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VECTOR MEASURES

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1. Introduction.

In this paper we shall deal with the properties of vector valued finitely additive set functions defined on an algebra.

If m is a finitely additive set function taking values in a locally convex Hausdorff space E, then $(x',m(\cdot))$ is a scalar valued finitely additive set function for every continuous linear functional x' on E. So the set function m gives us a certain set of scalar valued finitely additive set functions. This set reflects many properties of m. Hence as a preparation we discuss in Section 3 some properties of sets of scalar valued set functions. In particular we shall characterize the weakly compact sets in certain spaces of scalar valued set functions (see Theorems 1, 2, and 3). Theorem 1 is a slight modification of a theorem due to Grothendieck, see Theorem 2, p. 146–147, in [6].

In Section 4 we show that weak σ-additivity (or weak* σ-additivity) under certain restrictions implies σ-additivity (see Theorem 4).

In Section 5 we discuss the atomic structure of vector valued set functions. In particular, we prove that under mild conditions (see Theorem 6) every vector measure can be decomposed uniquely in an atomic part and an atomless part.

In Section 6 we study the range of vector valued finitely additive set functions. First we give necessary and sufficient conditions for the range to be weakly conditionally compact (see Theorems 7 and 8). Secondly we give necessary and sufficient conditions for the range to be (conditionally) compact (see Theorems 9 and 10). Thirdly we prove an analogue to a theorem of Liapounov. Liapounov's theorem (see [13]) states that the range of a finite dimensional atomless vector measure is closed and convex. In [13] it is also shown that this is not true in general for infinite dimensional vector measures. We shall prove, however, that the range of an atomless vector measure is weakly dense in its convex hull (see Theorem 11).

In Section 7 we give eight examples (or rather counter-examples) of vector measures, each of them disproving a natural conjecture.

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2. Definitions and notation.

In all that follows S denotes a nonempty set, Σ_0 an algebra of subsets of S, Σ the σ-abgebra generated by Σ_0, E a locally convex Hausdorff space, and E' the dual of E, that is, E' is the space of all continuous scalar valued linear functions on E.

As far as the notations of special spaces are concerned, such as $\mathrm{ba}(S,\Sigma_0)$, $\mathrm{ca}(S,\Sigma)$, $B(S,\Sigma_0)$, $L_p(S,\Sigma,a)$, l_1, l_∞, c_0 et cetera, we follow [4, Chapter IV].

Let $\mathscr{S}$ be a class of subsets of S, m a map from $\mathscr{S}$ into E, and M a family of maps from $\mathscr{S}$ into E. We then define:

(1) m is *s-bounded*, if and only if for every sequence $\{A_n\}$ of mutually disjoint sets from $\mathscr{S}$, we have $\lim_{n\to\infty} m(A_n)=0$.

(2) M is *uniformly s-bounded*, if and only if $\lim_{n\to\infty} m(A_n)=0$ uniformly for $m \in M$, whenever $A_1,A_2,\ldots$ are disjoint subsets of $\mathscr{S}$.

The notion of s-boundedness was introduced by Rickart in [10].

If a is a scalar valued finitely additive set function, then $|a|$ denotes the total variation of a (see for example Definition 4 of Chapter III in [4]).

Let m, M and $\mathscr{S}$ be as above. If a is a map from $\mathscr{S}$ into $\mathsf{R}_+=[0,\infty)$, we define:

(3) $m \ll a$, if and only if for any neighbourhood U of zero in E, there is a $d>0$ such that $m(A) \in U$ whenever $a(A)<d$, $A \in \mathscr{S}$.

(4) $M \ll a$, if and only if for any neighbourhood U of zero in E, there is a $d>0$ such that $m(A) \in U$ whenever $m \in M$, $A \in \mathscr{S}$, and $a(A)<d$.

An E-valued map m on Σ_0 is called *finitely additive*, if

$$m(A \cup B) = m(A)+m(B)\,,$$

whenever A,B are disjoint sets in Σ_0. The map m is called *σ-additive*, if

$$m\left(\textstyle\bigcup_{n=1}^\infty A_n\right) = \textstyle\sum_{n=1}^\infty m(A_n)$$

whenever $A_1,A_2,\ldots$ are mutually disjoint sets from Σ_0, such that $\bigcup_{n=1}^\infty A_n \in \Sigma_0$.

An E-valued σ-additive set function, m, defined on a σ-algebra is called a *vector measure*, or an *E-valued measure*.

Let m be a finitely additive set function defined on Σ_0 taking values in E; then we define:

$$N(m) = \{A \in \Sigma_0 \mid m(A\cap B)=0 \;\forall B \in \Sigma_0\}\,. \tag{5}$$

(6) $A \in \Sigma_0$ is an *m-atom*, if and only if for all $B \in \Sigma_0$ either $A \cap B \in N(m)$ or $A \setminus B \in N(m)$.

(7) $A \in \Sigma_0$ is a *proper m-atom*, if and only if A is an m-atom and $A \notin N(m)$.

(8) m is called *atomless*, if and only if m has no proper atoms.

(9) m is called *atomic*, if and only if there exists a sequence $\{A_n\}$ of m-atoms, such that $S = \bigcup_{n=1}^{\infty} A_n$.

Let m be an E-valued measure on Σ. Then m is called *absolutely continuous*, if there exists a positive finite measure, a, on Σ, such that $N(a) \subseteqq N(m)$, or equivalently such that $m \ll a$. Notice that

$$N(m) = \bigcap_{x' \in E'} N(x'm) .$$

So by Exercise 23, p. 141, in [14] we find that, if m is absolutely continuous, then there exists a positive finite measure a on Σ, such that $N(a) = N(m)$.

Let m be an E-valued finitely additive set function on Σ_0, and let q be a seminorm defined on E. Then the *q-variation* $|m|_q$ is defined by

$$(10) \qquad |m|_q(A) = \sup\{\textstyle\sum_{j=1}^{n} q(m(A_j))\}, \quad A \in \Sigma_0 ,$$

where the supremum is taken over all disjoint sets $A_1, \ldots, A_n$ from Σ_0 with $A = A_1 \cup \ldots \cup A_n$. The *q-semivariation* $q(m)$ is defined by

$$(11) \qquad q(m)(A) = \sup\{q(\textstyle\sum_{j=1}^{n} Z_j m(A_j))\}, \quad A \in \Sigma_0 ,$$

where the supremum is taken over all disjoint sets $A_1, \ldots, A_n$ from Σ_0, such that $A = A_1 \cup \ldots \cup A_n$, and all scalars $Z_1, \ldots, Z_n$, with $|Z_j| \leqq 1$ $\forall j = 1, \ldots, n$.

Let $A \subseteqq E$. Then $\mathrm{co}(A)$, $\overline{\mathrm{co}}(A)$, $\Gamma(A)$ and $\overline{\Gamma}(A)$ denote the convex hull of A, the closed convex hull of A, the convex balanced hull of A, and the closed convex balanced hull of A, respectively. The *polar* of A is denoted by A°, and is defined by

$$A^\circ = \{x' \in E' \mid |(x, x')| \leqq 1 \ \forall x \in A\} .$$

A locally convex space, E, is said to be (*sequentially*) *complete* if every (ordinary Cauchy sequence) generalized Cauchy sequence is convergent.

3. Weak compactness in $\mathrm{ca}(S, \Sigma)$ and $\mathrm{ba}(S, \Sigma)$.

In this section we give a charcterization of the weakly conditionally compact sets in $\mathrm{ba}(S, \Sigma)$ and $\mathrm{ca}(S, \Sigma)$. We prove that the weakly conditionally compact subsets of $\mathrm{ba}(S, \Sigma_0)$ are exactly the subsets, which are uniformly s-bounded on Σ_0. Further we prove that this criterion also holds in $\mathrm{ca}(S,\Sigma)$, still with Σ_0. The importance of the last criterion lies in the fact that weak conditional compactness in $\mathrm{ca}(S,\Sigma)$ is determined by the values of the measures on an algebra generating Σ.

First we prove a slight modification of a theorem due to Grothendieck (see [6, Theorem 2, p. 146–147]).

THEOREM 1. *Let T be a compact Hausdorff space, and $\mathscr{U}$ a base for the topology in T with the property:*

(a) *If $U_1, U_2 \in \mathscr{U}$, then $U_1 \cup U_2 \in \mathscr{U}$.*

If M is a bounded subset of $\mathrm{rca}(T)$, *then M is weakly conditionally compact, if and only if M is uniformly s-bounded on $\mathscr{U}$.*

PROOF. Let $\mathscr{V}$ be the family of all open subsets of T. Then Theorem 2, p. 146–147, in [6] states that the theorem holds if $\mathscr{V}=\mathscr{U}$. Hence it suffices to prove that if M is s-bounded on $\mathscr{U}$, then M is s-bounded on $\mathscr{V}$.

Suppose that M is not s-bounded on $\mathscr{V}$. Then there exist disjoint $\{V_n\} \subseteq \mathscr{V}$, and measures $\{a_n\} \subseteq M$, such that for some number $d>0$, we have

$$|a_n(V_n)| > d \quad \forall n \geqq 1 .$$

By regularity of $|a_n|$ we can find a compact set $K_n \subseteq V_n$, such that

$$|a_n|(V_n - K_n) < |a_n(V_n)| - d \quad \forall n \geqq 1 .$$

Since $\mathscr{U}$ is a base for the topology, which is closed under formations of finite unions, we can find $U_n \in \mathscr{U}$, such that

$$K_n \subseteq U_n \subseteq V_n \quad \forall n \geqq 1 .$$

Hence

$$|a_n(U_n)| \geqq |a_n(V_n)| - |a_n|(V_n - K_n) > d \quad \forall n \geqq 1 .$$

But $U_1, U_2, \ldots$ are mutually disjoint, and we see that M is not s-bounded on $\mathscr{U}$. Hence the theorem is proved.

