $\prod_{u=0}^{\infty}[1+q^{\alpha+u}]$.

3. A transformation.

We now proceed to deduce the following transformation:

$$\text{(3.1)} \quad \prod \begin{bmatrix} (b), 1-(c), 1-x-\alpha, x+\alpha; q \\ (a), 1-(a) \end{bmatrix} {}_C\Psi_B \begin{bmatrix} (c); (-q^{-1})^{A-C} q^x; A-C \\ (b) \end{bmatrix}$$

$$= qq^{-a_1} \prod \begin{bmatrix} 1+(b)-a_1, 2-x-\alpha-a_1, a_1-(c), a_1-1+x+\alpha; q \\ (a)'+1-a_1, 1-a_1, a_1-(a)', a_1 \end{bmatrix} \times$$

$$\times {}_C\Psi_B \begin{bmatrix} 1+(c)-a_1; (-q^{-a_1})^{A-C} q^x; A-C \\ 1+(b)-a_1 \end{bmatrix} +$$

$$+ \text{idem}(a_1; a_2, \ldots a_A) ,$$

provided

(i) $A>C$ or $A=C$ and $|q^x|<1$,
(ii) $A>B$ or $A=B$ and $|q^{\Sigma(b)-\Sigma(c)}|<|q^x|<1$,

where $\alpha=\sum(c)-\sum(a)$.

To prove (3.1) consider the following transformation of Slater [7;7.2.5]:

$$\text{(3.2)} \quad \prod \begin{bmatrix} x+\alpha, 1-x-\alpha, (b_A), 1-(c_A); q \\ (a), 1-(a) \end{bmatrix} {}_A\Psi_A \begin{bmatrix} (c_A); q^x \\ (b_A) \end{bmatrix}$$

$$= qq^{-a_1} \prod \begin{bmatrix} a_1+x+\alpha-1, 2-a_1-x-\alpha, a_1-(c_A), 1+(b_A)-a_1; q \\ a_1, 1-a_1, a_1-(a)', 1+(a)'-a_1 \end{bmatrix} \times$$

$$\times {}_A\Psi_A \begin{bmatrix} 1+(c_A)-a_1; q^x \\ 1+(b_A)-a_1 \end{bmatrix} + \text{idem}(a_1; a_2, \ldots a_A) ,$$

where $\alpha = \sum(c_A) - \sum(a_A)$, $|q^x| < 1$.

If in (3.2) x is replaced by $x - c_{C+1} - c_{C+2} - \ldots - c_A$ and $c_{C+1}, c_{C+2}, \ldots c_A$ tend to ∞ while $b_{B+1}, b_{B+2}, \ldots, b_A$ tend to ∞, we get (3.1).

(3.1) can also be derived by evaluating the integral

$$\frac{1}{2\pi i} \int_{-i\pi/t}^{i\pi/t} \prod \begin{bmatrix} 1-s-(c), (b)+s, 1+s-x-\alpha, x+\alpha-s; q \\ (a)+s, 1-s-(a), 1+s, -s \end{bmatrix} ds ,$$

where $\alpha = \sum(c) - \sum(a)$ and there are A of the "a" parameters, B of the "b" and C of the "c" parameters.

If in (3.1) we take $A = C$, we get the following transformation between $(A+1)\ {}_A\Psi_B$-series:

$$\prod \begin{bmatrix} (b), 1-(c), 1-x-\alpha, x+\alpha; q \\ (a), 1-(a) \end{bmatrix} {}_A\Psi_B \begin{bmatrix} (c_A); q^x \\ (b) \end{bmatrix}$$

$$= qq^{-a_1} \prod \begin{bmatrix} 1+(b)-a_1, 2-x-\alpha-a_1, a_1-c_A, a_1-1+x+\alpha; q \\ (a)'+1-a_1, 1-a_1, a_1-(a)', a_1 \end{bmatrix} \times$$

$$\times\ {}_A\Psi_B \begin{bmatrix} 1+(c_A)-a_1; q^x \\ 1+(b)-a_1 \end{bmatrix} + \text{idem}(a_1; a_2, \ldots a_A) ,$$

where $\alpha = \sum(c) - \sum(a)$, provided $A > B$, and if $A = B$, $|q^{\Sigma(b)-\Sigma(c)}| < |q^x| < 1$.

