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CONFORMALITY AND PSEUDO-RIEMANNIAN MANIFOLDS

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Preface.

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Introduction.

Conformal mappings of Riemannian manifolds were investigated by several authors in the local and global formulation as well (cf. e.g. [4], [9], and [12]), and some results were also obtained in the case of pseudo-riemannian manifolds, however, in the local formulation only (cf. e.g. [4], [10], and [8]). Quasiconformal mappings of Riemannian manifolds were introduced and investigated in [17].

In the present paper we are concerned with conformal mappings of pseudo-riemannian manifolds in the global formulation.

We begin our study with preliminaries. We first introduce some notation and terminology, in particular the notion of an essentially pseudo-riemannian manifold, develop measurability and integration (Theorems 1 and 2), introduce the notion of an angle, and define its inner measure. We then deal with curves; especially we distinguish some kinds of curves: space-like, time-like, regular, and rectifiable, define the length of a regular curve, introduce some kinds of mappings: type-preserving and type-reversing, and give a basic theorem on these mappings (Theorem 3). Next we introduce the notion of the p-modulus of a family of regular curves and study basic properties of these moduli (Theorems 4–8).

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In the second part of the paper we are concerned with conformal mappings of essentially pseudo-riemannian manifolds. We introduce the notion of conformality which, roughly speaking, means that the isotropic cone is preserved at each point of the manifold in question. We give then a necessary and sufficient condition for conformality in terms of quadratic forms determined by the metrics of the manifolds in question (Theorem 9). Now we give a characterization of conformal mappings in terms of angles and their inner measure (Theorems 10 and 11), and, finally, in terms of families of regular curves and their moduli (Theorems 12 and 13).

In the last section we define regular quasiconformal mappings and conclude the paper with the result that in the case of essentially pseudo-riemannian manifolds there is no analogue of regular quasiconformal mappings other than conformal. Here we mention that the problem of the existence of some irregular quasiconformal mappings remains open. We also pose some other natural problems, some of them being planned to be discussed in a subsequent paper.

Part I.
Preliminaries.

1. Notation and terminology.

Throughout this paper the set of all points (resp. vectors) of a manifold (resp. vector space) X is denoted by $\operatorname{supp} X$. If f is a mapping from a set (resp. manifold or vector space) X into a set (resp. manifold or vector space) Y, we write $f\colon X \to Y$, and denote the image of any subset E of X (resp. $\operatorname{supp} X$) by $f[E]$. If, in particular, f is a homeomorphism, $\to$ means "onto", that is, $Y = f[X]$ (resp. $\operatorname{supp} Y = f[\operatorname{supp} X]$).

The n-dimensional Euclidean space is denoted by R^n, and the subspace of it that consists of points with the last component positive by R^n_+. In the case where $n = 1$ we drop the index n.

By a *pseudo-riemannian* manifold we mean a C^∞-differentiable paracompact connected manifold endowed with a *pseudo-riemannian metric*, i.e. a symmetric C^∞ tensor field of type $(0, 2)$ which is nondegenerate and has the same index at each point. Let g be the metric in question. Denote by n and p its dimension and index, respectively. Clearly, there is no loss of generality if we assume that $p \leqq \frac{1}{2}n$, i.e. if we replace, if convenient, g by $-g$. We say that a pseudo-riemannian manifold is *essentially pseudo-riemannian* if $1 \leqq p \leqq \frac{1}{2}n$. For the definition and properties of C^∞-differentiable manifolds as well as tensor fields we refer to [2].

Given a pseudo-riemannian manifold M and an $x \in \operatorname{supp} M$, $T_x M$ denotes the tangent space to M at x, while

$$\begin{aligned} I_x^0 M &= \{v \in \operatorname{supp} T_x M : g(v,v) = 0\}\,, \\ I_x^+ M &= \{v \in \operatorname{supp} T_x M : g(v,v) > 0\}\,, \\ I_x^- M &= \{v \in \operatorname{supp} T_x M : g(v,v) < 0\}\,. \end{aligned}$$

In other words $I_x^0 M$ is the collection of vectors of all isotropy subspaces of $T_x M$, while $I_x^+ M$ and $I_x^- M$ are the collections of vectors of all positive and negative definite subspaces of $T_x M$, respectively. Further, TM denotes the tangent bundle of M. Finally, if N is another pseudo-riemannian manifold and $f\colon M \to N$ a diffeomorphism, then $Df\colon TM \to TN$ denotes the derivative of f.

2. Measurability.

Throughout this section X, Y, and Z are C^∞-differentiable paracompact connected manifolds, while M and N are pseudo-riemannian manifolds with metrics g and g', respectively. Under a *Borel measure* on X we mean a measure which is defined on the collection of Borel subsets of $\operatorname{supp} X$. A mapping $f\colon X \to Y$ is said to be a *Borel function* if the preimage $f^{-1}[E]$ of each open set $E \subset \operatorname{supp} Y$ is a Borel set.

We need the following lemmas proved in [17, pp. 7–9].

LEMMA 1. *If $f\colon X \to Y$ is a Borel function, then the preimage $f^{-1}[E]$ of each Borel set $E \subset \operatorname{supp} Y$ is a Borel set. If, in addition, $h\colon Y \to Z$ is Borel, then $h \circ f$ is also Borel.*

LEMMA 2. *The product map $h\colon X \to Y \times Z$ of $f\colon X \to Y$ and $f^*\colon X \to Z$ is a Borel function if and only if f and f^* are Borel.*

LEMMA 3. *If V is a finite-dimensional real vector space and $f\colon X \to V$, $h\colon X \to V$ are Borel functions, then $f+h$ and af are Borel for each Borel function $a\colon X \to \mathsf{R}$.*

Let $L(\mathsf{R}^m, \mathsf{R}^n)$ denote the linear space formed by the set of linear mappings from R^m into R^n. It is well known (cf. e.g. [2, pp. 71–73]) that this set may be canonically represented by matrices of type (m,n) with real entries. Hence to each mapping from X into $L(\mathsf{R}^m, \mathsf{R}^n)$ corresponds a matrix function $[a_{ij}]$, $a_{ij}\colon X \to \mathsf{R}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$.

LEMMA 4. *If $[a_{ij}]$ is the matrix function that corresponds to some $f\colon X \to L(\mathsf{R}^m, \mathsf{R}^n)$, then the following conditions are equivalent:*

(i) f *is Borel,*
(ii) a_{ij} *are Borel,*
(iii) $h: X \times \mathsf{R}^m \to \mathsf{R}^n$, *defined by* $h(x,v) = f(x)(v)$, $x \in \operatorname{supp} X$, $v \in \operatorname{supp} \mathsf{R}^m$, *is Borel.*

As in [17, p. 8], we say that a set $E \subset \operatorname{supp} X$ is a *null set* if for each coordinate neighbourhood $U \subset \operatorname{supp} X$ and each coordinate C^∞-mapping $\mu: U \to \operatorname{supp} \mathsf{R}^n$ the set $\mu[E \cap U]$ has Lebesgue measure zero. A condition is said to hold for *almost every* $x \in \operatorname{supp} X$, or *almost everywhere* on X, if it holds everywhere except perhaps for a null set. In our considerations as derivatives of functions differentiable almost everywhere we shall meet functions which are not defined on a Borel null set. If such a function is Borel on its set of definition, then its extension by a constant value will also be Borel. We will carry out always such an extension by the value 0. Hence we may regard all functions as defined everywhere.

