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THE GENERAL RIESZ DECOMPOSITION AND THE SPECIFIC ORDER OF EXCESSIVE FUNCTIONS

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Introduction.

P. A. Meyer [4, p. 162], [5, pp. 165–170] has proved that the class of all excessive functions with respect to a resolvent forms a lattice in the specific order, under a certain hypothesis on the resolvent (= "hypothesis of absolute continuity [5; p. 159]"). In the case of excessive measures (since they are σ-finite by definition), Meyer's proof does work without any hypothesis. We also mention that R. M. Hervé [1; p. 89] has proved the same result for the class of positive superharmonic functions in the axiomatic potential theory. This lattice property is useful: For example, it enables us to obtain the unique decomposition of excessive functions or measures into extreme elements under very mild conditions (see Hervé [2], Meyer [6], the author [7]).

In this note we will use another method to prove that Meyer's theorem is valid "without any hypothesis" for the class of excessive functions with respect to a single kernel or a resolvent. The key result is that for a single kernel; the case of a resolvent is proved by a routine argument of "passage to the limits". (Actually, Meyer proved that, under his hypothesis, the class of excessive functions is a *complete* lattice in the specific order. But the completeness probably breaks down, in general.)

Let N be a submarkov kernel over a measurable space, G the potential kernel of N and $\mathscr{E}$ the set of all excessive functions. The specific order $u \gg v$ $(u,v \in \mathscr{E})$ is defined by the relation $u=v+w$ for some $w \in \mathscr{E}$. The problem is to find the join $u \overset{s}{\vee} v$ and the meet $u \underset{s}{\wedge} v$ of excessive functions u and v for the specific order $\gg$. Suppose that u and v are finite (or more generally, $N^\infty u = \lim_{n\to\infty} N^n u < \infty$, $N^\infty v < \infty$). Consider the Riesz decomposition of u and v:

$$u = Gf + N^\infty u, \quad v = Gg + N^\infty v .$$

Then it is not difficult to see that

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$$u \overset{s}{\vee} v = G(f \vee g) + N^{\infty}[(N^{\infty}u) \vee (N^{\infty}v)] ,$$
$$u \underset{s}{\wedge} v = G(f \wedge g) + N^{\infty}[(N^{\infty}u) \wedge (N^{\infty}v)] .$$

This proof is not applied to the general case as it stands. For example, the function $N^{\infty}u$ may no longer be an invariant function, and the function $N^{\infty}[(N^{\infty}u) \vee (N^{\infty}v)]$ may not even be well-defined. To cover these points we will introduce the notions of general invariant function and anti-excessive function, and then prove a generalization of the Riesz decomposition. Using this general Riesz decomposition we can prove the existence of $u \overset{s}{\vee} v$ and $u \underset{s}{\wedge} v$ for arbitrary excessive functions u, v by means of the same basic idea as in the case of finite excessive functions.

1. Excessive functions with respect to a single kernel. The general Riesz decomposition.

All terms and notations are taken from the book of Meyer [3; chap. 9] without reference, except the notions of "general invariant function" and "anti-excessive function".

Let E be a measurable space N, a kernel over E, and G the potential kernel of N,

$$G = \sum_{n \geq 0} N^n ,$$

where N^n $(N^0 = I)$ is the n-composite kernel of N. (It is not assumed that N is proper or submarkov, and the kernel G may not be proper or finite.) Throughout the following, a *positive function* will stand for a nonnegative measurable function, finite or not. An *excessive function* (*with respect to* N) and a *potential* are defined as usual. A positive function u, finite or not, is said to be a *general invariant function* if it satisfies

$$(1.1) \qquad u = Nu .$$

We here drop the usual finiteness assumption on an invariant function. A positive function u is said to be *anti-excessive* (or *submedian*) if

$$(1.2) \qquad u \leq Nu .$$

The *infinity support* of a positive function u is the set $\{x \mid u(x) = \infty\}$ and is denoted by $E^{\infty}[u]$. For a given positive function u, if $N^n u$ converges in each point of E, this limit function is denoted by $N^{\infty}u$. As usual, $u \vee v(x) = \max(u(x), v(x))$. We often write $\varepsilon_x N$ for $N(x, \cdot)$.