4. Relations between 'complete' extended mock theta functions.

The following identities between $f_1(\alpha; q)$ and $\omega_1(\alpha; q)$ shall now be established:

$$\text{(4.1)} \qquad \prod \left[-[c]^*; q\right] f_1(q^c; q) = \prod \left[-[2-c]^*; q\right] f_1(q^{2-c}; q) .$$

$$\text{(4.2)} \qquad \prod \begin{bmatrix} -\frac{1}{2}, \frac{3}{2}, -[1]^*; q \\ -[\frac{1}{2}]^*, -[\frac{1}{2}]^*, -[1-c]^* \end{bmatrix} f_1(q^c; q) +$$

$$+ q^{\frac{1}{2}} \prod \begin{bmatrix} -[-1]^*, -[2]^*, \frac{1}{2}; q \\ -[\frac{1}{2}]^*, -[\frac{1}{2}]^*, \frac{1}{2}-c \end{bmatrix} (1-q^{c-\frac{1}{2}}) \omega_1(q^{\frac{1}{2}c}; q^{\frac{1}{2}}) +$$

$$+ qq^{-c} \prod \begin{bmatrix} -[c-\frac{3}{2}]^*, -[\frac{5}{2}-c]^*, 1; q \\ -[c]^*, -[1-c]^*, c-\frac{1}{2}, \frac{3}{2}-c \end{bmatrix} = 0 .$$

$$\text{(4.3)} \qquad q^c \prod \left[c-\tfrac{1}{2}; q\right] \omega_1(q^{\frac{1}{2}c}; q^{\frac{1}{2}}) = q \prod \left[\tfrac{3}{2}-c; q\right] \omega_1(q^{1-\frac{1}{2}c}; q^{\frac{1}{2}}) .$$

(4.1) can be deduced from (3.1) by taking $B = 2$, $C = 1$, $A = 2$, replacing c_1 by $c_1 - x$ and letting $x \to \infty$ and then putting $b_2 = 1 + \pi i/t$, $b_1 = c_1 + \pi i/t$, $a_1 = \frac{1}{2} + \pi i/t$ and $a_2 = c_1$, where $q = e^{-t}$.

On the other hand if in (3.1) $B=2$, $C=1$, $A=2$, c_1 is replaced by c_1-x and $x\to\infty$ and then b_2, b_1, a_1 and a_2 are replaced by $1+\pi i/t$, $c_1+\pi i/t$, $\frac{1}{2}+\pi i/t$ and $c_1+\pi i/t$ respectively, we get (4.2) because

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}q^{-c_1 n}}{[q;q]_n[q^{2-c_1};q]_n}=\frac{1}{\prod[q^{2-c_1};q]}.$$

For (4.3) take $B=2$, $C=1$, $A=2$ and replace c_1 by c_1-x+1, let $x\to\infty$ and then set $b_1=c_1+\frac{1}{2}$, $b_2=\frac{3}{2}$, $a_1=\frac{3}{2}+\pi i/t$ and $a_2=c_1$ in (3.1).

5. A general transformation.

In this section we shall obtain the following general transformation between "bibasic" ${}_A\Psi_B$ type of series which extends (3.1):

$$\text{(5.1)}\quad 1/t\sum_{u=0}^{r-1}\prod\begin{bmatrix}(a)+1-f_1,f_1-(b),2-x-\alpha-f_1,x+\alpha+f_1-1;q\\1,(f)+1-f_1,2-f_1,f_1-(f)',f_1-1\end{bmatrix}\times$$

$$\times\prod\begin{bmatrix}(c)-m[f_1]+1-u_1,-(d)+m[f_1]+u_1;q_1\\(h)+1-m[f_1]-u_1,-(h)+m[f_1]+u_1\end{bmatrix}\times$$