LEMMA 5. *If* $f: X \to Y$ *is continuous and differentiable almost everywhere, then* Df *is Borel.*

LEMMA 6. *Suppose that* $f: M \to N$ *is a continuous function and* $u: TM \to TN$ *a Borel function which maps each* $T_x M$, $x \in \operatorname{supp} M$, *linearly into* $T_{f(x)}N$. *Given an* $x \in \operatorname{supp} M$ *consider arbitrary bases* $(e_i)(x)$ *and* $(\varepsilon_i) \circ f(x)$ *of* $T_x M$ *and* $T_{f(x)}N$, *respectively. Let* $[a_{ij}]$ *be the matrix function which corresponds to* u. *Then*
(i) *the quantities*

$$\|u\|(x) = \sup |g'(u(x)(v), u(x)(v))|^{\frac{1}{2}}, \quad x \in \operatorname{supp} M,$$

where the supremum is taken over all $v \in T_x M$ *such that* $|g(v,v)| \leqq 1$, *and*

$$(\det u)(x) = \det a_{ij}(x) \left| \frac{\det g'(\varepsilon_i, \varepsilon_j) \circ f(x)}{\det g(e_i, e_j)(x)} \right|^{\frac{1}{2}}, \quad x \in \operatorname{supp} M,$$

do not depend on the choice of (e_i) *and* (ε_i),
(ii) *the functions* $\|u\|: X \to \mathsf{R}$ *and* $\det u: X \to \mathsf{R}$ *are Borel.*

Lemma 6 is a simple consequence of Lemmas 2, 3, and 1. Its proof is analogous to that given in [17, p. 8], in the case of Riemannian manifolds.

THEOREM 1. *Suppose that* $f: M \to N$ *is continuous and differentiable almost everywhere. For any* $x \in \operatorname{supp} M$ *consider arbitrary coordinate* C^∞-

mappings $\mu = (\mu^i)$ *on* M *at* x *and* $\nu = (\nu^i)$ *on* N *at* $f(x)$ *whose dimensions are equal to the dimensions of the corresponding manifolds. Then*

(i) *the quantities*

$$\|Df\|(x) = \sup |g'(Df(x)(v), Df(x)(v))|^{\frac{1}{2}}, \quad x \in \operatorname{supp} M ,$$

where the supremum is taken over all $v \in T_x M$ *such that* $|g(v,v)| \leqq 1$, *and*

$$(\det Df)(x) = \det(\nu^j \circ f \circ \mu^{-1})_{|i} \circ \mu(x) \left| \frac{\det g'_{ij} \circ \nu \circ f(x)}{\det g_{ij} \circ \mu(x)} \right|^{\frac{1}{2}}, \quad x \in \operatorname{supp} M ,$$

where $_{|i}$ *denotes partial differentiation with respect to* μ^i, *do not depend on the choice of* μ *and* ν,

(ii) *the functions* $\|Df\|: M \to \mathsf{R}$ *and* $\det Df: M \to \mathsf{R}$ *are Borel.*

Theorem 1 is a simple consequence of Lemmas 5 and 6. Its proof is analogous to that given in [17, p. 9], in the case of Riemannian manifolds.

3. Integration.

We begin with quoting a lemma proved in [17, p. 10].

LEMMA 7. *Let* θ *and* τ *be Borel measures on a* C^∞*-differentiable paracompact connected manifold* X. *Further let* φ *be a nonnegative Borel function on* X *such that*

$$\theta(E) = \int_E \varphi \, d\tau$$

for each Borel set $E \subset \operatorname{supp} X$. *Then a Borel function* $\varrho : X \to \mathsf{R}$ *is* θ*-integrable if and only if* $\varrho\varphi$ *is* τ*-integrable and*

$$\int_X \varrho \, d\theta = \int_X \varrho\varphi \, d\tau .$$

We introduce then the notion of jacobian. If M and N are pseudo-riemannian manifolds and $f: M \to N$ is a C^1-diffeomorphism, then, by Theorem 1, $\det Df$ is a real-valued Borel function of $x \in \operatorname{supp} M$, that is, $\det Df: M \to \mathsf{R}$. Moreover, as it is easily seen, it is continuous. The function

$$J_f = |\det Df|$$

is called the *jacobian* of f. Analogously, as in the case of Riemannian manifolds, we prove

LEMMA 8. (i) *If f is the identity mapping, then $J_f(x)=1$ identically.*

(ii) *If $f\colon M \to N$ and $h\colon N \to L$ are C^1-diffeomorphisms of pseudo-riemannian manifolds M, N, and L, then $J_{h\circ f}=(J_h\circ f)J_f$.*

(iii) *If $f\colon M \to M'$ and $h\colon N \to N'$ are C^1-diffeomorphisms of pseudo-riemannian manifolds M, M', N, N', then $J_{f\times h}(x,y)=J_f(x)J_g(y)$ identically.*

We now give a theorem which enables us to define the Lebesgue measure on a pseudo-riemannian manifold.

THEOREM 2. *With each pseudo-riemannian manifold M we can associate a unique Borel measure $\tau(M)$ so that the following conditions are satisfied:*

(i) *if N is an open pseudo-riemannian submanifold of M, then $\tau(M)(E)=\tau(N)(E)$ for all Borel sets $E\subset \operatorname{supp} N$,*

(ii) *if $f\colon M \to N$ is a C^1-diffeomorphism, then*

$$\tau(N)(f[E]) = \int\limits_E J_f\, d\tau(M)$$

for all Borel sets $E\subset \operatorname{supp} M$,

(iii) *if $M=\mathsf{R}^m$ or R^m_+, $m=1,2,\ldots$, then $\tau(M)$ is the Lebesgue measure.*

The proof uses Lemmas 7 and 8. It is rather long, however it does not differ from the proof given by Suominen [17, pp. 10–12] in the riemannian case: it is sufficient to replace everywhere in his proof the adjective "riemannian" by "pseudo-riemannian".

Now we define the *Lebesgue measure* on a pseudo-riemannian manifold M as the measure $\tau(M)$ determined in Theorem 2.

We conclude this section by two corollaries which are direct counterparts of theorems given in [17, p. 12]. In each of them M and N are pseudo-riemannian manifolds.

COROLLARY 1. *The Lebesgue measure on the pseudo-riemannian product manifold $M\times N$ is the product of the Lebesgue measures on M and N.*

COROLLARY 2. *If $f\colon M \to N$ is a C^1-diffeomorphism, then a Borel function $\varrho\colon N \to \mathsf{R}$ is $\tau(N)$-integrable if and only if $(\varrho\circ f)J_f$ is $\tau(M)$-integrable and*

$$\int\limits_N \varrho\, d\tau(N) = \int\limits_M (\varrho\circ f)J_f\, d\tau(M)\,.$$

4. Angles and their inner measure.

Let M be an essentially pseudo-riemannian manifold with metric g. For a real number a, $a \neq 0$, let

$$I_x^a M = \{v \in \operatorname{supp} T_x M : g(v,v) = a\}\,.$$

We say that a set E *forms an ordinary angle* $\arg(x,E)$ at a point x of M if E is a Borel subset of some $I_x^a M$, $a \neq 0$. We say that a set E *forms a topological angle* $\arg(x,E)$ at a point x of M if E is a Borel subset of either $I_x^+ M$ or $I_x^- M$.

Let $x \in \operatorname{supp} M$. Given a set E that forms a topological angle at x, let

$$I_x E = \{bv : v \in E,\ 0 < b < 1/|g(v,v)|^{\frac{1}{2}}\}\,.$$

It is easily seen that $I_x E$ is Lebesgue-measurable on $T_x M$. According to Section 3, we denote its Lebesgue measure by $\tau(T_x M)(I_x E)$ and the volume element by $d\tau(T_x M)$. We then define the *inner measure* $A(x,E)$ of $\arg(x,E)$ by

$$A(x,E) = \tau(T_x M)(I_x E)\,.$$

From the above definition it follows that various properties of the Lebesgue-measurable sets in Euclidean spaces can be translated into the language of inner measures of topological angles. Since such results are rather trivial there is no necessity to formulate them here.

It is clear that the above definitions make also sense in the case where M is pseudo-riemannian but not essentially, and that they agree with the definitions we already know, but these questions are not essential for us, so we prefer to leave them aside.