LEMMA 1. *Let u, v be anti-excessive. Then $N^n(u \vee v)$ increases to a* *general invariant function* $N^{\infty}(u \vee v)$. *This function is the smallest general invariant majorant of* u *and* v.

The proof is easy.

LEMMA 2. *Let* u, v *be general invariant functions and let* $u \geqq v$. *Then there exists a general invariant function* w *such that*

$$u = v + w . \tag{1.3}$$

PROOF. Define

$$\begin{aligned} w_0(x) &= u(x) - v(x) && \text{if } v(x) < \infty, \\ &= 0 && \text{otherwise (that is, on } E^{\infty}[v]) . \end{aligned}$$

One claims that w_0 is anti-excessive. Suppose that $v(x) < \infty$. Since $v(x) = Nv(x) = (\varepsilon_x N, v) < \infty$, it follows that $\varepsilon_x N(E^{\infty}[v]) = 0$. Hence one has

$$\begin{aligned} w_0(x) = u(x) - v(x) &= \int_{\{v(y)<\infty\}} N(x,dy)\,[u(y) - v(y)] \\ &= \int_{\{v(y)<\infty\}} N(x,dy)\, w_0(y) = \int_E N(x,dy)\, w_0(y) = N w_0(x) . \end{aligned}$$

If $x \in E^{\infty}[v]$, obviously $Nw_0(x) \geqq 0 = w_0(x)$. Therefore, $N^n w_0$ increases to a general invariant function $w = N^{\infty} w_0$. But since $u = v + w_0$, one has

$$u = N^{\infty} u = N^{\infty} v + N^{\infty} w_0 = v + w .$$

Let u be excessive. One defines

$$N^{\infty} u = \lim_{n \to \infty} N^n u , \tag{1.4}$$

$$\begin{aligned} \tilde{u}_{\infty}(x) &= N^{\infty} u(x) && \text{if } N^{\infty} u(x) < \infty , \\ &= 0 && \text{otherwise (that is, on } E^{\infty}[N^{\infty} u]) . \end{aligned} \tag{1.5}$$

Since $\tilde{u}_{\infty}$ is anti-excessive by the subsequent lemma, the function

$$u_{\infty} := \lim_{n \to \infty} N^n \tilde{u}_{\infty} \tag{1.6}$$

is well-defined and a general invariant function. This function u_{∞} is called the *general invariant part* of u.

LEMMA 3. *The function* $\tilde{u}_{\infty}$ *is anti-excessive. The general invariant function* u_{∞} *satisfies*

$$(1.7)\qquad u_\infty \leqq N^\infty u\,,$$

$$(1.8)\qquad u_\infty = N^\infty u \quad \textit{on the set } \{N^\infty u<\infty\}\,.$$

Moreover it is the smallest general invariant function among those which dominate $N^\infty u$ on the set $\{N^\infty u<\infty\}$.

PROOF. It is obvious that $N\tilde{u}_\infty(x)\geqq 0=\tilde{u}_\infty(x)$, $x\in E^\infty[N^\infty u]$. Suppose that $N^\infty u(x)<\infty$. Take k so that $N^k u(x)<\infty$. Then the function $N^{k-1}u$ is integrable for the measure $\varepsilon_x N$. Since $N^n u$ decreases to $N^\infty u$, by the dominated convergence theorem, one has

$$N(N^\infty u)(x) = N(\lim_{n\to\infty} N^n u)(x) = \lim_{n\to\infty} N^{n+1}u(x) = N^\infty u(x) < \infty\,.$$

In particular, $\varepsilon_x N(E^\infty[N^\infty u]) = 0$. Similarly to Lemma 2, one has

$$(1.9)\qquad \tilde{u}_\infty(x) = N^\infty u(x) = N(N^\infty u(x)) = N\tilde{u}_\infty(x)\,.$$

Since $N^\infty u$ is excessive and $\tilde{u}_\infty$ is anti-excessive,

$$(1.10)\qquad \tilde{u}_\infty \leqq u_\infty = N^\infty \tilde{u}_\infty \leqq N^\infty[N^\infty u] \leqq N^\infty u\,,$$

which proves (1.7) and (1.8).