We shall now deduce the criterions for weak conditional compactness in $\mathrm{ba}(S,\Sigma_0)$ and in $\mathrm{ca}(S,\Sigma)$ from this theorem by constructing some isometries between certain spaces of set functions.

LEMMA 1. *Let a be a complex valued finitely additive set function on Σ_0, and let Σ_1 be a subalgebra of Σ_0, such that*

(a) $$\forall \varepsilon > 0 \ \forall A \in \Sigma_0 \ \exists B \in \Sigma_1 \text{ such that } |a|(A \Delta B) < \varepsilon .$$

If a_0 is the restriction of a to Σ_1, then $|a_0|(A) = |a|(A)$ for all $A \in \Sigma_1$.

PROOF. Let $A \in \Sigma_1$. Then $|a_0|(A) \leqq |a|(A)$ for obvious reasons. Let $d < |a|(A)$. Then we can find disjoint sets $A_1,\ldots,A_n$ from Σ_0, such that $A_j \subseteqq A$ for all $j=1,\ldots,n$, and

$$\sum_{j=1}^{n} |a(A_j)| > d .$$

Let $\varepsilon = \sum_{j=1}^{n} |a(A_j)| - d > 0$. Then from (a) we can find $B_j \in \Sigma_1$ with

$$|a|(A_j \Delta B_j) \leqq n^{-2}\varepsilon \quad \forall j=1,\ldots,n .$$

Since $A \in \Sigma_1$ we may assume that $B_j \subseteqq A$ for all $j=1,\ldots,n$. Let

$$C_j = B_j \setminus \bigcup_{v=1}^{j-1} B_v \quad \forall j=1,\ldots,n .$$

Then $C_1,\ldots,C_n$ are disjoint sets in Σ_1 all contained in A, and obviously we have

$$|a|(A_j \Delta C_j) \leqq \sum_{v=1}^{j} |a|(A_v \Delta B_v) \leqq n^{-1}\varepsilon \quad \forall j=1,\ldots,n .$$

Hence for each $j=1,\ldots,n$ we find

$$|a(C_j)| \geqq |a(A_j)| - |a|(A_j \Delta C_j) \geqq |a(A_j)| - n^{-1}\varepsilon .$$

Summing over j we find

$$|a_0|(A) \geqq \sum_{j=1}^{n} |a(C_j)| \geqq \sum_{j=1}^{n} |a(A_j)| - \varepsilon = d .$$

Since this holds for any $d < |a|(A)$, we find that $|a_0|(A) \geqq |a|(A)$. And the Lemma 1 is proved.

LEMMA 2. *Let P be the restriction map from* $\mathrm{ca}(S,\Sigma)$ *to* $\mathrm{ca}(S,\Sigma_0)$, *that is, Pa is the restriction of a to Σ_0 for all $a \in \mathrm{ca}(S,\Sigma)$. Then P is an isometry from* $\mathrm{ca}(S,\Sigma)$ *onto* $\mathrm{ca}(S,\Sigma_0)$, *and* $|Pa| = P|a|$ *for all* $a \in \mathrm{ca}(S,\Sigma)$.

PROOF. P is obviously linear, and from the extension theorem for bounded complex valued measures we find that P maps $\mathrm{ca}(S,\Sigma)$ onto $\mathrm{ca}(S,\Sigma_0)$.

From Theorem D, § 13, in [7] and Lemma 1 it follows that $|Pa| = P|a|$ for all $a \in \mathrm{ca}(S,\Sigma)$, and so P is an isometry.

LEMMA 3. *Let T be a compact Hausdorff space, $\mathscr{B}$ the Borel σ-algebra and $\mathscr{B}_0$ the Baire σ-algebra. Let R be the restriction map from* $\mathrm{rca}(T)$ *into* $\mathrm{ca}(T,\mathscr{B}_0)$. *Then R is an isometry from* $\mathrm{rca}(T)$ *onto* $\mathrm{ca}(T,\mathscr{B}_0)$, *and* $|Ra| = R|a|$ *for all* $a \in \mathrm{rca}(T)$.

PROOF. From Theorem D, § 54, in [7] it follows that R maps $\mathrm{rca}(T)$ onto $\mathrm{ca}(T,\mathscr{B}_0)$. From Theorem H, §52, in [7] and Lemma 1 it follows that $|Ra| = R|a|$ for all $a \in \mathrm{rca}(T)$, and so R is an isometry.

LEMMA 4. *Let T be a compact totally disconnected Hausdorff space, and let $\mathscr{C}$ be the algebra of all open-closed sets in T. Let Q be the restriction map from* $\mathrm{rca}(T)$ *into* $\mathrm{ba}(T,\mathscr{C})$. *Then Q is an isometry from* $\mathrm{rca}(T)$ *onto* $\mathrm{ba}(T,\mathscr{C})$, *and* $|Qa|=Q|a|$ *for all* $a\in\mathrm{rca}(T)$.

PROOF. If $\{C_n\}$ is a sequence in $\mathscr{C}$ which decreases to $\emptyset$, then necessarily there exists an integer $k\geqq 1$, such that $C_n=\emptyset$ for all $n\geqq k$.

This argument shows that $\mathrm{ba}(T,\mathscr{C})=\mathrm{ca}(T,\mathscr{C})$. Let P be defined as in Lemma 2 with $S=T$, $\Sigma=\mathscr{B}_0$ and $\Sigma_0=\mathscr{C}$. From Theorem C, § 51, in [7] it follows that Σ is the σ-algebra generated by Σ_0, hence P is an isometry from $\mathrm{ca}(T,\mathscr{B}_0)$ onto $\mathrm{ba}(T,\mathscr{C})$. Let R be defined as in Lemma 3. Then $Q=RP$, hence Q is an isometry from $\mathrm{rca}(T)$ onto $\mathrm{ba}(T,\mathscr{C})$, and $|Qa|=Q|a|$ for all $a\in\mathrm{rca}(T)$.

THEOREM 2. *If M is a bounded subset of* $\mathrm{ba}(S,\Sigma_0)$, *then the following four statements are equivalent.*

(i) *M is weakly conditionally compact.*
(ii) *M is uniformly s-bounded on Σ_0.*
(iii) $\exists a\in\mathrm{ba}^+(S,\Sigma_0)$, *such that $M\ll a$.*
(iv) $M_0=\{|a| \mid a\in M\}$ *is weakly conditionally compact.*

PROOF. By Theorem 12, IV.9, in [4] (i) and (iii) are equivalent. Obviously (iii) implies (ii). If $M\ll a$, then necessarily $M_0\ll a$ and conversely. Hence (iii) and (iv) are equivalent. So the only implication, which is missing, is (ii) implies (i).

Suppose that M satisfies (ii). By a theorem of Kakutani (see for example Theorems 10 and 11, IV.9, in [4]) there exists a totally disconnected Hausdorff space T and a bijection from $\mathscr{C}$ onto Σ_0 satisfying

$$\begin{aligned} t(C\cup D) &= t(C)\cup t(D) \quad && \forall C,D\in\mathscr{C}\,,\\ t(T\setminus C) &= S\setminus t(C) && \forall C\in\mathscr{C}\,,\\ t(T) &= S\,, \end{aligned}$$

where $\mathscr{C}$ is the family of all open-closed subsets of T. Let V be defined by

$$(Va)(C) = a(t(C)) \quad \forall C\in\mathscr{C}\ \forall a\in\mathrm{ba}(S,\Sigma_0)\,.$$

Then V is an isometry from $\mathrm{ba}(S,\Sigma_0)$ onto $\mathrm{ba}(T,\mathscr{C})$. Let Q be defined as in Lemma 4. Then $L=Q^{-1}V$ is an isometry from $\mathrm{ba}(S,\Sigma_0)$ onto $\mathrm{rca}(Y)$, such that

$$(La)(C) = a(t(C)) \quad \forall C\in\mathscr{C}\ \forall a\in\mathrm{ba}(S,\Sigma_0)\,.$$

Since M is bounded, $L(M)$ is bounded. And the equation above shows that $L(M)$ is uniformly s-bounded on $\mathscr{C}$. So by Theorem 1, $L(M)$ is weakly conditionally compact, and since L is a surjective isometry, M is weakly conditionally compact.

THEOREM 3. *Let M be a bounded subset of* $\mathrm{ca}(S,\Sigma)$, *and let P be the restriction map from* $\mathrm{ca}(S,\Sigma)$ *into* $\mathrm{ba}(S,\Sigma_0)$. *Then the following 7 statements are equivalent.*

(i) *M is weakly conditionally compact.*
(ii) *M is uniformly s-bounded on Σ.*
(iii) $\exists a \in \mathrm{ca}^+(S,\Sigma)$ *such that* $M \ll a$.
(iv) *PM is weakly conditionally compact in* $\mathrm{ba}(S,\Sigma_0)$.
(v) *M is uniformly s-bounded on Σ_0.*
(vi) $\exists a \in \mathrm{ba}^+(S,\Sigma_0)$ *such that* $PM \ll a$.
(vii) $M_0 = \{|a| \mid a \in M\}$ *is weakly conditionally compact.*

PROOF. Theorems 2 and 1, IV.9, in [4] show that (i), (ii), (iii) and (vii) are equivalent, and (iv), (v) and (vi) are equivalent.