5. Curves and arc length.

Let M be an essentially pseudo-riemannian manifold with metric g. By a *curve* on M we understand a continuous mapping c from a closed interval $[a;b]$, $a \leqq b$, to M. If c is differentiable, we identify the derivative $Dc(t)$, $t \in [a;b]$, with a tangent vector to M at $c(t)$. This determines a curve Dc in the tangent bundle TM.

A curve c is called *space-like* (resp. *time-like*) if it is absolutely continuous and $Dc(t)$ is a vector of a positive (resp. negative) definite subspace of $T_{c(t)}M$ at every point of differentiability. If c is either space-like or time-like, it is called *regular*.

The length of a regular curve c is defined by

$$l(c) = \int_{[a;b]} |g(Dc(t),Dc(t)|^{\frac{1}{2}}\,dt\,.$$

If $l(c)$ is finite, c is said to be *rectifiable*. Now let $\varrho: M \to \mathsf{R}$ be a Borel function, c_0 the parametrization of c by arc length, and ds the arc length element. The integral of ϱ along c is defined by

$$\int_c \varrho\, ds = \int_0^{l(c)} \varrho \circ c_0\, ds\,,$$

provided that the latter integral exists. Otherwise the integral of ϱ along c is undefined.

Finally, suppose that N is an essentially pseudo-riemannian manifold and $f: M \to N$ a C^1-diffeomorphism. Then f is said to be *type-preserving* (resp. *type-reversing*) if it transforms space-like curves onto space-like (resp. time-like) curves and time-like curves onto time-like (resp. space-like) curves. Here we confine ourselves to one theorem needed later on:

THEOREM 3. *Suppose that $f: M \to N$ is either type-preserving or type-reversing, $c: [a; b] \to M$ is rectifiable, while $\varrho: N \to \mathsf{R}$ is Borel and non-negative. Then $f(c)$ is rectifiable and*

$$\int_{f(c)} \varrho\, ds \leqq \int_c (\varrho \circ f)\, \|Df\|\, ds\,.$$

The proof is analogous to that given in [17, p. 14], in the case of Riemannian manifolds.

6. Moduli.

Here we give an analogue of the p-moduli discussed in [17, pp. 15–20]. Our composition and proofs, however, follow rather [5] or [13]. Throughout the whole section M is an essentially pseudo-riemannian manifold, while $C, C_0, C_1, C_2, \ldots$ are families of regular curves on M.

Denote by $\operatorname{adm} C$ the class of all nonnegative Borel functions ϱ on M which satisfy

$$\int_c \varrho\, ds \geqq 1$$

for all rectifiable $c \in C$. Here we do not assume that the integrals in question are finite. If $\varrho \in \operatorname{adm} C$, ϱ is said to be an *admissible metric* for C. For each positive number p we define the *p-modulus* $\operatorname{mod}_p C$ of C by

$$\operatorname{mod}_p C = \inf \int_M \varrho^p\, d\tau\,,$$

where the infimum is taken over all $\varrho \in \operatorname{adm} C$. If $\operatorname{adm} C$ is empty, we put $\operatorname{mod}_p C = \infty$. The quantity $1/\operatorname{mod}_p C$ is called the *p-extremal length* of C.

If in $\operatorname{adm} C$ there is a metric ϱ_0 such that

$$\operatorname{mod}_p C = \int_M \varrho_0^p \, d\tau ,$$

then ϱ_0 is called *p-extremal.* It has the following important property:

THEOREM 4 (uniqueness of an extremal metric). *If, for some positive integer p, $\operatorname{mod}_p C$ is finite and ϱ_0, ϱ_0^* are p-extremal, then $\varrho_0^* = \varrho_0$ almost everywhere on M.*

PROOF. Since, clearly, $\frac{1}{2}(\varrho_0 + \varrho_0^*) \in \operatorname{adm} C$, then

$$\int_M [\tfrac{1}{2}(\varrho_0 + \varrho_0^*)]^p \, d\tau \geqq \operatorname{mod}_p C .$$

On the other hand, if $0 < k < p$, then, by the Hölder inequality, we have

$$\int_M \varrho_0^{p-k} \, \varrho_0^{*k} \, d\tau \leqq \left(\int_M \varrho_0^p \, d\tau \right)^{1-k/p} \left(\int_M \varrho_0^{*p} \, d\tau \right)^{k/p}$$
$$= (\operatorname{mod}_p C)^{1-k/p} \, (\operatorname{mod}_p C)^{k/p} = \operatorname{mod}_p C .$$

Hence

$$\int_M [\tfrac{1}{2}(\varrho_0 + \varrho_p^*)]^p \, d\tau = (\tfrac{1}{2})^p \sum_{k=0}^{p} \int_M \binom{p}{k} \varrho_0^{p-k} \varrho_0^{*k} \, d\tau$$
$$\leqq (\tfrac{1}{2})^p \operatorname{mod}_p C \sum_{k=0}^{p} \binom{p}{k} = \operatorname{mod}_p C .$$

Consequently,

$$\int_M [\tfrac{1}{2}(\varrho_0 + \varrho_0^*)]^p \, d\tau = \operatorname{mod}_p C ,$$

whence, by

$$\int_M \varrho_0^{p-k} \, \varrho_0^{*k} \, d\tau \leqq \operatorname{mod}_p C ,$$

we conclude that the quotient

$$(\varrho_0^{p-k})^{p/(p-k)} / (\varrho_0^{*k})^{p/k}$$

is a constant function almost everywhere on M (cf. e.g. [1, pp. 19–20]). Thus $\varrho_0^* = \varrho_0$ almost everywhere on M, as desired.

Now we formulate and prove other basic properties of p-moduli. Thereafter p is a positive number and $\sum$, $\bigcup$ denote summation over all positive integers k.

THEOREM 5 (monotoneity of moduli). *If $C_1 \subset C_2$ or, more generally, each c_1 of C_1 contains a c_2 of C_2, then*

$$\operatorname{mod}_p C_1 \leqq \operatorname{mod}_p C_2 .$$

PROOF. Suppose first that $C_1 \subset C_2$. Hence, by the definition of an admissible metric, we conclude that the relation $\varrho \in \operatorname{adm} C_2$ implies $\varrho \in \operatorname{adm} C_1$. Therefore $\operatorname{adm} C_2 \subset \operatorname{adm} C_1$. Since an infimum over a subset does not exceed the infimum over the whole set, we obtain

$$\operatorname{mod}_p C_2 = \inf_{\varrho \in \operatorname{adm} C_2} \int_M \varrho^p \, d\tau \geqq \inf_{\varrho \in \operatorname{adm} C_1} \int_M \varrho^p \, d\tau = \operatorname{mod}_p C_1 .$$

Suppose now that each c_1 of C_1 contains a c_2 of C_2. By the definition of $\operatorname{adm} C_1$ we may, without any loss of generality, assume that c_1 is rectifiable. Then, for $\varrho \in \operatorname{adm} C_2$ we get

$$\int_{c_1} \varrho \, ds \geqq \int_{c_2} \varrho \, ds \geqq 1 ,$$

whence $\varrho \in \operatorname{adm} C_1$ and the assertion follows by the reason given in the preceding case.

In order to prove the forthcoming theorem we need the following lemma (cf. [15], [11], or [1, p. 18]):

LEMMA 9. *If $0 < p \leqq q$ and $M_k \geqq 0$, $k = 1, 2, \dots$, then*

$$(\sum M_k^p)^{1/p} \geqq (\sum M_k^q)^{1/q} .$$

THEOREM 6 (the principle of composition for extremal lengths). *Suppose that C_k, $k = 1, 2, \dots$, consist of curves lying in disjoint Borel subsets E_k of* $\operatorname{supp} M$, *respectively, and that any c of C contains some curve of C_k for each k. Then*

$$\sum 1/\operatorname{mod}_p^{1/(p-1)} C_k \leqq 1/\operatorname{mod}_p^{1/(p-1)} C, \quad p > 1 , \tag{1}$$

$$\sum 1/\operatorname{mod}_p C_k \leqq 1/\operatorname{mod}_p C, \quad p \geqq 2 . \tag{2}$$

PROOF. If $1/\operatorname{mod}_p C = \infty$, (1) and (2) are obvious. Next, if $1/\operatorname{mod}_p C_k = \infty$ for some k, then, by Theorem 5, also $1/\operatorname{mod}_p C = \infty$. Finally, if

$1/\operatorname{mod}_p C_k = 0$ for some k, then the corresponding term on the left-hand side of (1) (resp. (2)) may be neglected.