The proof of the last statement is quite easy. Define

$$(1.11)\qquad \begin{aligned} f_u(x) &= u(x)-Nu(x) && \text{if } u(x)<\infty,\\ &= \infty && \text{otherwise (that is, on } E^\infty[u])\,. \end{aligned}$$

THEOREM 1 (General Riesz decomposition). (a) *A decomposition of u into the sum of a potential and a general invariant function is given by*

$$(1.12)\qquad u = Gf_u + u_\infty\,.$$

(b) *Consider any decomposition of the form*

$$(1.13)\qquad u = Gg+h\,,$$

where g is a positive function and h, a general invariant function. Then

$$(1.14)\qquad g \leqq f_u,\quad u_\infty \leqq h \leqq N^\infty u\,,$$

$$(1.15)\qquad g = f_u \textit{ on } \{u<\infty\},\quad h = u_\infty \textit{ on } \{N^\infty u<\infty\}\,.$$

DEFINITION. The formula (1.12) is called the (*Riesz*) *canonical decomposition of u.*

PROOF OF THEOREM 1. (a) It is easy to see that

$$(1.16)\qquad u = \textstyle\sum_{k\leqq n} N^k f_u + N^{n+1}u\,,$$

so that

$$u = Gf_u + N^{\infty}u . \tag{1.17}$$

If $u(x) < \infty$, then $N^{\infty}u(x) < \infty$. Hence $N^{\infty}u(x) = u_{\infty}(x)$. If $u(x) = \infty$, then $Gf_u(x) \geqq f_u(x) = \infty$. Hence one has (1.12).

Suppose that one has a decomposition (1.13). Then

$$N^{\infty}u = N^{\infty}[Gg] + h . \tag{1.18}$$

If $N^{\infty}u(x) < \infty$, $N^{\infty}[Gg](x) = 0$ by a theorem of Doob [3; p. 180, T 18]. Hence $N^{\infty}u = h$ on the set $\{N^{\infty}u < \infty\}$. By Lemma 3, $h \geqq u_{\infty}$. It is obvious that $h = u_{\infty}$ on $\{N^{\infty}u < \infty\}$.

Suppose that $u(x) < \infty$. Since

$$\begin{aligned} u(x) &= g(x) + NGg(x) + h(x) \\ &= g(x) + N[Gg + h](x) = g(x) + Nu(x) , \end{aligned}$$

one has

$$g(x) = u(x) - Nu(x) = f_u(x) .$$

Then the inequality $g \leqq f_u$ is also proved.

Definition. Let $\mathscr{E}$ be the set of all excessive functions (with respect to N). The *specific* (or *intrinsic*, or *strong*) *order* "$\gg$" *in the cone* (or rather, *wedge*) $\mathscr{E}$ is defined by

$$(u \gg v) \Leftrightarrow (u = v + w \text{ for some } w \in \mathscr{E}) \quad \text{for } u, v \in \mathscr{E} . \tag{1.19}$$

Theorem 2. *The set $\mathscr{E}$ of all excessive functions for the kernel N is a lattice in the specific order.*

Proof. (a) *Existence of the specific join.* Let u, v be excessive. Consider their canonical decompositions

$$u = Gf_u + u_{\infty}, \quad v = Gf_v + v_{\infty} . \tag{1.20}$$

Define an excessive function w_0 by

$$w_0 = G(f_u \vee f_v) + N^{\infty}[u_{\infty} \vee v_{\infty}] . \tag{1.21}$$

We will prove that w_0 is the specific join $u \overset{s}{\vee} v$ of u and v.