Since P is an isometry from $\mathrm{ca}(S,\Sigma)$ onto $\mathrm{ca}(S,\Sigma_0)$ (see Lemma 2) and $\mathrm{ca}(S,\Sigma_0)$ is weakly closed in $\mathrm{ba}(S,\Sigma_0)$, we find that (i) and (iv) are equivalent.

4. Weak σ-additivity.

Let m be a finitely additive set function on (S,Σ_0). We shall then deal with the following question: Knowing that $x'm$ is σ-additive for x' in a certain subspace F of E', can we then conclude that m itself is σ-additive?

A theorem of Pettis (see for example Theorem 1, IV.10, in [4]) states that, if Σ_0 is a σ-algebra, E is a Banach space, and $F=E'$, then our question has a positive answer. This theorem has an immediate extension to general locally convex spaces which is due to Metivier [8]. In Example 7 we show that this does not hold if Σ_0 is not a σ-algebra.

LEMMA 5. *Let q be a continuous seminorm on E and m a finitely additive set function on (S,Σ_0). Let U be the q-unit ball, that is, $U = \{x \in E \mid q(x) \leqq 1\}$. If F is a subspace of E' satisfying*

(a) $$q(x) = \sup\{|(x,x')| \mid x' \in U^\circ \cap F\},$$

then the q-semivariation is given by

$$q(m)(A) = \sup\{|x'm|(A) \mid x' \in U^\circ \cap F\} \qquad \forall A \in \Sigma_0 .$$

REMARK. (a) is satisfied, if and only if U is $\sigma(E,F)$-closed. Hence if $F=E'$, then (a) holds for any continuous seminorm q on E.

For suppose that U is $\sigma(E,F)$-closed. Then $U^\circ \cap F$ is the polar of U in F, and since U is convex and balanced, we find from Theorem 4, p. 35, in [10] that $U=(U^\circ \cap F)^\circ$. But from this (a) follows immediately.

Now suppose that (a) holds. Then obviously $U=(U^\circ \cap F)^\circ$, which shows that U is $\sigma(E,F)$-closed.

PROOF OF LEMMA 5. If $a \in \mathrm{ba}(S,\Sigma_0)$, then it is well known that

$$|a|(A) = \sup\{|\textstyle\sum_{j=1}^n t_j a(A_j)|\} \quad \forall A \in \Sigma_0 ,$$

where the supremum is taken over all Σ_0-partitions $A_1,A_2,\ldots,A_n$ of A and over all scalars $t_1,\ldots,t_n$ with $|t_j| \leqq 1$ for all $j=1,\ldots,n$.

Hence the lemma is an immediate consequence of (a) and the definition of $q(m)$.

PROPOSITION 1. *Let m be a bounded finitely additive set function on (S,Σ_0). Then the following four statements are equivalent.*

(i) *m is s-bounded on Σ_0.*

(ii) *$\{x'm \mid x' \in U^\circ\}$ is uniformly s-bounded on Σ_0, for every neighbourhood U of zero in E.*

(iii) *$q(m)$ is s-bounded for all $q \in \mathscr{P}$.*

(iv) *$\forall q \in \mathscr{P}\ \exists a \in \mathrm{ba}^+(S,\Sigma_0)$ such that $q(m) \ll a$.*

Here $\mathscr{P}$ is a family of continuous seminorms on E, which generates the topology in E.

REMARK. From Corollary 1, p. 507, in [15] it then follows that, if m is σ-additive on Σ_0 and E is sequentially complete, then m has a σ-additive extension to Σ, if and only if m is s-bounded.

PROOF OF PROPOSITION 1. Let $q \in \mathscr{P}$. Then by Lemma 5 we have

$$q(m(A)) = \sup\{|x'm(A)| \mid x' \in U^\circ\} \quad \forall A \in \Sigma ,$$
$$q(m)(A) = \sup\{|x'm|(A) \mid x' \in U^\circ\} \quad \forall A \in \Sigma ,$$

where U is the q-unit ball. Hence Proposition 1 follows easily from Theorem 2.

PROPOSITION 2. *Let m be a s-bounded finitely additive E-valued set function on (S,Σ_0). Let q be a continuous seminorm on E whose unit ball is $\sigma(E,F)$-closed, where F is a given subspace of E'.*

If $x'm$ is σ-additive on Σ_0 for all $x' \in F$, then $q(m)$ is continuous at $\emptyset$.

Proof. By Proposition 1 and Theorem 3, the family

$$M = \{|x'm| \mid x' \in F \cap U^\circ\}$$

is weakly conditionally compact in $\mathrm{ca}(S,\Sigma_0)$. Hence by Theorem 3 there exists $a \in \mathrm{ca}^+(S,\Sigma_0)$ such that $M \ll a$. Now let $\{A_n\}$ be a sequence in Σ_0 which decreases to $\emptyset$. Then $\lim_{n\to\infty} a(A_n)=0$, and so

$$|x'm|(A_n) \to 0 \text{ uniformly for } x' \in U^\circ \cap F \text{ as } n \to \infty .$$

Hence $\lim_{n\to\infty} q(m)(A_n)=0$ by Lemma 5. Which proves Proposition 2.

Proposition 3. *Let m be a finitely additive s-bounded set function on (S,Σ_0) taking values in E. If E is metrizable, then there exists $a \in \mathrm{ba}^+(S,\Sigma_0)$ such that $q(m) \ll a$ for every continuous seminorm q on E.*

Proof. Since E is metrizable there exist continuous seminorms $q_1 \leqq q_2 \leqq \ldots$ such that $\{q_n \mid n \geqq 1\}$ generates the topology of E. By Proposition 1 we can find $a_n \in \mathrm{ba}^+(S,\Sigma_0)$ such that $q_n(m) \ll a_n$. Now let

$$a(A) = \Sigma_{n=1}^{\infty} 2^{-n}\, a_n(A)/a_n(S), \quad A \in \Sigma_0 .$$

Then $a \in \mathrm{ba}^+(S,\Sigma_0)$, and $q_n(m) \ll a \ \forall n \geqq 1$.

If q is a continuous seminorm on E, then there exist an integer k and a number $M>0$, such that $q \leqq Mq_k$. Hence $q(m) \leqq Mq_k(m)$, and so $q(m) \ll a$.

Proposition 4. *Let m be an E-valued measure on (S,Σ). If there exists a metrizable locally convex Hausdorff topology on E which is weaker than the original topology, then m is absolutely continuous.*

Remark. The hypothesis of the proposition is particularly satisfied, if there exists a countable subset of E' which is $\sigma(E',E)$-dense in E', or equivalently, which separates points in E.

Proof of Proposition 4. By hypothesis there exist continuous seminorms $q_1 \leqq q_2 \leqq \ldots$ such that $q_n(x)=0 \ \forall n \geqq 1$ implies $x=0$. By Proposition 1 and Theorem 3, there exist $a_n \in \mathrm{ca}^+(S,\Sigma)$, such that $q_n(m) \ll a_n$ $\forall n \geqq 1$. Now let

$$a(A) = \Sigma_{n=1}^{\infty} 2^{-n}\, a_n(A)/a_n(S) \quad \forall A \in \Sigma .$$

Then $a \in \mathrm{ca}^+(S,\Sigma)$ and $q_n(m) \ll a \;\forall n \geqq 1$.

If $A \in N(a)$, then $q_n(m(B)) \leqq q_n(m)(B) = 0$ for all $n \geqq 1$ and all $B \in \Sigma$ with $B \subseteqq A$. Hence $m(B) = 0$ for $B \in \Sigma$ with $B \subseteqq A$, that is, $A \in N(m)$, and so $N(a) \subseteqq N(m)$.

THEOREM 4. *Let m be an E-valued s-bounded finitely additive set function on (S,Σ_0), and let F be a linear subspace of E', such that*
(a) *$x'm$ is σ-additive on Σ_0 $\forall x' \in F$,*
(b) *E has a base at zero, consisting of $\sigma(E,F)$-closed convex balanced sets.*
Then m is σ-additive on Σ_0.

PROOF. (b) states that there exists a family $\mathscr{P}$ of continuous seminorms such that the q-unit ball $U(q) = \{x \in E \mid q(x) \leqq 1\}$ is $\sigma(E,F)$-closed for all $q \in \mathscr{P}$, and such that $\mathscr{P}$ generates the topology of E. Hence Theorem 4 is a consequence of Proposition 2.

COROLLARY 1. *Let E be the dual space of the locally convex space F, and suppose that the topology τ in E is the topology of uniform convergence on A, $A \in \mathscr{A}$, where $\mathscr{A}$ is a class of bounded subsets of F with $\bigcup\{A \mid A \in \mathscr{A}\} = F$.*

If m is an E-valued s-bounded finitely additive set function on (S,Σ_0), such that $(y, m(\cdot))$ is σ-additive for all $y \in F$, then m is σ-additive.

PROOF. Let $\mathscr{A}^*$ denote the family of all subsets of sets of the form $\Gamma(\bigcup_{i=1}^n A_i)$ with $A_1, \ldots, A_n \in \mathscr{A}$. Since F may be considered a subset of E', and since $\{A^\circ \mid A \in \mathscr{A}^*\}$ is a base for the topology in E at zero consisting of $\sigma(E,F)$-closed convex balanced sets, Corollary 1 follows from Theorem 4.