Suppose then that $0 < 1/\operatorname{mod}_p C_k < \infty$. If $\varrho_k \in \operatorname{adm} C_k$, then, for any a_k satisfying

(3) $$0 \leqq a_k \leqq 1, \quad \sum a_k = 1\,,$$

we have

(4) $$\varrho = \sum a_k \varrho_k \in \operatorname{adm} C\,.$$

Indeed, take any c of C. By the definition of $\operatorname{adm} C$ we may, without any loss of generality, assume that c is rectifiable. By the hypotheses c contains some disjoint curves c_k of C_k for each k. Hence, by (3),

$$\int_c \varrho\, ds \geqq \int_{\Sigma c_k} \varrho\, ds = \sum \int_{c_k} \varrho\, ds = \sum a_k \int_{c_k} \varrho_k\, ds \geqq \sum a_k = 1\,.$$

On the other hand, since E_k are disjoint, we may assume that

(5) $$\varrho_k(x) = 0, \quad x \in \operatorname{supp} M \setminus E_k \supset \bigcup_{i \neq k} E_i\,.$$

Let now

$$M_k = \int_{E_k} \varrho^p\, d\tau, \quad a_k = M_k^{-1/(p-1)} / \sum M_i^{-1/(p-1)}\,.$$

Then, by (4) and (5), we get

$$\begin{aligned} \operatorname{mod}_p C &\leqq \int_M \varrho^p\, d\tau = \int_M (\sum a_k \varrho_k)^p\, d\tau = \sum a_k^p \int_{E_k} \varrho_k^p\, d\tau \\ &= \sum (M_k^{-1/(p-1)} / \sum M_i^{-1/(p-1)})^p M_k = (\sum M_k^{-1/(p-1)})^{1-p}\,, \end{aligned}$$

whence

$$1/\operatorname{mod}_p^{1/(p-1)} C \geqq \sum M_k^{-1/(p-1)}\,.$$

Since $1/M_k$ may be chosen arbitrarily near to $1/\operatorname{mod}_p C_k$, respectively, and $p > 1$, then (1) follows.

Estimate (2) is a consequence of (1) and Lemma 9, provided that $p \geqq 2$.

THEOREM 7 (subadditivity of moduli). *If $C = \bigcup C_k$, then*

(6) $$\operatorname{mod}_p C \leqq \sum \operatorname{mod}_p C_k\,.$$

PROOF. If the right-hand side of (6) is infinite, (6) is obvious. Suppose then that

$$\sum \operatorname{mod}_p C_k < \infty\,,$$

and take any $\varepsilon > 0$ and $\varrho_k \in \operatorname{adm} C_k$ so that

$$\int_M \varrho_k^p \, d\tau \leqq \operatorname{mod}_p C_k + 2^{-k}\varepsilon . \tag{7}$$

Clearly,

$$\varrho = (\Sigma \varrho_k^p)^{1/p} \tag{8}$$

is a Borel function and $\varrho \in \operatorname{adm} C$. Indeed, let $c \in C$. Then there is a k_0 such that $c \in C_{k_0}$, whence

$$\int_c \varrho \, ds \geqq \int_c \varrho_{k_0} \, ds \geqq 1$$

and, consequently, $\varrho \in \operatorname{adm} C$. Furthermore, by (7) and (8),

$$\operatorname{mod}_p C \leqq \int_M \varrho^p \, d\tau = \Sigma \int_M \varrho_k^p \, d\tau \leqq \Sigma \operatorname{mod}_p C_k + \varepsilon .$$

Since ε can be chosen arbitrarily near to 0, then (6) follows.

Theorems 5 and 7 yield

COROLLARY 3. *If* $\operatorname{mod}_p C_0 = 0$, *then* $\operatorname{mod}_p (C \cup C_0) = \operatorname{mod}_p C$.

THEOREM 8 (superadditivity of moduli). (i) *Suppose that* $\bigcup C_k \subset C$ *and that all* C_k *consist of curves lying in disjoint Borel subsets* E_k *of* $\operatorname{supp} M$, *respectively. Then*

$$\Sigma \operatorname{mod}_p C_k \leqq \operatorname{mod}_p C . \tag{9}$$

(ii) *The inequality* (9) *remains valid if the condition* $\bigcup C_k \subset C$ *is replaced by the requirement for each* c_k *of* C_k, $k = 1, 2, \ldots$, *to contain some curve of* C.

PROOF. Consider first (i). If $\operatorname{mod}_p C = \infty$, (9) is obvious. Suppose then that $\operatorname{mod}_p C < \infty$ and

$$\int_M \varrho^p \, d\tau \leqq \operatorname{mod}_p C + \varepsilon , \tag{10}$$

where $\varrho \in \operatorname{adm} C$ and ε, $\varepsilon > 0$, is chosen arbitrarily. It is easily seen that the functions ϱ_k, defined by $\varrho_k(x) = \varrho(x)$ for $x \in E_k$ and by $\varrho_k(x) = 0$ for $x \in \operatorname{supp} M \setminus E_k$, belong to $\operatorname{adm} C_k$, respectively. Consequently, by (10),

$$\sum \operatorname{mod}_p C_k \leqq \sum \int_M \varrho_k^p \, d\tau = \sum \int_{E_k} \varrho^p \, d\tau$$

$$= \int_{\cup E_k} \varrho^p \, d\tau \leqq \int_M \varrho^p \, d\tau \leqq \operatorname{mod}_p C + \varepsilon \, .$$

Hence (9) follows.

In case (ii), the proof remains almost unchanged.

REMARK 1. Theorems 4–8 are valid also in the case where M is pseudo-riemannian but not essentially. If the index p of M equals 1, i.e. in the riemannian case, $C, C_0, C_1, C_2, \ldots$ denote just families of curves on M (cf. [17, pp. 15–20]). If $\frac{1}{2}n < p < n$, we can establish the same results as those given above, if we replace the metric g of M by $-g$.

Part II.

Conformal mappings of essentially pseudo-riemannian manifolds.

Throughout Sections 7–10 we always assume that M and N are essentially pseudo-riemannian manifolds with metrics g and g', respectively, while $f: M \to N$ is a C^1-diffeomorphism.

Moreover, a sum over the empty set of indices is interpreted as 0. We do not apply the Einstein summation convention since in various places of our paper it may lead to misunderstanding.

7. Conformality.

A C^1-diffeomorphism $f: M \to N$ is said to be *conformal*, if

(11) $$Df(x)[I_x^0 M] = I_{f(x)}^0 N, \quad x \in \operatorname{supp} M \, ,$$

in the case where the index of g is less than $\frac{1}{2}n$, while

(12) $$Df(x)[I_x^+ M] = I_{f(x)}^+ N, \quad x \in \operatorname{supp} M \, ,$$

in the case where the index of g equals $\frac{1}{2}n$, n being the dimension of g.

We begin our considerations with two lemmas and then prove a theorem that gives a necessary and sufficient condition for conformality. This condition agrees with the usual definition applied in the case of Riemannian manifolds (cf. [9, p. 106], [12, vol. I, p. 309], and [17, p. 16]) as well as in the local formulation in the case of pseudo-riemannian manifolds (cf. [4, p. 89] and [10, p. 5]).