Since $N^{\infty}[u_{\infty} \vee v_{\infty}] \geqq u_{\infty}$, there is a general invariant function h' such that

$$N^{\infty}[u_{\infty} \vee v_{\infty}] = u_{\infty} + h' .$$

Choose a positive function f' such that $f_u \vee f_v = f_u + f'$. It then is obvious that

(1.22) $$w_0 = u+u' \quad \text{with} \quad u' = Gf'+h' \in \mathscr{E}\,,$$

so that $w_0 \gg u$. By the same reason, $w_0 \gg v$.

Let w be an excessive function which majorizes both u and v in the specific order. Consider the canonical decomposition of w,

$$w = Gf_w + w_\infty\,.$$

On the other hand, there is a $u' \in \mathscr{E}$ such that

$$\begin{aligned} w = u+u' &= Gf_u + u_\infty + Gf_{u'} + u'_\infty \\ &= G(f_u + f_{u'}) + (u_\infty + u'_\infty)\,, \end{aligned}$$

and a $v' \in \mathscr{E}$ such that

$$w = v+v' = G(f_v + f_{v'}) + (v_\infty + v'_\infty)\,.$$

Define a general invariant function h by

$$h = N^\infty[(u_\infty + u'_\infty) \vee (v_\infty + v'_\infty)]\,.$$

By Theorem 1 it follows that

$$w_\infty \leqq h \leqq N^\infty w\,.$$

Hence

$$w = Gf_w + h\,.$$

Again, by Theorem 1, $f_w \geqq f_u + f_{u'}$, $f_w \geqq f_v + f_{v'}$. Hence,

$$f_w \geqq f_u \vee f_v\,.$$

It is obvious that $h \geqq N^\infty[u_\infty \vee v_\infty]$. Choose $f' \geqq 0$ and a general invariant function h' such that

$$f_w = f_u \vee f_v + f', \quad h = N^\infty[u_\infty \vee v_\infty] + h'\,.$$

Then

$$w = w_0 + w' \quad \text{with} \quad w' = Gf' + h' \in \mathscr{E}\,,$$

which proves $w \gg w_0$.

(b) *Existence of the specific meet.* If u and v are finite excessive functions, it is easy to see that the function

(1.23) $$w_1 = u + v - (u \overset{s}{\vee} v)$$

is the specific meet $u \underset{s}{\wedge} v$, as in the proof of the general relation in a vector lattice

(1.24) $$u + v = u \overset{s}{\vee} v + u \underset{s}{\wedge} v\,.$$

However, since u and v are not finite in general, such a proof breaks down.

Consider the canonical decompositions (1.20) of u and v. Then, consider the canonical decomposition of the excessive function $\varphi = (N^\infty u) \wedge (N^\infty v)$,

$$\varphi = Gf_\varphi + \varphi_\infty . \tag{1.25}$$

We will prove that the excessive function

$$w_1 = G(f_u \wedge f_v) + \varphi_\infty \tag{1.26}$$

is the specific meet $u \underset{s}{\wedge} v$.

To prove $u \gg w_1$, define the general invariant function h by

$$h = N^\infty[u_\infty \vee \varphi_\infty] \geqq \varphi_\infty .$$

Since $u_\infty \vee \varphi_\infty \leqq N^\infty u$, $h \leqq N^\infty u$. By $u_\infty \leqq h$, one has

$$u = Gf_u + h .$$

Then, take $f' \geqq 0$ and a general invariant function h' such that

$$f_u = f_u \wedge f_v + f', \quad h = \varphi_\infty + h' .$$

It is obvious that

$$u = w_1 + w' \quad \text{with} \quad w' = Gf' + h' \in \mathscr{E} .$$

In the same way, $v \gg w_1$.