5. The atomic structure of vector measures.

In this section we shall study the atoms of vector measures. The main result states that, under mild restriction, every vector measure m can be decomposed uniquely as the sum of an atomic and an atomless vector measure; and this decomposition arises from a decomposition of S into two complementary sets such that m is atomic on the one and atomless on the other.

First we need some preparatory lemmas.

LEMMA 6. *Let m_1 and m_2 be two E-valued measures on (S,Σ). If $N(m_2) \subseteqq N(m_1)$, then every m_2-atom is an m_1-atom.*

Let a_1 and a_2 be two complex valued measures on (S,Σ). If $N(a_2) \subseteq N(a_1)$, then every proper a_1-atom contains a proper a_2-atom.

Proof. The first statement follows immediately from the definition. Now let A be a proper a_1-atom. By the Radon-Nikodym theorem there exists a nonnegative $|a_2|$-integrable function f such that

$$|a_1|(B) = \int_B f\, d|a_2| \qquad \forall B \in \Sigma .$$

Let $A(t) = \{s \in A \mid f(s) \geqq t\}$ for $t \geqq 0$. Then $A(0) = A$ and

$$\lim_{t\to\infty} |a_1|(A(t)) = 0 .$$

Since A is a proper a_1-atom, we find that if

$$t_0 = \sup\{t \geqq 0 \mid |a_1|(A(t)) > 0\} ,$$

then $0 < t_0 < \infty$. Now $|a_1|(A(t)) = 0$ if $t > t_0$. Then

$$|a_2|(s \in A \mid f(s) > t_0) = 0 .$$

By definition of $A(t)$ we see that $|a_1|(A(\cdot))$ is left continuous. Hence $|a_1|(A \setminus A(t_0)) = 0$, and so

$$|a_2|(s \in A \mid 0 < f(s) < t_0) = 0 .$$

Now $|a_1|(A) > 0$, so f is not zero a.e. in A with respect to $|a_2|$. Hence the set

$$A_0 = \{s \in A \mid f(s) = t_0\}$$

must have positive $|a_2|$-measure, and since

$$|a_1|(B) = t_0 |a_2|(B) \quad \forall B \in \Sigma \text{ such that } B \subseteq A_0 ,$$

we see that A_0 is a proper a_2-atom.

Corollary 2. *Let m_1 and m_2 be two E-valued measures such that $N(m_2) \subseteq N(m_1)$. If m_2 is atomic, then so is m_1.*

If a_1 and a_2 are two complex-valued measures such that $N(a_2) \subseteq N(a_1)$ and a_2 is atomless, then a_1 is atomless.

Proposition 5. *If m is an E-valued measure on (S,Σ), then $A \in \Sigma$ is an m-atom, if and only if A is an $x'm$-atom for all $x' \in E'$.*

If A is a proper m-atom, then for some $x' \in E'$ the set A is a proper $x'm$-atom.

If m is absolutely continuous and A is a proper $x'm$-atom for some $x' \in E'$, then A contains a proper m-atom.

PROOF. Let $M=\{x'm \mid x' \in E'\}$, then M is a convex subset of $\mathrm{ca}(S,\Sigma)$. Since $N(m) \subseteqq N(b)$, every m-atom is a b-atom for all b in M.

Now let A be a b-atom for all $b \in M$, and let $B \in \Sigma$, $B \subseteqq A$, such that $B \notin N(m)$. We shall then show, that $A \setminus B \in N(m)$. Suppose this was not true. Then there exists $b_1 \in M$, such that $A \setminus B \notin N(b_1)$. We know that $B \notin N(m)$. Hence $B \notin N(b_2)$ for some $b_2 \in M$. Let $b=b_1+b_2$. Then $b \in M$. Since A is a b_1-atom and a b_2-atom, $B \in N(b_1)$ and $A \setminus B \in N(b_2)$. Thus

$$\begin{aligned} b(C) &= b_2(C) \neq 0 \qquad \text{for } C \subseteqq B,\ C \in \Sigma, \\ b(A \setminus C) &= b_1(A \setminus C) \neq 0 \quad \text{for } C \subseteqq B,\ C \in \Sigma. \end{aligned}$$

But this contradicts the fact that A is a b-atom.

Now suppose that A is a proper m-atom. Then for some $b \in M$, $A \notin N(b)$, and since A is a b-atom, A is a proper b-atom.

Now suppose that A is a proper b-atom and m is absolutely continuous. Then there exists $a \in \mathrm{ca}^+(S,\Sigma)$, such that $N(a)=N(m) \subseteqq N(b)$. Hence by Lemma 6, A contains a proper a-atom, but this must necessarily be a proper m-atom, since $N(a)=N(m)$.

COROLLARY 3. *Let m be an E-valued measure on (S,Σ).*

If $x'm$ is atomless for all $x' \in E'$, then m is atomless.

If m is atomic, then $x'm$ is atomic for all $x' \in E'$.

If m is absolutely continuous and m is atomless, then $x'm$ is atomless for all $x' \in E'$.

If m is absolutely continuous and $x'm$ is atomic for all $x' \in E'$, then m is atomic.

REMARK. In Example 1 in Section 7 we shall construct a vector measure $m \neq 0$ such that m is atomless and $x'm$ is atomic for all $x' \in E'$. That is, the hypothesis of absolute continuity of m in the last two statements in Corollary 3 cannot be suppressed in general. But it should be noticed that by Proposition 4 this cannot happen in "nice" spaces.

THEOREM 5. *Let m_1 and m_2 be two E-valued measures on (S,Σ).*

If m_1 and m_2 are atomic, then m_1+m_2 is atomic.

If m_1 and m_2 are absolutely continuous and atomless, then m_1+m_2 is atomless.

REMARK. In Example 2 we shall construct two atomless measures, m_1 and m_2, such that $m_1+m_2 \neq 0$ and m_1+m_2 is atomic. Hence the hypothesis of absolute continuity cannot be suppressed in general. But it should be noticed that by Proposition 4 this cannot happen if E is a "nice" space.

Proof of Theorem 5. Let $m = m_1 + m_2$ and suppose that A is an atom for m_1 and for m_2. We then show that A is a union of at most two m-atoms. We divide the discussion in three cases.

Case 1. $\exists C \in \Sigma,\ C \subseteqq A$, such that $C \in N(m_1) \setminus N(m_2)$. Then $D = A \setminus C \in N(m_2)$, and hence we find

$$\begin{aligned} m(B) &= m_1(B) \quad \forall B \subseteqq D,\ B \in \Sigma\,, \\ m(B) &= m_2(B) \quad \forall B \subseteqq C,\ B \in \Sigma\,. \end{aligned}$$

But this shows that C and D are atoms for m.

Case 2. $\exists C \in \Sigma,\ C \subseteqq A$, such that $C \in N(m_2) \setminus N(m_1)$. This case is treated similarly to Case 1.

Case 3. $\{C \mid C \subseteqq A, C \in N(m_1)\} = \{C \mid C \subseteqq A, C \in N(m_2)$. Let $C \in \Sigma$, $C \subseteqq A$, if $C \notin N(m)$. Then either $C \notin N(m_1)$ or $C \notin N(m_2)$, so by assumption, $C \notin N(m_1) \cup N(m_2)$. Since A is an m_1-atom and an m_2-atom,

$$A \setminus C \in N(m_1) \cap N(m_2) \subseteqq N(m_1 + m_2)\,.$$

That is, A is an m-atom.

Hence we have proved our statement.

Now suppose that m_1 and m_2 are atomic. Then we can find disjoint m_1-atoms $S_1, S_2, \ldots$, and disjoint m_2-atoms $T_1, T_2, \ldots$, such that

$$S = \textstyle\bigcup_{n=1}^{\infty} S_n = \bigcup_{n=1}^{\infty} T_n\,.$$

Since $S_n \cap T_k$ is an m_1-atom and an m_2-atom for all $n, k \geqq 1$, the above argument shows that $S_n \cap T_k$ is a disjoint union of two m-atoms, and since

$$S = \textstyle\bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} S_n \cap T_k\,,$$

we have proved that m is atomic.

Now let m_1 and m_2 be absolutely continuous and atomless. Then by Corollary 3, $x'm_1$ and $x'm_2$ are atomless for all $x' \in E'$. So by 5.14 in [12] the sum $x'm = x'm_1 + x'm_2$ is atomless for all $x' \in E'$. Hence by Corollary 3, m is atomless.

Theorem 6. *Let m be an absolutely continuous E-valued measure on (S, Σ). Then there exist unique absolutely continuous E-valued measures m_1 and m_2 on (S, Σ) satisfying:*

(i) *m_1 is atomless, m_2 is atomic.*
(ii) $m = m_1 + m_2$.

Furthermore there exists $S_0 \in \Sigma$ such that S_0 is a countable union of disjoint proper m-atoms and $S \setminus S_0$ contains no proper m-atom. The measures m_1 and m_2 are given by
(iii) $m_1(A) = m(A \setminus S_0)$, $A \in \Sigma$,
(iv) $m_2(A) = m(A \cap S_0)$, $A \in \Sigma$.

REMARK. Examples 2 and 3 in Section 7 show that neither the uniqueness part nor the existence part is true in general (that is, without the assumption of absolute continuity of m, m_1 and m_2). But it should be noticed that by Proposition 4 this cannot happen, when E is a "nice" space.

PROOF OF THEOREM 6. Let $a \in \mathrm{ca}^+(S,\Sigma)$, such that $N(a) = N(m)$. From 5.4.55 in [12] we deduce that there exists $S_0 \in \Sigma$ such that S_0 is a countable union of disjoint proper a-atoms and $S \setminus S_0$ contains no proper a-atoms. Since a and m have the same atoms and the same nullsets, the last part of Theorem 6 is proved.

