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HOMOLOGICAL DIMENSIONS OF FINITELY PRESENTED MODULES

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Relationships between homological dimensions of Noetherian rings and duality conditions on finitely generated modules have been studied in [1], [4]. In this paper we consider generalizations of these results by relaxing the Noetherian conditions on the ring while restricting the modules to be finitely presented. For this it is convenient to introduce homological dimensions of rings that are defined in terms of only the finitely presented modules.

1. Preliminary notions and notations.

All rings under consideration have an identity and all modules are unitary. If R is a ring, a left R-module is *finitely presented* if there exists an epimorphism $\varphi: R^n \to M$ such that $\operatorname{Ker}\varphi$ is finitely generated. A module is called *coherent* if it is finitely generated and every finitely generated submodule is finitely presented. A ring is left coherent if it is a coherent module when considered as a left module over itself [2]. The class of left coherent rings is considerably more extensive than the class of left Noetherian rings, for it includes all valuation rings, von Neumann regular rings, Boolean rings, in fact, all semihereditary rings. Also, a polynomial ring in an infinite number of indeterminates over a Noetherian ring is a coherent ring.

The projective and weak dimensions of an R-module will be denoted by $\operatorname{Pd}(M)$ and $\operatorname{w.dim}(M)$, respectively; and the left, right and weak global dimensions will be denoted $\operatorname{l.gl.dim}(R)$, $\operatorname{r.gl.dim}(R)$ and $\operatorname{w.gl.dim}(R)$.

DEFINITION. If R is a ring, let

$$\operatorname{l.f.p.gl.dim}(R) = \sup\{\operatorname{Pd}(M) \mid M \text{ a finitely presented left } R\text{-module}\}.$$

Similary, for finitely presented right R-modules we have $\operatorname{r.f.p.gl.dim}(R)$.

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Since the weak dimension of a module does not exceed the projective dimension and since the weak global dimension is attained by taking the supremum of the weak dimension of finitely presented cyclic modules, we have the first part of the following

PROPOSITION 1.1. (i) *If R is any ring, then*

$$\text{w.gl.dim}(R) \leqq \text{l.f.p.gl.dim}(R) \leqq \text{l.gl.dim}(R).$$

(ii) *If R is a left coherent ring, then*

$$\text{w.gl.dim}(R) = \text{l.f.p.gl.dim}(R).$$

The second part follows from the

LEMMA 1.2. *If R is left coherent and M is a finitely presented left R-module, then* $\text{w.dim}(M) = \text{Pd}(M)$.

This lemma is a consequence of the following two facts: (1) for a left coherent ring, any finitely presented left R-module is coherent and therefore has a free resolution of finite type; and (2) for any ring, a finitely presented flat module is projective.

Of course, for a left Noetherian ring the three dimensions in 1.1.(i) coincide. Perhaps, we should also keep in mind that Noetherian rings are characterized by the condition that finitely generated modules are finitely presented.

In Proposition 1.1(i) it is possible that

$$\text{w.gl.dim}(R) < \text{l.f.p.gl.dim}(R).$$

Small [6] has given an example of a ring T which is right coherent but not left coherent and for which

$$\text{w.gl.dim}(T) = \text{r.f.p.gl.dim}(T) = \text{r.gl.dim}(T) = 1$$

and

$$\text{l.f.p.gl.dim}(T) = \text{l.gl.dim}(T) = 3.$$

It is not essentially a non-commutative phenomenon that the inequality $\text{w.gl.dim}(R) \leqq \text{l.f.p.gl.dim}(R)$ may be strict. In fact, any (commutative) non-semihereditary ring of weak global dimension 1 will do. (A simple example is the subring of the complete product $\prod_1^\infty Q[X]$ generated by the sequence $(X, 0, X^2, 0, X^3, \ldots)$ and all sequences consisting eventually of constants. The author wishes to thank the referee for this example.)

Before continuing our investigation of these dimensions, we recall some terminology. If M is a left R-module, the dual of M, denoted M^*, is the right R-module $\mathrm{Hom}_R(M,R)$, where the module multiplication is given by "post multiplication"

$$(fr)(m) = (f(m))r \quad \text{for } f \in M^*,\ m \in M \text{ and } r \in R\,.$$

Analogously, the dual of a right R-module is a left R-module. The map $\sigma_M\colon M \to M^{**}$ defined by $\sigma_M(m)(f^*) = f^*(m)$ for $m \in M$ and $f^* \in M^*$ is a natural transformation from the identity functor on the category of (left) R-modules to the "double dual" functor. A module M is called *torsionless* if σ_M is a monomorphism and is called *reflexive* if σ_M is an isomorphism.