Let w be an excessive function which is majorized by both u and v in the specific order. Consider the canonical decomposition of w,

$$w = Gf_u + w_\infty .$$

On the other hand, there is a $w' \in \mathscr{E}$ such that

$$u = w + w' = G(f_w + f_{w'}) + (w_\infty + w'_\infty) ,$$

and a $w'' \in \mathscr{E}$ such that

$$v = w + w'' = G(f_w + f_{w''}) + (w_\infty + w''_\infty) .$$

Therefore, $f_u \geqq f_w + f_{w'}$, $f_v \geqq f_w + f_{w''}$, so that

$$f_u \wedge f_v \geqq f_w . \tag{1.27}$$

On the other hand,

$$w_\infty \leqq N^\infty \varphi , \tag{1.28}$$

since $w_\infty \leqq (N^\infty u) \wedge (N^\infty v) = \varphi$ by Theorem 1. Define a general invariant function h by

$$h = N^\infty[w_\infty \vee \varphi_\infty] \geqq w_\infty . \tag{1.29}$$

By (1.28) it follows that

(1.30) $$\varphi_\infty \leqq h \leqq N^\infty \varphi .$$

One claims that

(1.31) $$w_1 = G(f_u \wedge f_v) + h .$$

This is obvious if $x \in E^\infty[u] \cap E^\infty[v]$. Suppose that $u(x) < \infty$. Then, since $\varphi(x) < \infty$, it follows by Theorem 1 that $\varphi_\infty(x) = N^\infty \varphi(x)$. By (1.30), one has

(1.32) $$\varphi_\infty(x) = h(x) .$$

In the same way, (1.32) is true also if $v(x) < \infty$. One has proved (1.31).

By (1.27) and (1.29). there are $f' \geqq 0$ and a general invariant function h' such that

$$f_u \wedge f_v = f_w + f', \quad h = w_\infty + h' .$$

Hence, by (1.31),

$$w_1 = w + w' \quad \text{with } w' = Gf' + h' \in \mathscr{E} ,$$

which proves that $w_1 \gg w$.

REMARK. We are not sure if the formula (1.24) is a quite general fact or not, after knowing that a cone (or wedge) forms a lattice in the specific order. Note that the difference of two excessive functions is not defined in general and hence $\mathscr{E}$ cannot be extended to a vector lattice containing $\mathscr{E}$ as the positive cone. Hence the usual proof of (1.24) in a vector lattice is not applicable.

However, we note that the formula (1.24) is valid for the general case where u and v are not finite. In fact, that formula is obviously true if $x \in E^\infty[u] \cup E^\infty[v]$. Suppose that $u(x) < \infty$ and $v(x) < \infty$. Then, for each n,

$$\varepsilon_x N^n(E^\infty[u] \cup E^\infty[v]) = 0 .$$

Hence, by Theorem 1, $N^\infty u = u_\infty$, $N^\infty v = v_\infty$ almost everywhere for the measure $\varepsilon_x N^n$. Therefore, one has

$$\begin{aligned} \varphi_\infty(x) = N^\infty \varphi(x) &= \lim_{n\to\infty} N^n[(N^\infty u) \wedge (N^\infty v)](x) \\ &= \lim_{n\to\infty} N^n[u_\infty \wedge v_\infty](x) . \end{aligned}$$

By the definitions (1.21) and (1.26) of $w_0 = u \overset{s}{\vee} v$ and $w_1 = u \underset{s}{\wedge} v$, it follows that

$$\begin{aligned} w_0(x) + w_1(x) &= G(f_u \vee f_v + f_u \wedge f_v)(x) + N^\infty[u_\infty \vee v_\infty](x) + \varphi_\infty(x) \\ &= G(f_u + f_v)(x) + \lim_{n\to\infty} N^n[u_\infty \vee v_\infty + u_\infty \wedge v_\infty](x) \\ &= G(f_u + f_v)(x) + N^\infty[u_\infty + v_\infty](x) \\ &= Gf_u(x) + u_\infty(x) + Gf_v(x) + v_\infty(x) \\ &= u(x) + v(x) . \end{aligned}$$

2. Excessive functions with respect to a resolvent.

Let $\{V_\alpha\}_{\alpha<0}$ be a submarkov resolvent. (The condition "submarkov" can be replaced by the condition "proper". See Remark at the end of this section.) Excessive functions and supermedian functions with respect to $\{V_\alpha\}$ are defined as usual.