$$\times{}_{B+D}\Psi_{A+C}\begin{bmatrix}(b)+1-f_1:1+(d)-m[f_1]-u_1;Q_1\\(a)+1-f_1:1+(c)-m[f_1]-u_1;q^{F-B}q_1^{H-D}\end{bmatrix}+$$

$$+\operatorname{idem}(f_1;f_2,\ldots,f_F)+$$

$$+1/t\sum_{u=0}^{r-1}\prod\begin{bmatrix}(a),1-(b),1-x-\alpha,x+\alpha;q\\1,1,(f),1-(f)\end{bmatrix}\times$$

$$\times\prod\begin{bmatrix}(c)+m[0]+u_1,1-(d)-m[0]-u_1,;q_1\\(h)+m[0]+u_1,1-(h)-m[0]-u_1\end{bmatrix}\times$$

$$\times{}_{B+D}\Psi_{A+C}\begin{bmatrix}(b):(d)+m[0]+u_1;z\\(a):(c)+m[0]+u_1;q^{F-B}q_1^{H-D}\end{bmatrix}+$$

$$+1/t_1\prod\begin{bmatrix}(a)+1-M[h_1],2-x-\alpha-M[h_1],x+\alpha+M[h_1]-1,M(h_1)-(b);q\\(f)+1-M[h_1],2-M[h_1],M[h_1]-(f),M[h_1]-1\end{bmatrix}\times$$

$$\times\prod\begin{bmatrix}(c)+1-h_1,h_1-(d);q_1\\(h)+1-h_1,h_1-(h)',1\end{bmatrix}\times$$

$$\times{}_{B+D}\Psi_{A+C}\begin{bmatrix}(b)+1-M[h_1]:1+(d)-h_1;Q_2\\(a)+1-M[h_1]:1+(c)-h_1;q^{F-B}q_1^{H-D}\end{bmatrix}+$$

$$+\operatorname{idem}(h_1;h_2,\ldots,h_H)=0,$$

where $u_1 = 2\pi i u/t$, $\alpha = \sum(b) - \sum(f)$,

$$z = (-1)^{B+D-F-H} q^{B-F+x} q_1^{D-H-\Sigma(d)+\Sigma(h)+(H-D)(m[0]+u_1)}$$

$$Q_\mu = (-1)^{B+D-F-H} q^{B-F-x+(F-B)\xi_\mu} q_1^{D-H-\Sigma(d)+\Sigma(h)+(H-D)\eta_\mu}$$

with $\mu = 1, 2$,

$$\xi_1 = 1 - f_1,\ \xi_2 = 1 - M[h_1], \quad \eta_1 = 1 - m[f_1] - u_1, \quad \eta_2 = 1 - h_1,$$

r is a positive integer such that

$$rt_1 \leqq t < (r+1)t_1$$

and $m_{(\alpha)}$ is the sequence of least integers for which

$$-\pi/t_1 \leqq \operatorname{Im}\{m[(f)], M[(h)]\} < \pi/t_1,$$

where the numbers $(\alpha) + 2\pi i m_{(\alpha)}/t$ have been abbreviated to $m[(\alpha)]$ and $(\alpha) + 2\pi i m_{(\alpha)}/t_1$ to $M[(\alpha)]$. The parameters in (5.1) are subjected to the restrictions

(i) $F > B$ or $F = B$ and $\operatorname{Re} x > 0$,
(ii) $H > D$ or $H = D$ and $\operatorname{Re}[\sum(h) - \sum(d)] > 0$,
(iii) $F > A$ or $F = A$ and $\operatorname{Re}[\sum(a) - \sum(b) - x] > 0$,
(iv) $H > C$ or $H = C$ and $\operatorname{Re}[\sum(c) - \sum(h)] > 0$.