2. Dimensions 0 and 1.

In this section we see that if the l.f.p.gl.dim$(R) \leqq 1$, then the ring is left coherent. Regular rings (in the sense of von Neumann) are characterized by the condition that the weak global dimension equals 0. It is therefore immediate from Proposition 1.1 that we have

PROPOSITION 2.1. *If R is a ring, then the following are equivalent*:
(i) l.f.p.gl.dim$(R) = 0$,
(ii) w.gl.dim$(R) = 0$,
(iii) r.f.p.gl.dim$(R) = 0$.

The next proposition is also immediate from well-known properties of semi-hereditary and coherent rings.

PROPOSITION 2.2. *If R is a ring, the following are equivalent*:
(i) l.f.p.gl.dim$(R) \leqq 1$;
(ii) *R is left semihereditary*;
(iii) w.gl.dim$(R) \leqq 1$ *and R is left coherent*;
(iv) *all torsionless right R-modules are flat.*

The equivalence of parts (ii), (iii) and (iv) was shown in [3], we have included them here only because they provide some motivation for the results in the next section on higher dimensions.

As a consequence of these propositions and a result of Small, we have

PROPOSITION 2.3. *If R is either right or left Noetherian, then the following are equivalent*:
(i) l.f.p.gl.dim$(R) = 1$;
(ii) r.f.p.gl.dim$(R) = 1$.

PROOF. It suffices to consider only the case for R a right Noetherian ring. If $\text{l.f.p.gl.dim}(R)=1$, then R is left coherent in addition to being right Noetherian, and therefore,

$$1 = \text{l.f.p.gl.dim}(R) = \text{w.gl.dim}(R) = \text{r.f.p.gl.dim}(R) .$$

For (ii) $\Rightarrow$ (i), since R is right Noetherian,

$$\text{r.gl.dim}(R) = \text{r.f.p.gl.dim}(R) = 1 ,$$

which says that R is right hereditary. Small [7] has shown that a right Noetherian and right hereditary ring is left semihereditary. We conclude that $\text{l.f.p.gl.dim}(R)=1$.

2. Higher dimensions.

In [1] a special case of this next result appeared for left and right Noetherian rings and for the case $n=0$.

THEOREM 3.1. *For any ring R and any non-negative integer n, the following are equivalent*:

(i) $\text{r.f.p.gl.dim}(R) \leqq n+2$;

(ii) *the dual of any finitely presented left R-module has projective dimension $\leqq n$.*

PROOF. (i) $\Rightarrow$ (ii). Let M be a finitely presented left R-module, then from

$$R^m \xrightarrow{\alpha} R^n \to M \to 0$$

we get

$$0 \to M^* \to (R^n)^* \xrightarrow{\alpha^*} (R^m)^* \to \text{Coker}\,\alpha^* \to 0 .$$

But $\text{Coker}\,\alpha^*$ is finitely presented, so it has projective dimension $\leqq n+2$ and thus $\text{Pd}(M^*) \leqq n$.

(ii) $\Rightarrow$ (i). Let M be a finitely presented right R-module. In the exact commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \text{Ker}\,\alpha & \longrightarrow & R^n & \xrightarrow{\alpha} & R^m \to M \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to (\text{Coker}\,\alpha^*)^* & \to & (R^n)^{**} & \xrightarrow{\alpha^{**}} & (R^n)^{**} \end{array}$$

the vertical maps are isomorphisms and $\text{Coker}\,\alpha^*$ is finitely presented. Therefore, $\text{Pd}(M) \leqq n+2$ and the proof is complete.