Let $\mathscr{S}$ be the set of all supermedian functions and $\mathscr{E}$, the set of all excessive functions. The specific orders "$\succcurlyeq$" in $\mathscr{S}$ and $\mathscr{E}$ are defined as in (1.19). More precisely, the specific order in $\mathscr{S}$ is defined by

$$(2.1) \qquad (u \succcurlyeq v) \Leftrightarrow (u = v + w \text{ for some } w \in \mathscr{S})$$

for $u, v \in \mathscr{S}$, and the one in $\mathscr{E}$, by

$$(2.2) \qquad (u \succcurlyeq v) \Leftrightarrow (u = v + w \text{ for some } w \in \mathscr{E})$$

for $u, v \in \mathscr{E}$.

THEOREM 3. *Both $\mathscr{S}$ and $\mathscr{E}$ are lattices in their specific orders.*

PROOF. (a) *Case of $\mathscr{S}$.* Write N_β for a submarkov kernel βV_β and $\mathscr{E}_\beta$, for the class of all excessive functions with respect to the single kernel N_β. By Theorem 2, each $\mathscr{E}_\beta$ forms a lattice in its specific order "$\succcurlyeq_\beta$". One claims that

$$(2.3) \qquad \mathscr{E}_\beta \supset \mathscr{E}_{\beta'} \quad \text{for} \quad \beta < \beta' ,$$

$$(2.4) \qquad \mathscr{S} = \bigcap_\beta \mathscr{E}_\beta .$$

Suppose that $u \in \mathscr{E}_{\beta'}$ is bounded. Then

$$\begin{aligned} V_\beta u &= [I + (\beta' - \beta) V_\beta] V_{\beta'} u \\ &\leqq [I + (\beta' - \beta) V_\beta](u/\beta') , \end{aligned}$$

whence it follows that

$$(2.5) \qquad \beta V_\beta u \leqq u .$$

Then it follows that (2.5) is valid for every $u \in \mathscr{E}_{\beta'}$. The relation (2.4) is obvious.

Let $u, v \in \mathscr{S}$. Since both u and v are in $\mathscr{E}_\beta$, there is the specific join w_β of u and v in $\mathscr{E}_\beta$. By (2.3) and (2.4), w_β increases to a function $w_0 \in \mathscr{S}$. One claims that w_0 is the specific join $u \overset{s}{\vee} v$ in $\mathscr{S}$, that is,

$$(2.6) \qquad w_0 = u \overset{s}{\vee} v \quad \text{in } \mathscr{S} .$$

In fact, since

$$w_\beta = u + u_\beta' , \quad u_\beta' \in \mathscr{E}_\beta ,$$

one has

$$w_0 = u + u', \quad \text{where } u' = \liminf_{\beta_n \to \infty} u'_{\beta_n} \in \mathcal{S}, \tag{2.7}$$

which proves that $w_0 \gg u$. In the same way, $w_0 \gg v$.

Next take any specific majorant $w \in \mathcal{S}$ of u and v. Obviously, w specifically majorizes w_β in $\mathcal{E}_\beta$, that is,

$$w = w_\beta + w_\beta', \quad w_\beta' \in \mathcal{E}_\beta.$$

Therefore,

$$w = w_0 + w_0' \quad \text{with} \quad w_0' = \liminf_{\beta_n \to \infty} w'_{\beta_n} \in \mathcal{S}, \tag{2.8}$$

which proves that $w \gg w_0$.