Let m_1 and m_2 be defined by (iii) and (iv) and define a_1 and a_2 similarly, that is,

$$a_1(A) = a(A \setminus S_0), \quad A \in \Sigma,$$
$$a_2(A) = a(A \cap S_0), \quad A \in \Sigma.$$

Then obviously $N(a_1) = N(m_1)$ and $N(a_2) = N(m_2)$. Hence m_1 and m_2 are absolutely continuous. Since a_1 is atomless and a_2 is atomic, m_1 is atomless and m_2 is atomic. Obviously, $m = m_1 + m_2$, so the existence of the decomposition is proved.

If a vector measure is atomic and atomless at the same time, then it necessarily must vanish identically. Hence the uniqueness follows from Theorem 5.

6. The range of a vector measure.

Let m be a finitely additive set function on (S,Σ_0). If m is bounded, then the range $m(\Sigma_0)$ of m is bounded. If E is finite-dimensional, then $m(\Sigma_0)$ is conditionally compact. In this section we shall study the properties of the range of m. In particular we shall prove that $m(\Sigma_0)$ is weakly conditionally compact under fairly mild restrictions on m and E.

In order to study the range of m, we consider the integral operator going along with m. Let $\mathscr{S}(S,\Sigma_0)$ be the space of all Σ_0-simple scalar valued functions on S, and let $B(S,\Sigma_0)$ be the closure of $\mathscr{S}(S,\Sigma_0)$ under the norm

$$\|f\| = \sup\{|f(s)| \mid s \in S\}.$$

If $f=\sum_{j=1}^{n} t_j 1_{A_j}$ is a Σ_0-simple function, we define the integral in the usual way, that is

$$\int_S f\,dm = I_m(f) = \sum_{j=1}^{n} t_j m(A_j).$$

There is no difficulty in proving that I_m is well-defined (that is, $I_m(f)$ is independent of the particular representation of f).

Then I_m is a linear map from $\mathscr{S}(S,\Sigma_0)$ into E. If q is a continuous seminorm on E, then by the very definition of $q(m)$,

$$q(m)(A) = \sup\left\{q\Big(\int_A f\,dm\Big) \,\Big|\, f \in \mathscr{S}(S,\Sigma_0), \|f\| \leqq 1\right\}. \tag{12}$$

Hence if m is bounded, then I_m is continuous. If E is sequentially complete and m is bounded, then I_m has a unique extension to $B(S,\Sigma_0)$ since $\mathscr{S}(S,\Sigma_0)$ is dense in $B(S,\Sigma_0)$ and I_m is continuous. This extension will still be denoted by I_m.

The dual space of $\mathscr{S}(S,\Sigma_0)$ is $\mathrm{ba}(S,\Sigma_0)$ (see for example Theorem 1, IV.5, in [4]). If m is bounded, then $x'm \in \mathrm{ba}(S,\Sigma_0)$ for $x' \in E'$, and the transposed I_m' of I_m is given by

$$I_m' x' = x'm \qquad \forall x' \in E'.$$

Lemma 7. *Let B^+ be the positive part of the unit ball in $\mathscr{S}(S,\Sigma)$, that is,*

$$B^+ = \{f \in \mathscr{S}(S,\Sigma_0) \mid f \geqq 0, \|f\| \leqq 1\}.$$

If m is a finitely additive set function on (S,Σ_0), then

$$I_m(B^+) = \mathrm{co}\big(m(\Sigma_0)\big).$$

Proof. Let $f \in B^+$, then we can find disjoint nonempty sets $A_1,\dots,A_n$ in Σ_0 and numbers $0 \leqq a_1 \leqq a_2 \leqq \dots \leqq a_n \leqq 1$ such that

$$f = \sum_{j=1}^{n} a_j 1_{A_j}.$$

Now let $b_j = a_j - a_{j-1}$ for $j=1,\dots,n+1$, where $a_0=0$ and $a_{n+1}=1$, and let

$$B_j = \bigcup_{v=j}^{n} A_v,\ j=1,\dots,n+1, \qquad B_{n+1}=\emptyset.$$

Then

$$f = \sum_{j=1}^{n+1} b_j 1_{B_j},$$

and so we find

$$I_m(f) = \sum_{j=1}^{n+1} b_j m(B_j) \in \mathrm{co}\big(m(\Sigma_0)\big)$$

since $b_j \geqq 0$ and $\sum_{j=1}^{n+1} b_j = 1$. This means that $I_m(B^+) \subseteq \text{co}(m(\Sigma_0))$, and since the converse inclusion is obvious, the lemma is proved.

PROPOSITION 6. *Let m be an E-valued finitely additive set function on (S,Σ_0) such that $\text{co}(m(\Sigma_0))$ is weakly conditionally compact. Then m is s-bounded, and I_m is a weakly compact linear operator.*

PROOF. From Lemma 7 it follows immediately that I_m is weakly compact.

From Lemma 7, p. 153, in [10] it follows that $I_m'(U^\circ)$ is weakly compact in $\text{ba}(S,\Sigma_0)$ for every neighbourhood, U, of zero in E. Hence from Proposition 1 and Theorem 2 we deduce that m is s-bounded.

THEOREM 7. *Let m be a bounded E-valued finitely additive set function. If E is quasicomplete (that is every bounded closed set in E is complete), then the following three statements are equivalent.*

(i) *m is s-bounded.*
(ii) *$m(\Sigma_0)$ is weakly conditionally compact.*
(iii) *I_m is a weakly compact operator.*

REMARK. Example 4 in Section 7 shows that, even if E is weakly sequentially complete, then $m(\Sigma_0)$ may not be weakly conditionally compact. Hence the assumption of quasicompleteness of E cannot be suppressed in general.

PROOF. Suppose that m is s-bounded. Let $F = (\mathscr{S}(S,\Sigma_0), \|\cdot\|)$, $F' = (\text{ba}(S,\Sigma_0), \|\cdot\|)$, and F'' the dual of F' equipped with the $\tau(F'',F')$-topology. Let $G = (E')^*$, that is, G is the algebraic dual of E', and let ξ be the topology on G of uniform convergence on equicontinuous subsets of E'. Then F is a subspace of F'', and the transposed of I_m', I_m'', is an extension of I_m to F''; and I_m'' maps F'' into G.

Let A' be an equicontinuous subset of E'. If A'^0 is the polar of A' taken in G, then

$$(I_m'')^{-1}(A'^\circ) = (I_m'(A'))^\circ ,$$

where the polar on the right side is taken in F''. By Proposition 1, $I_m'(A')$ is contained in a convex $\sigma(F',F'')$-compact set. Hence $(I_m'')^{-1}(A'^\circ)$ is a $\tau(F'',F')$ neighbourhood in F''. That is, I_m'' is a continuous linear map from $(F'',\tau(F'',F'))$ into (G,ξ).

Let B be the unit ball in F and B'' the unit ball in F''. Then B is $\sigma(F'',F')$-dense in B'' (see for example Theorem 5, V.4, in [4]). Since B is convex, B is also $\tau(F'',F')$-dense in B''. So if $z'' \in B''$, then we can find a generalized sequence $\{z_\alpha\}\subseteqq B$ which converges to z'' in $\tau(F'',F')$. Hence $\{I_m(z_\alpha)\}$ converges to $I_m''(z'')$ in ξ.

Let U be a closed convex balanced neighbourhood of zero in E. Then U° is equicontinuous, and so by definition of ξ we can find α_0, such that

$$|(I_m(z_\alpha)-I_m(z_\beta),x')| \leqq 1 \quad \forall x' \in U^\circ \ \forall \alpha,\beta \geqq \alpha_0 .$$

Hence $I_m(z_\alpha)-I_m(z_\alpha) \in U \ \forall \alpha,\beta \geqq \alpha_0$. This shows that $\{I_m(z_\alpha)\}$ is a generalized Cauchy sequence in E, and it is bounded since $I_m(B)$ is bounded. So by quasicompleteness of E, there exists $x_0 \in E$ such that $x_0 = \lim_\alpha I_m(z_\alpha) = I_m''(z'')$.

This shows that $I_m''(B'')\subseteqq E$, and since $F''=\bigcup_{n=1}^\infty nB''$, we find that $I_m''(F'')\subseteqq E$. Hence by Lemma 7, p. 153, in [10], I_m is weakly compact.

So we have proved that (i) implies (iii). Since $m(\Sigma_0)\subseteqq I_m(B)$, we find that (iii) implies (ii). If $m(\Sigma_0)$ is weakly conditionally compact, then so is $\mathrm{co}(m(\Sigma_0))$ (see for example (4′), p. 328, in [11]), and so by Proposition 6, m is s-bounded. That is, (ii) implies (i), and the theorem is proved.

THEOREM 8. *Let m be an absolutely continuous E-valued measure on (S,Σ). If E is sequentially complete, then $\overline{\mathrm{co}}(m(\Sigma))$ is weakly compact.*

PROOF. Let $a \in \mathrm{ca}^+(S,\Sigma)$, such that $N(a)\subseteqq N(m)$. If f and g are Σ-simple function such that $f=g$ a.e. with respect to a, then obviously $I_m(f) = I_m(g)$.

Since E is sequentially complete, I_m is defined on all of $B(S,\Sigma)$, and by the argument above we find that $I_m(f)=I_m(g)$ whenever f and g are functions in $B(S,\Sigma)$ such that $f=g$ a.e. with respect to a.