LEMMA 10. *Suppose that a^i and A_i^j, $i,j = 1, \ldots, n$, $n \geqq 2$, are arbitrary real numbers and*

(13) $$b^j = \sum_i A^j_i a^i ,$$

(14) $$-\bar{g} = \sum_{i\leqq p}(a^i)^2 - \sum_{i\geqq p+1}(a^i)^2 ,$$

(15) $$-\bar{g}' = \sum_{j\leqq p}(b^j)^2 - \sum_{j\geqq p+1}(b^j)^2 ,$$

where $1\leqq p\leqq n-1$. *If for every system* (a^i) *such that* $\bar{g}=0$ *we also have* $\bar{g}'=0$, *then*

(16) $$\bar{g}' = a\bar{g} ,$$

where a *is real and does not depend on* (a^i). *If, in addition,* $p\neq\frac{1}{2}n$, *then* $a\geqq 0$.

PROOF[1]. We shall consider, separately, three cases: $1\leqq p\leqq n-2$, $p=n-1\neq 1$, and $p=n-1=1$.

Suppose first that $1\leqq p\leqq n-2$. From (15) and (13) we get

$$-\bar{g}' = \sum_{j\leqq p}(\sum_i A^j_i a^i)^2 - \sum_{j\geqq p+1}(\sum_i A^j_i a^i)^2 .$$

Next we put

(17) $$\bar{g}' = \sum_{i,j} B_{ij} a^i a^j .$$

Clearly, B_{ij} do not depend on (a^i) and we may assume that

(18) $$B_{ji} = B_{ij} .$$

After a rearrangement, (17) becomes

$$\sum_{i<n} B_{ii}(a^i)^2 + B_{nn}(a^n)^2 + 2\sum_{i<j<n} B_{ij}a^i a^j - \bar{g}' = -2\sum_{i<n} B_{in} a^i a^n .$$

Now, applying (14), we get

$$[\sum_{i\leqq p}(B_{ii}+B_{nn})(a^i)^2 + \sum_{p<i<n}(B_{ii}-B_{nn})(a^i)^2 + 2\sum_{i<j<n} B_{ij}a^i a^j - \bar{g}' + B_{nn}\bar{g}]^2$$
$$= 4(\sum_{i<n} B_{in}a^i)^2[\sum_{j\leqq p}(a^j)^2 - \sum_{p<j<n}(a^j)^2 + \bar{g}] .$$

Since for every system (a^i) such that $\bar{g}=0$ also $\bar{g}'=0$, we obtain

$$\sum_{i\leqq p}(B_{ii}+B_{nn})^2(a^i)^4 + \sum_{p<i<n}(B_{ii}-B_{nn})^2(a^i)^4 + 4\sum_{i<j<n} B^2_{ij}(a^i)^2(a^j)^2 +$$
$$+ 2\sum_{i<j\leqq p}(B_{ii}+B_{nn})(B_{jj}+B_{nn})(a^i)^2(a^j)^2 +$$
$$+ \sum_{p<i<j<n}(B_{ii}-B_{nn})(B_{jj}-B_{nn})(a^i)^2(a^j)^2 +$$

---

[1] Prof. H. Haahti communicated to the authors that he had independently obtained a proof of this lemma (unpublished).

$$+8\sum_{\substack{i<j<n\\k<l<n\\i+j<k+l}} B_{ij}B_{kl}a^i a^j a^k a^l + 2\sum_{i\leqq p<j<n}(B_{ii}+B_{nn})(B_{jj}-B_{nn})(a^i)^2(a^j)^2 +$$

$$+4\sum_{\substack{i<j<n\\k\leqq p}} B_{ij}(B_{kk}+B_{nn})a^i a^j (a^k)^2 + 4\sum_{\substack{i<j<n\\p<k<n}} B_{ij}(B_{kk}-B_{nn})a^i a^j (a^k)^2$$

$$= 4\sum_{\substack{i<n\\j\leqq p}} B_{in}^2(a^i)^2(a^j)^2 - 4\sum_{\substack{i<n\\p<j<n}} B_{in}^2(a^i)^2(a^j)^2 +$$

$$+8\sum_{\substack{i<j<n\\k\leqq p}} B_{in}B_{jn}a^i a^j (a^k)^2 - 8\sum_{\substack{i<j<n\\p<k<n}} B_{in}B_{jn}a^i a^j (a^k)^2$$

identically with respect to $(a^1,\ldots,a^{n-1})$. Hence we conclude that the polynomials on both sides of the above relation have the same coefficients. Consequently, by (18),

$$(B_{ii}+B_{nn})^2 = 4B_{in}^2, \qquad i\leqq p, \tag{19}$$

$$(B_{ii}-B_{nn})^2 = -4B_{in}^2, \qquad p<i<n, \tag{20}$$

$$4B_{ij}^2+2(B_{ii}+B_{nn})(B_{jj}+B_{nn}) = 4B_{in}^2+4B_{jn}^2, \qquad i<j\leqq p, \tag{21}$$

$$4B_{ij}^2+2(B_{ii}+B_{nn})(B_{jj}-B_{nn}) = 4B_{jn}^2-4B_{in}^2, \qquad i\leqq p<j<n, \tag{22}$$

$$4B_{ij}^2+2(B_{ii}-B_{nn})(B_{jj}-B_{nn}) = -4B_{in}^2-4B_{jn}^2, \qquad p<i<j<n, \tag{23}$$

$$8\eta_{ijk}B_{ik}B_{jk}+4B_{ij}(B_{kk}+B_{nn}) = 8B_{in}B_{jn}, \qquad i<j<n,\ k\leqq p, \tag{24}$$

$$8\eta_{ijk}B_{ik}B_{jk}+4B_{ij}(B_{kk}-B_{nn}) = -8B_{in}B_{jn}, \qquad i<j<n,\ p<k<n, \tag{25}$$

$$8B_{ij}B_{kl} = 0, \qquad i<j<n,\ k<l<n,\ i+j<k+l,\ k\neq i,j,\ l\neq j, \tag{26}$$

where $\eta_{ijk}=0$ for $k=i,j$, and $\eta_{ijk}=1$ otherwise. From (19) and (20) we get

$$B_{in} = \tfrac{1}{2}\eta_i(B_{ii}+B_{nn}), \qquad \eta_i=1 \text{ or } -1,\ i\leqq p, \tag{27}$$

$$B_{in} = 0, \qquad p<i<n, \tag{28}$$

$$B_{ii} = B_{nn}, \qquad p<i<n. \tag{29}$$

On the other hand, by our hypothesis, $1\leqq p\leqq n-2$. Therefore (21)–(23) become

$$4B_{ij}^2+8\eta_i\eta_jB_{in}B_{jn} = 4B_{in}^2+4B_{jn}^2, \qquad i<j\leqq p,$$

$$4B_{ij}^2 = -4B_{in}^2, \qquad i\leqq p<j<n,$$

$$4B_{ij}^2 = 0, \qquad p<i<j<n,$$

respectively. Hence

$$B_{in} = 0, \qquad i \leqq p, \tag{30}$$

and, consequently,

$$B_{ii} = -B_{nn}, \quad i \leqq p, \tag{31}$$

$$B_{ij} = 0, \qquad i<j. \tag{32}$$

One can easily check that (28)–(32) is a solution of (19)–(26) and this solution is unique. Applying (18) and (28)–(32) to (17) we obtain

$$-\bar{g}' = B_{nn}\Sigma_{i\leqq p}(a^i)^2 - B_{nn}\Sigma_{i\geqq p+1}(a^i)^2,$$

that is, (16) with $a=B_{nn}$. Since, as remarked before, the B_{ij} do not depend on (a^i), then a does not depend on (a^i) either. If, in addition, $p \neq \frac{1}{2}n$, relations (15) and (13) yield $B_{nn} \geqq 0$, whence $a \geqq 0$.