(5.1) can be proved by evaluating the following integral (see Section 3 of [2] and [3]):

$$\frac{1}{2\pi i} \int_{-i\pi/t_1}^{i\pi/t_1} \prod \begin{bmatrix} (a)+s, 1-(b)-s, 1-x-\alpha+s, x+\alpha-s; q \\ (f)+s, 1-(f)-s, 1+s, -s \end{bmatrix} \times$$
$$\times \prod \begin{bmatrix} (c)+s, 1-(d)-s; q_1 \\ (h)+s, 1-(h)-s \end{bmatrix} ds,$$

where $\alpha = \sum(b) - \sum(f)$.

By taking $B = F = A$ and $D = H = C$ in (5.1) we get (3.3) of [3].

Further by taking $A = B = C = H = F = 1$, $D = 0$, replacing x by $x - h - m[0]$, b by $b - x$ and $x \to \alpha$, setting $a = b + \pi i/t$, $c = 1 + \pi i/t_1 - m[0]$, $f = \frac{1}{2} + \pi i/t$, $h = b - m[0]$ we get

$$1/t \sum_{u=0}^{r-1} \prod \begin{bmatrix} -[b]^*, -[\frac{3}{2}]^*, -[-\frac{1}{2}]^*; q \\ 1, -[\frac{1}{2}]^*, 1, -[\frac{1}{2}]^* \end{bmatrix} \prod \begin{bmatrix} 1+\pi i/t_1+u_1; q_1 \\ b+u_1, 1-b-u_1 \end{bmatrix} \times$$
$$\times \sum_{n=-\infty}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} q_1^{\frac{1}{2}n(n-1)} q_1^{(b+u_1)n}}{[-q^b; q]_n [-q_1^{1+u_1}; q_1]_n} +$$

$$+1/t\prod\begin{bmatrix}b+1-M[h]+\pi i/t,\frac{5}{2}-M[h]+\pi i/t,M[h]-\frac{3}{2}-\pi i/t;\ q\\ \frac{3}{2}+\pi i/t-M[h],2-M[h],M[h]-\frac{1}{2}-\pi i/t,M[h]-1\end{bmatrix}\times$$

$$\times\prod\begin{bmatrix}2-\pi i/t-b;\ q_1\\ 1,1\end{bmatrix}\sum_{n=-\infty}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}q_1{}^{\frac{1}{2}n(n+1)}q^{(1-M[h])n}}{[-q^{1+b-M[h]};\ q]_n[-q_1{}^{2-b};\ q_1]_n}=0\,,$$

where $h=b-m[0]$.

This result is an extension of (4.1) as seen by taking $q=q_1$ in it. Similar extensions can be obtained for (4.2) and (4.3) in the same manner.

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REFERENCES

1. R. P. Agarwal, *Certain basic hypergeometric identities associated with mock theta functions*, Quart. J. Math. (Oxford) 20 (1969), 121–128.
2. R. P. Agarwal and A. Verma, *Generalised basic hypergeometric series with unconnected bases*, Proc. Cambridge Philos. Soc. 63 (1967), 727–734.
3. R. P. Agarwal and A. Verma, *Generalised basic hypergeometric series with unconnected bases* II, Quart. J. Math. Oxford Ser. 18 (1967), 181–192.
4. G. E. Andrews, *On basic hypergeometric series, mock theta functions and partitions* (I), Quart. J. Math. Oxford Ser. 17 (1966), 64–80.
5. G. E. Andrews, *On basic hypergeometric series, mock theta functions and partitions* (II), Quart. J. Math. Oxford Ser. 17 (1966), 132–143.
6. M. Jackson, *On Lerch's Transcendant and the basic bilateral hypergeometric series* ${}_2\Psi_2$. J. London Math. Soc. 25 (1950), 189–196.
7. L. J. Slater, *Generalised hypergeometric functions*, Cambridge University Press, 1966.
8. G. N. Watson, *The final problem: an account of the mock theta functions*, J. London Math. Soc. 11 (1936), 55–80.
9. G. N. Watson, *The mock theta functions* (II), Proc. London Math. Soc. (2), 42 (1937), 272–304.

DEPARTMENT OF MATHEMATICS, LUCKNOW UNIVERSITY, LUCKNOW, INDIA