This means that we can consider I_m as defined on $L_\infty(S,\Sigma,a)$.

Let $x_0' \in E'$. Then $N(a)\subseteqq N(x_0'm)$, and so by the Radon–Nikodym theorem there exists $g_0 \in L_1(S,\Sigma,a)$, such that

$$\int_S f\,dx_0'm = \int_S fg_0\,da = (x_0',I_m(f)) \quad \forall f \in L_\infty(S,\Sigma,a) .$$

Hence I_m is a continuous map from $(L_\infty(S,\Sigma,a),\sigma(L_\infty,L_1))$ into $(E,\sigma(E,E'))$.

The unit ball B in $L_\infty(S,\Sigma,a)$ is $\sigma(L_\infty,L_1)$-compact. Hence $I_m(B)$ is weakly compact, and so the theorem follows from Lemma 7.

THEOREM 9. *Let m be a bounded finitely additive set function on (S,Σ_0). If E is quasicomplete, the following five statements are equivalent.*

(i) *$m(\Sigma_0)$ is conditionally compact.*
(ii) *$\overline{\text{co}}(m(\Sigma_0))$ is compact.*
(iii) *I_m is a compact linear operator.*
(iv) *For any $q \in \mathscr{P}$, there exist $a \in \text{ba}^+(S,\Sigma_0)$, and a sequence $\{f_n\}$ of E-valued Σ_0-simple functions, such that uniformly for $B \in \Sigma_0$,*

$$q\left(\int_B f_n da - m(B)\right) \to 0 \quad \text{as } n \to \infty .$$

(v) *$\{x'm \mid x' \in U^\circ\}$ is conditionally compact in* $\text{ba}(S,\Sigma_0)$.

Above, $\mathscr{P}$ is a family of continuous seminorms on E generating the topology of E.

REMARK. It is easily seen that (iv) implies that the total q-variation $|m|_q(S)$ is finite for every continuous seminorm q on E.

In Example 6 we shall see that even if E is a separable Banach space, m is σ-additive on Σ, and the total variation of m is finite, we cannot conclude that $m(\Sigma)$ is conditionally compact.

PROOF OF THEOREM 9. By the quasicompleteness of E and Lemma 7 one finds immediately that (i), (ii) and (iii) are equivalent (see also (4), p. 328, in [11]). By Lemma 7, p. 153, in [10] one finds that (iii) implies (v).

Now let us prove that (v) implies (iv). Let $q \in \mathscr{P}$ and let U be the q-unit ball. Then

$$\{x'm \mid x' \in U^\circ\} = M$$

is conditionally compact in $\text{ba}(S,\Sigma_0)$. So by Exercise 19, IV.13, in [4] there exist $a \in \text{ba}^+(S,\Sigma_0)$ and Σ_0-partitions $A_1^n,\ldots,A_{k(n)}^n$ of S, such that $a(A_j^n) > 0$, and

$$|\Sigma_{j=1}^{k(n)} x'\, m(A_j^n)\, a(B \cap A_j^n)/a(A_j^n) - a(B)| \leqq n^{-1}$$

$\forall x' \in U^\circ$, $\forall B \in \Sigma_0$ and $\forall n \geqq 1$. The function

$$f_n = \Sigma_{j=1}^{k(n)} a(A_j^n)^{-1}\, 1_{A_j^n}\, m(A_j^n)$$

is an E-valued Σ_0-simple function, such that

$$\int_B f_n\, da = \Sigma_{j=1}^{k(n)} m(A_j^n)\, a(B \cap A_j^n)\, a(A_j^n)^{-1} \quad \forall B \in \Sigma_0 .$$

Hence we find for all $n \geqq 1$

$$\left|\left(x', \int_B f_n da - m(B)\right)\right| \leqq n^{-1} \quad \forall B \in \Sigma_0 \;\; \forall x' \in U^\circ ,$$

and so

$$q\left(\int_B f_n da - m(B)\right) \leqq n^{-1} \quad \forall B \in \Sigma_0 \;\; \forall n \geqq 1 ,$$

which proves (iv).

Finally we prove that (iv) implies (i). Let $q \in \mathscr{P}$ and let $a \in \mathrm{ba}^+(S, \Sigma_0)$ and $\{f_n\}$ be chosen according to (iv). Then we define

$$m_n(B) = \int_B f_n da, \quad B \in \Sigma_0 .$$

Then m_n is bounded and $m_n(\Sigma_0)$ is finite dimensional. Hence $m_n(\Sigma_0)$ is conditionally compact. If $\varepsilon > 0$ we can choose an integer $k \geqq 1$, such that

$$q(m_k(B) - m(B)) < \tfrac{1}{2}\varepsilon \quad \forall B \in \Sigma_0 .$$

Since $m_k(\Sigma_0)$ is conditionally compact we can find $x_1, \ldots, x_p \in E$ such that

$$m_k(\Sigma_0) \subseteqq \bigcup_{v=1}^{p} (x_v + \tfrac{1}{2}\varepsilon U) ,$$

where U is the q-unit ball. But this obviously implies that

$$m(\Sigma_0) \subseteqq \bigcup_{v=1}^{p} (x_v + \varepsilon U) .$$

Since $\mathscr{P}$ generates the topology of E, this shows that $m(\Sigma_0)$ is precompact in E. But E is quasicomplete, and so $m(\Sigma_0)$ is conditionally compact.

THEOREM 10. *Let m be an atomic E-valued measure on (S, Σ), then $m(\Sigma)$ is compact.*

If E is sequentially complete, then $\overline{\mathrm{co}}(m(\Sigma))$ is compact.

PROOF. Let $S_1, S_2, \ldots$ be proper disjoint m-atoms in S, such that $S = \bigcup_{n=1}^{\infty} S_n$, and let $x_n = m(S_n)$. Let $C = \{0,1\}^\infty$ be the Cantor set. If $\xi = (\xi_n) \in C$, we define

$$f(\xi) = \sum_{n=1}^{\infty} \xi_n x_n = m\left(\bigcup_{n \in B} S_n\right) ,$$

where $B = \{n \mid \xi_n = 1\}$. Then f is a map from C into $m(\Sigma)$. Let $A \in \Sigma$ and define

$$\begin{aligned} \xi_n &= 1 \quad \text{if } A \cap S_n \notin N(m) , \\ &= 0 \quad \text{if } A \cap S_n \in N(m) . \end{aligned}$$

Then $f(\xi) = m(A)$, and so $f(C) = m(\Sigma)$.

Let $\xi, \eta \in C$, such that $\xi_j = \eta_j$ for $j = 1, \ldots, k$, and let q be a continuous seminorm on E. If we put $B_1 = \{n \mid \xi_n = 1\}$ and $B_2 = \{n \mid \eta_n = 1\}$, then

$$q(f(\xi) - f(\eta)) = q\big(m(\textstyle\bigcup_{n \in B_1 \setminus B_2} S_n) - m(\bigcup_{n \in B_2 \setminus B_1} S_n)\big) \leqq q(m)(A_k) ,$$

where $A_k = \bigcup_{n=k+1}^{\infty} S_n$, since $B_1 \Delta B_2 \subseteq \{k+1, k+2, \ldots\}$. By Proposition 2,

$$\lim_{k \to \infty} q(m)(A_k) = 0 .$$

Hence f is a continuous map from C onto $m(\Sigma)$. But this implies that $m(\Sigma)$ is compact.

Now suppose that E is sequentially complete. Let $D = [0,1]^{\infty}$ be the infinite dimensional cube equipped with the product topology. If $\xi = (\xi_n) \in D$ and q is a continuous seminorm, then

$$q(\textstyle\sum_{j=n+1}^{n+m} \xi_j x_j) \leqq q(m)(A_n) ,$$

so $\{\sum_{j=1}^{n} \xi_j x_j\}_{n=1}^{\infty}$ is a Cauchy sequence in E by Proposition 2. Therefore

$$g(\xi) = \textstyle\sum_{j=1}^{\infty} \xi_j x_j$$

exists for all $\xi \in D$. Let $\xi, \eta \in D$ and let q be a continuous seminorm on E. Then

$$q(g(\xi) - g(\eta)) \leqq \textstyle\sum_{j=1}^{n} |\xi_j - \eta_j| q(x_j) + 2q(m)(A_n) \quad \forall n \geqq 1 .$$

Since $\lim_{n \to \infty} q(m)(A_n) = 0$ by Proposition 2, this shows that g is continuous. Hence $g(D)$ is compact, and since g is an extension of f and $g(D)$ is convex, we find that $\overline{\text{co}}(m(\Sigma))$ is compact.

A theorem of Liapounov (see [13]), states that an atomless finite-dimensional vector measure has a compact convex range. Liapounov also shows in [13] that this is not true in general. We shall here show an analogue theorem, which states that the weak closure of the range of an atomless vector measure is convex.

THEOREM 11. *Let m be a bounded finitely additive set function on Σ_0, such that $x'm$ is σ-additive on Σ_0 for all $x' \in E'$. Let $\overline{x'm}$ be the unique extension of $x'm$ to Σ. If $\overline{x'm}$ is atomless for all x' in E', then the weak closure of $m(\Sigma_0)$ is equal to $\overline{\text{co}}(m(\Sigma_0))$.*

REMARK. In Example 5 we construct a vector measure m taking values in a separable Hilbert space such that m has finite total variation, m is atomless, the range of m is conditionally compact, but $m(\Sigma)$ is not convex and not closed.