Suppose next that $p=n-1 \neq 1$. Then we replace $\bar{g}$ by $-\bar{g}$ and $\bar{g}'$ by $-\bar{g}'$, and apply Lemma 10 which is already proved in the case we need: $p=1$, $n \geqq p+2=3$.

Finally, suppose that $p=n-1=1$. Then

$$b^1 = A^1_1 a^1 + A^1_2 a^2, \quad b^2 = A^2_1 a^1 + A^2_2 a^2,$$
$$-\bar{g} = (a^1+a^2)(a^1-a^2), \quad -\bar{g}' = (b^1+b^2)(b^1-b^2).$$

Since for every system (a^i) such that $\bar{g}=0$ we also have $\bar{g}'=0$, then $\bar{g}'$, treated as a polynomial of a^1 and a^2, must be divisible by a^1+a^2 and a^1-a^2. On the other hand, this polynomial is a quadratic form. Therefore it is of the form (16), where a is real and does not depend on (a^i).

This completes the proof.

Remark 2. One can easily check that under the hypotheses of Lemma 10 with the additional assumption $p=n-1=1$ we also have $B_{11}=-B_{22}$ and $B_{21}=B_{12}=0$, where B_{ij} are defined by (17) and (18), i.e. we have $(A^1_1)^2+(A^1_2)^2=(A^2_2)^2+(A^2_1)^2$ and $A^1_1A^1_2=A^2_2A^2_1$, or, what is the same,

$$A^1_1 = \tfrac{1}{2}(\eta_1+\eta_2)A^2_2 + \tfrac{1}{2}(\eta_1-\eta_2)A^2_1$$

and

$$A^1_2 = \tfrac{1}{2}(\eta_1-\eta_2)A^2_2 + \tfrac{1}{2}(\eta_1+\eta_2)A^2_1,$$

where $\eta_1, \eta_2 = 1$ or -1.

Lemma 11. *If $f\colon M \to N$ is conformal, then M and N are of the same dimension and of the same index.*

PROOF. Since f is a homeomorphism, M and N are of the same dimension n, say. Now, denote by p and q the indices of M and N, respectively, i.e. the indices of g and g', respectively. Applying the fact that the nullity of g as well as of g' is zero we conclude that the dimensions of maximal isotropy subspaces of T_xM and $T_{f(x)}N$ are $\frac{1}{2}(n-|n-2p|)$ and $\frac{1}{2}(n-|n-2q|)$, respectively (cf. e.g. [2, pp. 106–107]). Since f is conformal, we have

$$\tfrac{1}{2}(n-|n-2q|) = \tfrac{1}{2}(n-|n-2p|) .$$

On the other hand, since M and N are essentially pseudo-riemannian, we have $p \leqq \frac{1}{2}n$ and $q \leqq \frac{1}{2}n$. Therefore $q=p$, as desired.

THEOREM 9. *A C^1-diffeomorphism $f\colon M \to N$ is conformal if and only if*

$$(33) \quad g'(Df(x)(v), Df(x)(v)) = a(x)g(v,v) ,\quad a(x)>0,\ x \in \operatorname{supp} M,\ v \in \operatorname{supp} T_xM ,$$

where a does not depend on v.

PROOF. Since, as it is easily seen, the sufficiency of (33) is obvious, we have only to prove the necessity.

Suppose then that $f\colon M \to N$ is conformal. Given an $x \in \operatorname{supp} M$ choose two orthonormal bases: $(e_i)(x)$ of T_xM and $(\varepsilon_i)\circ f(x)$ of $T_{f(x)}N$. By Lemma 11, M and N are of the same dimension n, say, and of the same index p, say. Consequently, since g and g' are nondegenerate, we may assume that

$$(34) \quad \begin{aligned} g(e_i,e_i)(x) &= -1 \quad \text{if } i \leqq p ,\\ &= 1 \quad \text{if } i \geqq p+1 ,\end{aligned}$$

$$(35) \quad \begin{aligned} g'(\varepsilon_i,\varepsilon_i)\circ f(x) &= -1 \quad \text{if } i \leqq p ,\\ &= 1 \quad \text{if } i \geqq p+1 .\end{aligned}$$

Let now

$$(36) \quad v = \textstyle\sum_i a^i(x)\, e_i(x) ,$$

where $a^i(x)$ are real, and

$$(37) \quad Df(x)(v) = \textstyle\sum_j b^j(x)\, \varepsilon_j\circ f(x) .$$

Then, by (34) and (35),

$$-g(v,v) = \textstyle\sum_{i\leqq p}(a^i)^2(x) - \sum_{i\geqq p+1}(a^i)^2(x)$$

and

$$-g'(Df(x)(v), Df(x)(v)) = \textstyle\sum_{j\leqq p}(b^j)^2(x) - \sum_{j\geqq p+1}(b^j)^2(x) ,$$

respectively, where, since M is essentially pseudo-riemannian, we have $1 \leqq p \leqq n-1$. On the other hand, we have (cf. e.g. [2, pp. 56 and 50])

$$(38) \qquad Df(v)(v) = \sum_{i,j} a^i(x)\,(\nu^j \circ f \circ \mu^{-1})_{|i} \circ \mu(x)\,\varepsilon_j \circ f(x)\,,$$

where μ and ν are coordinate C^∞-mappings on M at x and on N at $f(x)$, respectively, and $_{|i}$ denotes partial differentiation with respect to μ^i. Therefore

$$b^j(x) = \sum_i A_i^j(x)\,a^i(x)\,,$$

where

$$(39) \qquad A_i^j(x) = (\nu^j \circ f \circ \mu^{-1})_{|i} \circ \mu(x)\,.$$

We notice that the $A_i^j(x)$ do not depend on v. Furthermore, since f is conformal, then for every system $(a^i)(x)$ such that $g(v,v)=0$ we also have $g(Df(x)(v),Df(x)(v))=0$.

Thus we conclude that the hypotheses of Lemma 10 are fulfilled. By this lemma we obtain

$$(40) \qquad g(Df(x)(v),Df(x)(v)) = a(x)\,g(v,v)\,,$$

where $a(x)$ is real and does not depend on $(a^i)(x)$. If, in addition, $p<\frac{1}{2}n$, then, by the same lemma, $a(x) \geqq 0$. Besides, since f is a diffeomorphism, then $a(x) \neq 0$ for any p in question. If, in particular, $p=\frac{1}{2}n$, then, since f is conformal, the relation $v \in I_x^+M$ implies $Df(x)(v) \in I_{f(x)}^+N$, whence also in this case $a(x)>0$. Therefore, by (40), we conclude that the relation (33) holds, as desired, with an a independent of v.

Theorem 9 implies

COROLLARY 4. *The conditions* (12) *and*

$$Df(x)[I_x^-M] = I_{f(x)}^-N, \quad x \in \operatorname{supp} M\,,$$

are both necessary and sufficient for a C^1-diffeomorphism $f\colon M \to N$ to be conformal.

8. Preservation of angles.

In this section we give a characterization of conformal mappings in terms of the inner measure of angles. To this end we need a lemma of a purely algebraic character.

LEMMA 12. *Suppose that a^i and A_i^j, $i,j=1,\ldots,n$, $n \geqq 2$, are arbitrary real numbers and that relations* (13)–(15) *hold, where $1 \leqq p \leqq n-1$. If $\bar{g}=0$ implies $\bar{g}'=0$ for every system (a^i), then*

$$\bar{g}' = \eta\,|\det A_i^j|^{2/n}\bar{g}\,, \tag{41}$$

where $\eta = 1$ *for* $p \neq \frac{1}{2}n$ *and* $\eta = 1$ *or* -1 *for* $p = \frac{1}{2}n$.