In the same way, the specific meet $u \underset{s}{\wedge} v$ in $\mathcal{S}$ is obtained as the (decreasing) limit w_1 of the specific meets of u and v in $\mathcal{E}_\beta$. The details are omitted.

The formula (1.24) is also valid; this is a simple consequence of the passage to the limits.

(b) *Case of* $\mathcal{E}$. Let $u, v \in \mathcal{E}$. Consider the regularization of w_0, denoted by $\mathrm{reg}[w_0]$. One claims that

$$\mathrm{reg}[w_0] = u \overset{s}{\vee} v \quad \text{in } \mathcal{E}. \tag{2.9}$$

In fact, taking the regularization on both sides in (2.7), one has

$$\mathrm{reg}[w_0] = u + \mathrm{reg}[u'], \tag{2.10}$$

so that $\mathrm{reg}[w_0] \gg u$ in $\mathcal{E}$. In the same way, if $w \gg u, v$ in $\mathcal{E}$, by (2.8),

$$w = \mathrm{reg}[w_0] + \mathrm{reg}[w_0'], \tag{2.11}$$

so that $w \gg \mathrm{reg}[w_0]$ in $\mathcal{E}$.

Similarly, the specific meet $u \underset{s}{\wedge} v$ in $\mathcal{E}$ is obtained as the regularization of w_1 in (a). Note that $w_0 = \mathrm{reg}[w_0]$, since $\mathrm{reg}[w_0] \gg u, v$ (in $\mathcal{S}$) and therefore $\mathrm{reg}[w_0] \gg w_0$ in $\mathcal{S}$. This is not the case of w_1.

REMARK. We will note that *Theorem* 3 *is valid for any proper resolvent.* In fact, Theorem 2 is valid for any single kernel. Then, only the relation (2.3) is not obvious if the submarkov resolvent is replaced by a proper resolvent; the rest of the proof of Theorem 3 needs no change.

Suppose first that $\{V_\beta\}_{\beta>0}$ is closed, i.e., the potential kernel V of $\{V_\beta\}$ is also a proper kernel. Let G_β be the potential kernel of N_β. Recall the resolvent identity [3; p. 193, T 55];

$$I + \beta V = \Sigma_{n \geqq 0} [\beta V_\beta]^n = \Sigma_{n \geqq 0} N_\beta{}^n = G_\beta.$$

Since G_β is proper, any excessive function with respect to N_β is the limit of an increasing sequence of finite G_β-potentials. Therefore, to prove (2.3) it is enough to show that (2.5) is valid for any finite $G_{\beta'}$-potential. If $u = G_{\beta'}f = [I + \beta' V]f < \infty$, it follows that

$$V_\beta f \leqq Vf < \infty, \quad V_\beta Vf = \beta^{-1}[Vf - V_\beta f] < \infty,$$

so that $V_\beta u = V_\beta[I + \beta' V]f < \infty$. Then, (2.5) follows from the preceding inequality $V_\beta u \leqq [I + (\beta' - \beta)V_\beta](u/\beta')$.

Next consider the case of a general proper resolvent. Then the above result is applied to the closed resolvent $\{V_{\alpha+\beta}\}_{\beta>0}$ for each $\alpha > 0$. Therefore, one sees that, for $\beta < \beta'$, the relation $\beta' V_{\alpha+\beta'} u \leqq u$ implies that $\beta V_{\alpha+\beta} u \leqq u$. Suppose that $u \in \mathscr{E}_{\beta'}$. Obviously,

$$\beta' V_{\alpha+\beta'} u \leqq \beta' V_{\beta'} u \leqq u \quad \text{for every } \alpha > 0.$$

Therefore, $\beta V_{\alpha+\beta} u \leqq u$ for every $\alpha > 0$, which implies that $\beta V_\beta u \leqq u$. Thus the relation (2.3) has been proved for any proper resolvent.