Notice that if $m(\Sigma_0)$ is conditionally compact and m satisfies the hypothesis of the theorem, then the closure of $m(\Sigma_0)$ is equal to $\overline{\text{co}}(m(\Sigma_0))$.

In Example 6 we construct a vector measure m taking values in a separable Banach space, such that m is atomless, m has finite total variation, the range of m is closed, but not convex and not weakly closed.

Proof of Theorem 11. It suffices to prove that $m(\Sigma_0)$ is weakly dense in $\mathrm{co}(m(\Sigma_0))$. So let $x_0 \in \mathrm{co}(m(\Sigma_0))$ and let $x_1',\ldots,x_n' \in E'$. We can then define the measure c by

$$c(A) = (\overline{x_1'm}(A),\ldots,\overline{x_n'm}(A)), \quad A \in \Sigma .$$

Then c is atomless and $((x_1',x_0),\ldots,(x_n',x_0)) \in \mathrm{co}(c(\Sigma))$, so by Liapounov's theorem (see [13]) there exists $A \in \Sigma$, such that

$$\overline{x_j'm}(A) = (x_j',x_0) \quad \forall j = 1,\ldots,n .$$

By Theorem D, § 13, in [7] there exist $B_j \in \Sigma_0$, such that

$$|\overline{x_j'm}|(A \Delta B_j) \leqq n^{-1} \quad \forall j = 1,\ldots,n .$$

If $B = \bigcup_{j=1}^n B_j$, then $B \in \Sigma_0$ and $A \Delta B \subseteqq \bigcup_{j=1}^n A \Delta B_j$. Hence

$$|\overline{x_j'm}|(A \Delta B) \leqq 1 \quad \forall j = 1,\ldots,n .$$

But from this we find that

$$|(x_j',m(B)-x_0)| = |\overline{x_j'm}(B)-\overline{x_j'm}(A)| \leqq |\overline{x_j'm}|(A \Delta B) \leqq 1$$

$\forall j = 1,\ldots,n$, which shows that x_0 belongs to the weak closure of $m(\Sigma_0)$.

7. Counter examples.

In this section we shall give some examples of additive vector valued set functions, which disprove many natural conjectures.

Example 1. We shall construct an atomless vector measure $m \neq 0$, such that $x'm$ is atomic for all $x' \in E'$.

If S is a set, a 0-1 *measure*, a, on S is a σ-additive probability measure defined on the power set, $\mathscr{P}(S)$, satisfying

(i) $a(\{s\}) = 0 \ \ \forall s \in S$,

(ii) a takes only the values 0 or 1.

The existence of a 0-1 measure on S clearly depends only of the cardinal of S. A cardinal number ξ is called *measurable* if there exists a 0-1 measure on a set (and henceforth on every set) with cardinal number equal to ξ.

It is not known whether there exist measurable cardinals, or rather whether it is consistent with the axioms of set theory to assume the existence of a measurable cardinal. It is known that it is consistent to assume the nonexistence of measurable cardinals, and it is generally believed that it is consistent to assume the existence of measurable cardinals too.

We shall here assume that there exist measurable cardinals.

Let ξ be the first measurable cardinal, and let S be a set, whose cardinal number is equal to ξ. Let $\Sigma = \mathscr{P}(S)$, and let M be the class of all 0-1 measures on S. If $A \in \Sigma$, we define

$$m(A) = \big(a(S)\big)_{a \in M} .$$

Then m is a set function on Σ, which takes values in $E = \mathsf{R}^M$. If E is equipped with the product topology, then obviously m is an E-valued measure on (S, Σ).

Every a in M is atomic (S is an a-atom for all $a \in M$), and since $x'm$ is a finite linear combination of elements from M for all $x' \in E'$, we find that $x'm$ is atomic for all $x' \in E'$.

Let $A \in \Sigma$, if the cardinal of A is less than ξ. Then $a(A) = 0 \; \forall a \in M$, by the minimality of ξ. Hence

$$A \in N(m) \qquad \forall A \subseteq S \text{ with } k(A) < \xi$$

where $k(A)$ is the cardinal number of A.

If $A \in \Sigma$, $k(A) = \xi$, then since ξ is measurable there exists an $a \in M$ with $a(A) = 1$. Hence we find that

$$N(m) = \{A \subseteq S \mid k(A) < \xi\} .$$

So if $A \notin N(m)$, then $k(A) = \xi$, and since ξ is infinite we can find a subset B of A such that $k(B) = k(A \setminus B) = \xi$. Hence $B \notin N(m)$ and $A \setminus B \notin N(m)$, that is, m has no proper atoms, and so m is atomless.

I strongly believe that such an example as the one above does depend on the existence of measurable cardinals. In Example 3 however we give an example of a vector measure m, such that $x'm$ is atomic for all $x' \in E'$, but m is not atomic, without assuming the existence of measurable cardinals.

EXAMPLE 2. We shall now construct two atomless vector measures m_1 and m_2, such that $m_1 + m_2$ is atomic and nonzero.

Let (S, Σ, m) be the vector measure space defined in Example 1. Let a_0 be an arbitrary 0-1 measure on S, and let

$$m_0(A) = \big(a_0(A)\big)_{a \in M}, \quad A \in \Sigma .$$

That is, all the coordinates of $m_0(A)$ are constantly equal to $a_0(A)$.

Then clearly m_0 is atomic, since S is an a_0-atom. Now one can show exactly as in Example 1 that m_0+m is atomless. So putting $m_1=m_0+m$ and $m_2=-m$, both m_1 and m_2 are atomless, and $m_1+m_2=m_0$ is atomic.

EXAMPLE 3. We construct a vector measure m, which cannot be decomposed as a sum of an atomic vector measure and an atomless vector measure.

Let S be an uncountable set and Σ a σ-algebra in S such that all singletons $\{s\}$, $s \in S$, belong to Σ. Let

$$m(A) = 1_A, \quad A \in \Sigma .$$

Then m is a finitely additive set function taking values in $E=\mathsf{R}^S$. If E is equipped with the product topology, then m becomes σ-additive.

Now suppose that $m=m_1+m_2$ with m_1 atomless. Since $\{s\}$ is an m_1-atom for all $s \in S$, we have $m_1\{s\}=0$ $\forall s \in S$. Hence

$$m_2\{s\} = m\{s\} \neq 0 \quad \forall s \in S .$$

Since S is uncountable m_2 cannot be atomic.

EXAMPLE 4. We construct an E-valued measure m, where E is sequentially complete, but the range of m is not weakly conditionally compact, nor is the range of m weakly sequentially compact.

Let $S=[0,1]$, Σ the Borel subsets of S, and define m by

$$m(A) = 1_A \quad \forall A \in \Sigma .$$

Then m is a finitely additive set function on (S,Σ) taking values in $E=M(S,\Sigma)$, where $M(S,\Sigma)$ denotes the space of all real Borel functions on S. If E is equipped with the product topology, then m becomes σ-additive, and E is sequentially complete.

Let A be a subset of S, such that $A \notin \Sigma$, and let P be the family of all finite subsets of A directed by inclusion. Then the generalized sequence $\{m(\pi) \mid \pi \in P\}$ converges in R^S to $1_A \notin E$, hence $\{m(\pi) \mid \pi \in P\}$ can have no convergent generalized subsequences in E. So $m(\Sigma)$ is not conditionally compact, and henceforth not weakly conditionally compact, since the topology in E is the weak topology $\sigma(E,E')$.

Let $I_j^n = (j\,2^{-n},(j+1)\,2^{-n}]$ for $0 \leqq j \leqq 2^n-1$, and put

$$A_n = \textstyle\bigcup_{j=1}^{2n-1} I_{2j-1}^n, \qquad f_n = 1_{A_n} = m(A_n) .$$

If A is an interval whose endpoints are dyadic rationals, then clearly

$$\int_A f_n\,dx \to \int_A \tfrac{1}{2}\,dx \quad \text{as } n \to \infty .$$

So from Exercise 6, IV.13, in [4] we conclude that $f_n \to \frac{1}{2}$ as $n \to \infty$ in the $\sigma(L_\infty, L_1)$-topology. Here L_∞ is the L_∞-space of the Lebesgue measure on S, and L_1 is the L_1-space of the Lebesgue measure on S.

So if $\{f_n\}$ has a subsequence which converges pointwise, then the limit function must necessarily be equal to $\frac{1}{2}$ almost everywhere with respect to the Lebesgue measure. But this is clearly impossible since f_n only takes the values 0 or 1. Hence $\{m(A_n)\}$ has no weakly convergent subsequences, and so $m(\Sigma)$ is not weakly sequentially compact.

EXAMPLE 5. We construct an l_2-valued vector measure m, such that m has finite total variation, m is atomless, $m(\Sigma)$ is conditionally compact, but $m(\Sigma)$ is not convex and not closed.

Let $S=[0,1]$, Σ the Borel subsets of S, and l the Lebesgue measure on (S,Σ). Then there exists an orthonormal base $\{l_n\}_1^\infty$ for $L_2(S,\Sigma,l)$, such that $|l_n(x)| \leqq 1 \; \forall x \in S \; \forall n \geqq 1$. Let $f_n = n^{-1} l_n$ and define

$$m_n(A) = \int_A f_n \, dx \quad \text{for } A \in \Sigma ,$$

$$m(A) = (m_n(A))_1^\infty \quad \text{for } A \in \Sigma .$$

Then m is an l_2-valued vector measure. This may be seen in the following way. Let

$$f(x) = (f_n(x))_1^\infty \quad \text{for } x \in S .$$

Then f is a measurable map from S into l_2 and

$$\|f(x)\|_2 = \{\textstyle\sum_{n=1}^\infty n^{-2} |l_n(x)|^2\}^{\frac{1}{2}} \leqq (\textstyle\sum_{n=1}^\infty n^{-2})^{\frac{1}{2}} < \infty .$$

So f becomes l-integrable and

$$\int_A f \, dx = m(A) \quad \forall A \in \Sigma .$$

See for example [4, III.2.19 and III.2.22]. Hence m is σ-additive and the total variation, $|m|$, of m is given by

$$|m|(A) = \int_A \|f(x)\|_2 \, dx .$$

So m has finite total variation.