PROOF. We have

$$(-1)^p(\det A_i^j)^2 = \begin{vmatrix} -A_1^1 \dots -A_1^p & A_1^{p+1} \dots A_1^n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ -A_n^1 \dots -A_n^p & A_n^{p+1} \dots A_n^n \end{vmatrix} \begin{vmatrix} A_1^1 \dots A_1^n \\ \dots\dots\dots \\ A_n^1 \dots A_n^n \end{vmatrix}$$

$$= \begin{vmatrix} -\sum_{k\leq p} A_1^k A_1^k + \sum_{k\geq p+1} A_1^k A_1^k \dots -\sum_{k\leq p} A_1^k A_n^k + \sum_{k\geq p+1} A_1^k A_n^k \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -\sum_{k\leq p} A_n^k A_1^k + \sum_{k\geq p+1} A_n^k A_1^k \dots -\sum_{k\leq p} A_n^k A_n^k + \sum_{k\geq p+1} A_n^k A_n^k \end{vmatrix}$$

$$= \det B_{ij}\,,$$

where the B_{ij} are defined by (17). Since the hypotheses of Lemma 10 are fulfilled, in the case where $1 \leqq p \leqq n-2$ we may apply relations (28)–(32). Hence, by (18), we obtain

$$\det B_{ij} = (-1)^p B_{nn}^n$$

and, consequently, the relation (41), where $\eta = 1$ for n odd and $\eta = 1$ or -1 for n even. Finally, by Lemma 10, we conclude that $\eta = 1$ in the case where $p \neq \frac{1}{2}n$.

If $p = n-1 \neq 1$, we replace $\bar{g}$ by $-\bar{g}$ and $\bar{g}'$ by $-\bar{g}'$, and apply the previous result. If $p = n-1 = 1$, we easily check (41) by direct calculation. It can also be derived from Remark 2.

THEOREM 10. *If* $f\colon M \to N$ *is conformal and* E *forms a topological angle at* $x \in \operatorname{supp} M$, *then*

(i) $Df(x)[E]$ *forms a topological angle at* $f(x)$ *and*

$$A(f(x), Df(x)[E]) = A(x,E)\,, \tag{42}$$

(ii) *the relation* $E \subset I_x^+M$ *implies* $Df(x)[E] \subset I_{f(x)}^+N$, *while* $E \subset I_x^-M$ *implies* $Df(x)[E] \subset I_{f(x)}^-N$.

If, in particular, E *forms an ordinary angle at* x, *then* $Df(x)[E]$ *forms an ordinary angle at* $f(x)$.

PROOF. By the definition of A, relation (42) may be rewritten in the equivalent form

$$(43) \qquad \int\limits_{I_{f(x)}Df(x)[E]} d\tau(T_{f(x)}N) = \int\limits_{I_xE} d\tau(T_xM)$$

provided that $Df(x)[E]$ forms a topological angle at $f(x)$.

Let us choose two orthonormal bases: $(e_i)(x)$ of T_xM and $(\varepsilon_i)\circ f(x)$ of $T_{f(x)}N$. By Lemma 11, M and N are of the same dimension n, say, and of the same index p, say. Consequently, since g and g' are nondegenerate, we may assume that relations (34) and (35) hold.

Let now $v \in T_xM$, and let v and $Df(x)(v)$ have the coordinate representations (36) and (37), respectively. Further, let $A_i^j(x)$ be defined by (39), where μ and ν are coordinate C^∞-mappings on M at x and on N at $f(x)$, respectively, and $_{|i}$ denotes partial differentiation with respect to μ^i. In the same way as in the proof of Theorem 9 one can verify that the hypotheses of Lemma 10 are fulfilled. On the other hand, they are identical with the hypotheses of Lemma 11. By these lemmas we obtain

$$(44) \qquad g'\big(Df(x)(v), Df(x)(v)\big) = \eta\,|\det A_i^j(x)|^{2/n} g(v,v)\,,$$

where $\eta = 1$ for $p \neq \frac{1}{2}n$ and $\eta = 1$ or -1 for $p = \frac{1}{2}n$. Furthermore, if $p = \frac{1}{2}n$, then, since f is conformal, the relation $v \in I_x^+M$ implies $Df(x)(v) \in I_{f(x)}^+N$, whence also in this case $\eta = 1$. Therefore, by (44) and the hypothesis that f is a diffeomorphism, we conclude that $Df(x)[E]$ forms a topological angle at $f(x)$ and that

$$\begin{aligned} I_{f(x)}Df(x)[E] &= \{b\,Df(x)(v) : v \in E,\ 0 < b < 1/(|\det A_i^j(x)|^{2/n}|g(v,v)|)^{\frac{1}{2}}\} \\ &= \{|\det A_i^j(x)|^{-1/n} b\,Df(x)(v) : v \in E,\ 0 < b < 1/|g(v,v)|^{\frac{1}{2}}\}\,. \end{aligned}$$

Next we observe that, since $Df(x)$ is a linear function (cf. e.g. formula (38)), then

$$I_{f(x)}Df(x)[E] = \{Df(x)\big(|\det A_i^j(x)|^{-1/n} b v\big) : v \in E,\ 0 < b < 1/|g(v,v)|^{\frac{1}{2}}\}\,.$$

Consequently, the left-hand side of (43) is equal to

$$\int\limits_{I_xE} \big(|(\det Df)(x)|/|\det A_i^j(x)|\big)\, d\tau(T_xM)\,.$$

Since the coefficients $A_i^j(x)$ correspond to some orthonormal bases of T_xM and $T_{f(x)}N$, Theorem 1 and relation (39) yield

$$|(\det Df)(x)| = |\det A_i^j(x)|\,,$$

whence (43) follows.

Furthermore, applying relations (44) and $\eta = 1$, we conclude that if

$E \subset I_x^+M$, then $Df(x)[E] \subset I_{f(x)}^+N$, while if $E \subset I_x^-M$, then $Df(x)[E] \subset I_{f(x)}^-N$.

The last conclusion of Theorem 10 is a straightforward consequence of (44).

THEOREM 11. *Suppose that $f\colon M \to N$ is a C^1-diffeomorphism and that if E forms an ordinary angle at $x \in \operatorname{supp} M$, then*

(i) *$Df(x)[E]$ forms a topological angle at $f(x)$,*

(ii) *the relation $E \subset I_x^+M$ implies $Df(x)[E] \subset I_{f(x)}^+N$, while $E \subset I_x^-M$ implies $Df(x)[E] \subset I_{f(x)}^-N$.*

Then f is conformal.

PROOF. If $v \in I_x^+M$, then the set $I_x^{g(v,v)}M$ forms an ordinary angle at x, so, by the hypotheses, $Df(x)[I_x^{g(v,v)}M]$ forms a topological angle at $f(x)$ and

$$Df(x)[I_x^{g(v,v)}M] \subset I_{f(x)}^+N .$$

Therefore

$$Df(x)[I_x^+M] \subset I_{f(x)}^+N . \tag{45}$$

On the other hand, if $w \in I_{f(x)}^+N$, then, since f is a diffeomorphism, there is a $v \in T_xM$ such that $Df(x)(v) = w$. If $v \in I_x^-M$, then the set $I_x^{g(v,v)}M$ forms an ordinary angle at x, so, by the hypotheses, $Df(x)[I_x^{g(v,v)}M]$ forms a topological angle at $f(x)$ and

$$Df(x)[I_x^{g(v,v)}M] \subset I_{f(x)}^-N .$$

Since this implies that $w \in I_{f(x)}^-N$, the relation $v \in I_x^-M$ is impossible. Now, if $v \in I_x^0M$, then, since f is a C^1-diffeomorphism, there is a $\check{v} \in I_x^-M$ such that $Df(x)(\check{v}) \in I_{f(x)}^+N$, but we have already proved that this is impossible. Hence we conclude that $v \in I_x^+M$ and therefore

$$Df(x)[I_x^+M] \supset I_{f(x)}^+N . \tag{46}$$

Relations (45) and (46) together with Corollary 4 imply that f is conformal, as desired.