3. Excessive measures.

Let N be a kernel and $\{V_\alpha\}_{\alpha>0}$, a proper resolvent. Let $\mathscr{M}^+$ be the set of all σ-finite measures over E. A measure $\nu \in \mathscr{M}^+$ is said to be *excessive with respect to* N (resp. *supermedian with respect to* $\{V_\alpha\}$), if

$$(3.1) \qquad \nu N \leqq \nu,$$

$$(3.2) \qquad [\text{resp. } \nu(\alpha V_\alpha) \leqq \nu \text{ for every } \alpha > 0].$$

The measure ν is said to be *excessive with respect to* $\{V_\alpha\}$, if it satisfies (3.2) and

$$(3.3) \qquad \lim_{\alpha\to\infty} \nu(\alpha V_\alpha) = \nu,$$

where convergence in $\mathscr{M}^+$ is defined as follows. A sequence $\{\mu_n\}$ of measures in $\mathscr{M}^+$ is said to converge to $\mu \in \mathscr{M}^+$ if every μ_n is dominated by a measure $\nu \in \mathscr{M}^+$ and if, for every measurable set A such that $\nu(A) < \infty$,

$$\lim_{n\to\infty} \mu_n(A) = \mu(A).$$

Actually, since $\nu(\alpha V_\alpha)$ is increasing, one has $\lim_{\alpha\to\infty} \nu(\alpha V_\alpha)(A) = \nu(A)$ for every measurable set A.

Consider the case of the single kernel N. A measure of the form $\nu = \mu G \in \mathscr{M}^+$, $\mu \in \mathscr{M}^+$, is called a potential of μ. An excessive measure ν is a potential if and only if $\nu N^\infty = 0$. An excessive measure ν is said to be *invariant* if $\nu N = \nu$.

It is not difficult to show that the "usual" Riesz decomposition is valid for excessive measures: Every excessive measure ν is written uniquely as the sum of a potential μG and an invariant measure λ with $\mu=(I-N)\nu$ and $\lambda=\nu N^{\infty}$.

THEOREM 4. *The cone of all excessive measures with respect to N is a lattice under the specific order.*

This is proved similarly to Theorem 2, using the Riesz decomposition. (Actually, the proof is much simpler than for Theorem 2, since every excessive measure is σ-finite.)

THEOREM 5. *Let $\mathscr{S}^*$ be the cone of all supermedian measures with respect to $\{V_\alpha\}$ and $\mathscr{E}^*$, the cone of all excessive measures with respect to $\{V_\alpha\}$. Then, $\mathscr{S}^*$ and $\mathscr{E}^*$ are lattices in their specific orders.*

The proof is the same as that of Theorem 3. (Use the argument of the remark at the end of Section 2, to prove the fact corresponding to (2.3).)

REFERENCES

1. M. Brelot, *Lectures on potential theory*, Tata Institute of Fundamental Research, Bombay, 1960.
2. R. M. Hervé, *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier (Grenoble) 12 (1962), 415–571.
3. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham, 1966.
4. P. A. Meyer, *Fonctionelles multiplicatives et additives de Markov*, Ann. Inst. Fourier (Grenoble) 12 (1962), 125–230.
5. P. A. Meyer, *Processus de Markov* (Lecture Notes in Mathematics 26), Springer-Verlag, Berlin · New York, 1967.
6. P.A.Meyer, *Processus de Markov; la frontière de Martin* (Lecture Notes in Mathematics 77), Springer-Verlag, Berlin · New York, 1968.
7. T. Watanabe, *Maximal measures on compact sets and some applications to probability theory* (Appendix to Lectures on Markov processes, Aarhus University 1969/70), Unpublished.
8. G. Mokobodzki, *Densité relative de deux potentiels comparables*, Séminaire de Probabilités (Strasbourg) IV (1968/69) (Lecture Notes in Mathematics 124), Springer-Verlag, Berlin · New York, 1970, 170–194.

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