We note also that $m(\Sigma)$ is conditionally compact in l_2, since

$$|m_n(A)| \leqq n^{-1} \quad \forall n \geqq 1 \;\; \forall A \in \Sigma .$$

Further we show that $tm(S) \notin m(\Sigma)$ for any $t \in (0,1)$. Suppose this was not true. Then for some $t \in (0,1)$ and some $A \in \Sigma$,

$$m_n(A) = tm_n(S) \quad \forall n = 1, 2, \ldots .$$

Let $g = t1_A - 1_S$. Then

$$\int_0^1 g(x) f_n(x)\, dx = tm_n(A) - m_n(S) = 0 \quad \forall n .$$

But $\{f_n\}$ is an orthogonal base in $L_2(S,\Sigma,l)$, and so $g=0$ a.e. in S. But this contradicts the fact that g only takes the values $t \neq 0$ and $t-1 \neq 0$.

We have now proved that $tm(S) \neq m(\Sigma)$ for any $t \in (0,1)$. Since 0 and $m(S)$ belong to $m(\Sigma)$, this means that $m(\Sigma)$ is not convex.

If $m(\Sigma)$ is closed, then $m(\Sigma)$ is compact and so $m(\Sigma)$ is weakly closed, but this is not possible by Theorem 11. Hence $m(\Sigma)$ is not closed.

This example is due to Liapounov (see [13]).

Example 6. We construct an E-valued vector measure m, such that $E = L_1[0,1]$, m has finite total variation, m is atomless, $m(\Sigma)$ is closed, but $m(\Sigma)$ is not convex, not compact, and not weakly closed.

Let $S = [0,1]$, Σ the Borel subsets of S, and l the Lebesgue measure on (S,Σ). If we define

$$m(A) = 1_A \quad \forall A \in \Sigma ,$$

then m is an $L_1[0,1]$-valued measure on (S,Σ), such that

$$|m|(A) = l(A) \quad \forall A \in \Sigma .$$

Hence m has finite total variation and m is atomless. It is easily seen that $m(\Sigma)$ is closed, but

$$\tfrac{1}{2}m(\emptyset) + \tfrac{1}{2}m(S) = \tfrac{1}{2}1_S \notin m(\Sigma) .$$

That is, $m(\Sigma)$ is not convex, so by Theorem 11, $m(\Sigma)$ is neither compact nor weakly closed.

Example 7. We construct a bounded finitely additive c_0-valued set function, m, on an algebra, such that $x'm$ is σ-additive $\forall x' \in c_0' = l_1$, but m is not s-bounded and not σ-additive.

Let $S = (0,1]$, let Σ_0 be the algebra generated by the intervals $(a,b]$, $0 \leqq a < b \leqq 1$, and define for $n = 1, 2, \ldots$

$$A_n = (0, \tfrac{1}{2}n^{-1}], \quad B_n = (\tfrac{1}{2}n^{-1}, n^{-1}],$$
$$f_n = 2n\,1_{A_n} - 2n\,1_{B_n},$$
$$m_n(A) = \int_A f_n\,dx \quad \forall A \in \Sigma,$$

where Σ is the Borel subsets of S. Let F_n be the function defined by

$$F_n(x) = \begin{cases} 2nx & \text{if } 0 < x \leqq \tfrac{1}{2}n^{-1}, \\ 1-(2nx-1) & \text{if } \tfrac{1}{2}n^{-1} \leqq x \leqq n^{-1}, \\ 0 & \text{if } n^{-1} \leqq x \leqq 1. \end{cases}$$

Then

$$m_n(a,b] = F_n(b) - F_n(a) \quad \forall\, 0 \leqq a \leqq b \leqq 1,$$

and so if

$$m(A) = (m_n(A))_1^\infty \quad \forall A \in \Sigma_0,$$

then m is a finitely additive c_0-valued set function on (S, Σ_0), since $\lim_{n\to\infty} F_n(x) = 0$ for all $x \in S$. If $A \in \Sigma$, then

$$|m_n(A)| \leqq \int_0^1 |f_n|\,dx = 2.$$

Hence m is bounded. Let $x' = (x_n) \in c_0' = l_1$. Then

$$x'm(A) = \Sigma_{n=1}^\infty x_n m_n(A) \quad \forall A \in \Sigma_0.$$

So by Corollary 4, III.7, in [4], $x'm$ is σ-additive for all $x' \in l_1$.

Now $A_n \in \Sigma_0$ for all n, and $\{A_n\}$ decreases to $\emptyset$. But $m_n(A_n) = 1$ for all n, and so

$$\|m(A_n)\| \geqq 1 \quad \forall n \geqq 1,$$

which shows that m is not σ-additive on Σ_0. So by Theorem 4, m is not s-bounded on Σ_0.

Example 8. We construct a c_0-valued bounded σ-additive set function, m, on an algebra, Σ_0, which is not s-bounded, and which has no σ-additive extension to the σ-algebra generated by Σ_0.

First we notice the following simple fact. Let $\mathscr{R}$ be a ring of subsets of S, such that $S \notin \mathscr{R}$, that is, $\mathscr{R}$ is not an algebra, and let m be a finitely additive set function on $\mathscr{R}$ taking values in E. Then

$$\Sigma_0 = \{A \subseteq S \mid A \in \mathscr{R} \text{ or } S \setminus A \in \mathscr{R}\}$$

is the least algebra containing $\mathscr{R}$.

Let x_0 be an arbitrary point in E. Since $S \notin \mathscr{R}$, it follows that $A \in \mathscr{R}$ implies $S \setminus A \notin \mathscr{R}$, and $S \setminus A \in \mathscr{R}$ implies $A \notin \mathscr{R}$. Hence we may define a set function m_0 by

$$m_0(A) = \begin{cases} m(A) & \text{if } A \in \mathscr{R}\,, \\ x_0 - m(S \setminus A) & \text{if } S \setminus A \in \mathscr{R}\,. \end{cases}$$

If A and B belong to Σ_0 and $A \cap B = \emptyset$, then either A or B belongs to $\mathscr{R}$. From this it follows easily that

(13) m_0 is finitely additive on Σ_0, and m_0 is an extension of m.

(14) If m is bounded, then so is m_0.

(15) If m is s-bounded, then so is m_0.

(16) If m is σ-additive, then so is m_0.

Now let us turn to the example. Let $S = [0,1]$, Σ the Borel subsets of S, and define for $n = 1, 2, \ldots$

$$\begin{aligned} A_n &= [0, 1/n!]\,, \\ f_n &= n!\, 1_{A_n}\,, \\ m_n(A) &= \int_A f_n\, dx \qquad \forall A \in \Sigma\,, \\ \mathscr{R} &= \{A \in \Sigma \mid \exists a > 0 \text{ such that } A \subseteqq [a,1]\}\,. \end{aligned}$$

Then $\mathscr{R}$ is a ring of subsets of S. Let $A \in \mathscr{R}$. Then we can find an integer k, such that $A \cap A_k = \emptyset$. Hence

$$m_n(A) = m_n(A \cap A_n) = 0 \qquad \forall n \geqq k\,.$$

This shows that the set function

$$m(A) = (m_n(A))_1^\infty, \qquad A \in \mathscr{R}\,,$$

is finitely additive and takes values in c_0. Suppose that $\{A_n\}$ is a sequence in $\mathscr{R}$, which decreases to $\emptyset$. Then we can find an integer k, such that $A_n \subseteqq A_1 \subseteqq (1/k!, 1]$ $\forall n \geqq 1$, hence

$$m(A_n) = (m_1(A_n), \ldots, m_k(A_n), 0, 0 \ldots) \qquad \forall n \geqq 1\,.$$

This shows that $\lim_{n\to\infty} m(A_n) = 0$ in c_0. That is, m is σ-additive on $\mathscr{R}$. Now let

$$\overline{m}(A) = \begin{cases} m(A) & \text{if } A \in \mathscr{R}\,, \\ -m(S \setminus A) & \text{if } S \setminus A \in \mathscr{R}\,. \end{cases}$$

Then by (14) and (16) $\overline{m}$ is a c_0-valued σ-additive bounded set function on Σ_0, where Σ_0 is the algebra

$$\Sigma_0 = \{A \subseteqq S \mid A \in \mathscr{R} \text{ or } S \setminus A \in \mathscr{R}\} .$$

Finally let $B_n = A_n \setminus A_{n+1}$ for $n = 1,2,\ldots$. Then $B_n \in \mathscr{R}$ for all n, and $\{B_n\}_1^\infty$ are mutually disjoint. By definition of m_n we find

$$m_n(B_n) = n! \quad \big((1/n!) - 1/(n+1)!\big) = 1 - 1/(n+1) .$$

Hence

$$\|m(B_n)\| \geqq 1 - 1/(n+1) \ \forall n \geqq 1 ,$$

and so m is not s-bounded, and a priori $\overline{m}$ is not s-bounded.

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