9. Preservation of moduli.

Finally we give a characterization of conformal mappings in terms of moduli.

THEOREM 12. *If f is conformal, then it is type-preserving. Furthermore, if C is a family of regular curves on M, then*

$$\operatorname{mod}_n f(C) = \operatorname{mod}_n C . \tag{47}$$

If, in particular,

$$0 < k \leqq \|Df(x)\| \leqq K < \infty, \quad x \in \operatorname{supp} M\,, \tag{48}$$

then

$$K^{n-p} \operatorname{mod}_p C \leqq \operatorname{mod}_p f(C) \leqq k^{n-p} \operatorname{mod}_p C \quad \textit{for } p \geqq n \tag{49}$$

and

$$k^{n-p} \operatorname{mod}_p C \leqq \operatorname{mod}_p f(C) \leqq K^{n-p} \operatorname{mod}_p C \quad \textit{for } p \leqq n\,. \tag{50}$$

PROOF. If f is conformal, then, by Theorem 9, it transforms space-like curves onto space-like curves and time-like curves onto time-like curves, so f is type-preserving.

Consider now a family C of regular curves on M and suppose that $\varrho \in \operatorname{adm} f(C)$. Hence, by Corollary 2,

$$\int_N \varrho^n \, d\tau(N) = \int_M (\varrho \circ f)^n J_f \, d\tau(M)\,. \tag{51}$$

Given an $x \in \operatorname{supp} M$ choose two orthonormal bases: $(e_i)(x)$ of $T_x M$ and $(\varepsilon_i)\circ f(x)$ of $T_{f(x)}N$. By Lemma 11, M and N are of the same dimension and of the same index. Consequently, since g and g' are nondegenerate, we may assume that relations (34) and (35) hold. Let now $v \in T_x M$, and let v and $Df(x)(v)$ have the coordinate representations (36) and (37), respectively. Further, let $A^j_i(x)$ be defined by (39), where μ and ν are coordinate C^∞-mappings on M at x and on N at $f(x)$, respectively, and $_{|i}$ denotes partial differentiation with respect to μ^i. In the same way as in the proof of Theorem 10 one can verify that the relation (44) with $\eta = 1$ holds. Therefore (cf. Theorem 1)

$$\|Df(x)\| = |\det A^j_i(x)|^{1/n}\,.$$

On the other hand, since the coefficients $A^j_i(x)$ correspond to some orthonormal bases of $T_x M$ and $T_{f(x)}N$, Theorem 1 and relation (39) yield

$$J_f(x) = |\det A^j_i(x)|\,.$$

Consequently,

$$J_f = \|Df\|^n \tag{52}$$

and (51) becomes

$$\int_N \varrho^n \, d\tau(N) = \int_M (\varrho \circ f)^n \, \|Df\|^n \, d\tau(M)\,. \tag{53}$$

Since, by Theorem 3, we have $(\varrho \circ f)\|Df\| \in \operatorname{adm} C$, relation (53) gives

$\operatorname{mod}_n f(C) \geqq \operatorname{mod}_n C$. On the other hand, since f^{-1} is also conformal, we have $\operatorname{mod}_n f(C) \leqq \operatorname{mod}_n C$, whence (47) follows.

Finally, suppose that f satisfies the additional conditon (48). Consider again a family C of regular curves on M and suppose that $\varrho \in \operatorname{adm} f(C)$. Hence, by Corollary 2, for each positive number p we have

$$\int_N \varrho^p \, d\tau(N) = \int_M (\varrho \circ f)^p J_f \, d\tau(M) .$$

Now, by (52), we get

$$\int_N \varrho^p \, d\tau(N) = \int_M (\varrho \circ f)^p \, \|Df\|^n \, d\tau(M) .$$

Therefore, by (48),

$$\int_N \varrho^p \, d\tau(N) \geqq K^{n-p} \int_M (\varrho \circ f)^p \, \|Df\|^p \, d\tau(M) \quad \text{for } p \geqq n \tag{54}$$

and

$$\int_N \varrho^p \, d\tau(N) \geqq k^{n-p} \int_M (\varrho \circ f)^p \, \|Df\|^p \, d\tau(M) \quad \text{for } p \leqq n . \tag{55}$$

Since, by Theorem 3, we have $(\varrho \circ f)\|Df\| \in \operatorname{adm} C$, relations (54) and (55) give

$$\operatorname{mod}_p f(C) \geqq K^{n-p} \operatorname{mod}_p C \quad \text{for } p \geqq n$$

and

$$\operatorname{mod}_p f(C) \geqq k^{n-p} \operatorname{mod}_p C \quad \text{for } p \leqq n ,$$

respectively. On the other hand, since f^{-1} is also conformal, we have

$$\operatorname{mod}_p f(C) \leqq k^{n-p} \operatorname{mod}_p C \quad \text{for } p \geqq n$$

and

$$\operatorname{mod}_p f(C) \leqq K^{n-p} \operatorname{mod}_p C \quad \text{for } p \leqq n ,$$

whence (49) and (50) follow.

THEOREM 13. *If $f \colon M \to N$ is type-preserving, then it is conformal.*

PROOF. If $v \in I_x^+ M$, then there is a space-like curve $c \colon [a; b] \to M$ and a number $t \in [a; b]$ such that $Dc(t) = v$. By the hypothesis, $f(c)$ is a space-like curve on N, so $Df(x)(v) \in I_{f(x)}^+ N$. Therefore (45) holds.

On the other hand, if $w \in I^+_{f(x)}N$, then, since f is a diffeomorphism, there is a $v \in T_xM$ such that $Df(x)(v) = w$. If $v \in I^-_xM$, then there is a time-like curve $c\colon [a; b] \to M$ and a number $t \in [a; b]$ such that $Dc(t) = v$. By the hypothesis, $f(c)$ is a time-like curve on N, so $Df(x)(v) \in I^-_{f(x)}N$. Since this implies that $w \in I^-_{f(x)}N$, the relation $v \in I^-_xM$ is impossible. Now, if $v \in I^0_xM$, then, since f is a C^1-diffeomorphism, there is a $\check{v} \in I^-_xM$ such that $Df(x)(\check{v}) \in I^+_{f(x)}N$, but we have already proved that this is impossible. Hence we conclude that $v \in I^+_xM$ and therefore (46) holds.

Relations (45) and (46) together with Corollary 4 imply that f is conformal, as desired.

10. Conclusions.

Suppose that $f\colon M \to N$ is type-preserving and that there is a constant Q, $1 \leqq Q < \infty$, such that

$$(1/Q)\operatorname{mod}_n C \leqq \operatorname{mod}_n f(C) \leqq Q \operatorname{mod}_n C \tag{56}$$

for some family C of regular curves. Then, by Theorem 13, f is conformal and consequently, by Theorem 12, we get (47). Hence we conclude that in the case of essentially pseudo-riemannian manifolds there is no analogue of regular quasiconformal mappings other than conformal (cf. [14, pp. 18, 179, and 222 (Theorems 3.2 and 4.2)], for the plane case; [18, pp. 18–19], for the euclidean case; and [17, pp. 24–25], for the riemannian case). Nevertheless, it is quite possible that if we properly weaken the hypotheses of Theorem 13 in the sense that we allow some less smooth mappings and assume, in addition, that f preserves the n-moduli, we will still be able to prove that f is conformal (cf. [6, pp. 388–390]). Then it will be natural to consider also the case where the preservation of the n-moduli is replaced by a quasi-preservation in the sense of (56) with some fixed Q, where C ranges over the class of all families of regular curves on M.

Other important problems that seem to be very natural are the convergence properties of sequences of conformal mappings, in particular, the problem of finding some conditions under which the limit mapping is conformal. These questions, including the problem of obtaining some analogue of the Carathéodory convergence theorem (cf. [3] and [7]), are essential for physical applications. They have not been solved even in the riemannian case.

The authors plan to discuss at least some of these problems in a subsequent paper.

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