In some of the remaining proofs we will make use of the following observations (cf. [5]). If R is a ring and X is a finitely generated left R-module, then we have the exact sequence of left R-modules

(a) $$0 \to \operatorname{Ker}\varrho \xrightarrow{j} R^n \xrightarrow{\varrho} X \to 0$$

which can be dualized to obtain the exact sequence

(b) $$0 \to X^* \to (R^n)^* \to \operatorname{Im} j^* \to 0$$

of torsionless right R-modules with $\operatorname{Im} j^*$ finitely generated. Next, (b) can be dualized and fitted together with (a) to obtain the exact commutative diagram of left R-modules

(c) $$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & \operatorname{Ker}\varrho & \xrightarrow{j} & R^n & \xrightarrow{\varrho} & X & \longrightarrow 0 \\ & \downarrow \tau & & \downarrow \sigma_{R^n} & & \downarrow \sigma_X & \\ 0 \to & (\operatorname{Im} j^*)^* & \xrightarrow{j^{**}} & (R^n)^{**} & \xrightarrow{\varrho^{**}} & \operatorname{Im}\varrho^{**} = \operatorname{Im}\sigma_X & \to 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

in which all entries are torsionless, except possibly X. In case X is finitely presented, $\operatorname{Ker}\varrho$ is finitely generated. If X is torsionless, then τ is an isomorphism (by the 5-lemma).

THEOREM 3.2. *If R is left coherent, the following are equivalent*:
(i) $\text{w.gl.dim}(R) \leqq n+2$;
(ii) *the dual of any left R-module has weak dimension* $\leqq n$;
(iii) *the dual of any torsionless left R-module has weak dimension* $\leqq n$.

PROOF. (i) $\Rightarrow$ (ii). Let M be an arbitrary left R-module and construct an exact sequence

$$P_1 \xrightarrow{\alpha} P_0 \to M \to 0,$$

where P_1, P_0 are projectives. Dualizing, we obtain the exact sequence of right R-modules

$$0 \to M^* \to P_0^* \xrightarrow{\alpha^*} P_1^* \to \operatorname{Coker}\alpha^* \to 0$$

in which P_0^* and P_1^* are flat since R is left coherent [3]. Hence, $\text{w.dim}(M^*) \leqq n$.

(ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). In computing the weak global dimension it suffices to consider only finitely presented right R-modules, so let M be a finitely presented right R-module. For M, we have an exact sequence

$$0 \to X \to R^m \to M \to 0$$

in which X is finitely generated and torsionless. And, we have an exact sequence

$$0 \to \operatorname{Ker}\varrho \xrightarrow{j} R^n \xrightarrow{\varrho} X \to 0$$

to which we can apply the observations preceding this theorem. We conclude that $\operatorname{Ker}\varrho$ is isomorphic to the dual of a torsionless module and therefore has weak dimension $\leqq n$. Hence, $\text{w.dim}(M) \leqq n+2$ and the proof is completed.

THEOREM 3.3. *If R is left and right coherent and n is a non negative integer, then the following are equivalent*:

(i) $\text{w.gl.dim}(R) \leqq n+1$;

(ii) *every torsionless left R-module has weak dimension* $\leqq n$;

(iii) *every finitely presented torsionless left R-module has projective dimension* $\leqq n$.

PROOF. (i) ⇒ (ii). If M is a torsionless left R-module, then M can be embedded in a direct product of copies of R and this product will be flat since R is right coherent. So $\text{w.dim}(\prod R/M) \leqq n+1$ implies $\text{w.dim}(M) \leqq n$.

(ii) ⇒ (iii). Since R is left coherent, the weak dimension and projective dimension of a finitely presented left R-module coincide.

(iii) ⇒ (i). Since R is left coherent, the finitely generated left ideals are finitely presented and torsionless, therefore $\text{w.gl.dim}(R) \leqq n+1$.

4. Reflexive finitely presented modules.

For M a finitely presented left R-module if R is right self injective we have

$$\begin{array}{ccccccc} R^n & \xrightarrow{\alpha} & R^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (R^n)^{**} & \xrightarrow{\alpha^{**}} & (R^m)^{**} & \to & M^{**} & \to & 0\,, \end{array}$$

and therefore M is reflexive. This proves

PROPOSITION 4.1. *If R is a right self injective ring, then every finitely presented left R-module is reflexive.*

Using methods of Jans [4], we get

PROPOSITION 4.2. *If R is a ring with right injective dimension $\leqq 1$, then every finitely presented torsionless left R-module is reflexive.*

In this proof we only need as a sufficient condition that $\mathrm{Ext}^2_R(N, R) = 0$ for all finitely presented N. This weaker sufficient condition is also a necessary condition under the hypothesis that R is left and right coherent. Under the hypothesis that R is left and right Noetherian the converse of 4.2 is true [4].

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