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THE PROJECTION MAPPING AND OTHER CONTINUOUS FUNCTIONS ON A PRODUCT SPACE[1])

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The theme of this note is that a number of conditions — eighteen, in fact — which the product of two topological spaces might satisfy are in fact equivalent. The conditions are stated in 1.1, 1.2, and 2.1 below. They have been considered by many authors in connection with diverse projects (some of which are described in § 3); several of the implications connecting them already appear in print, and surely others are known to various people. Our contribution is to furnish proofs for the new implications required to establish equivalence, and to add a few new and natural conditions to the list; our main purpose, however, is exposition and systematization.

In this collection of ideas are the theorem of Glicksberg on the Stone–Čech compactification of a product, and a theorem of Tamano on pseudocompactness of a product. In § 4, we give a new proof of Tamano's result and from it derive quickly Glicksberg's.

1. Mappings into the real line.

The statements of the results require the assumption that the topological spaces involved be completely regular and Hausdorff (uniformizable). *We shall make this assumption in* §§ 1–4; see § 5 concerning the possibility of weakening this hypothesis. Some terminology is required; see Gillman and Jerison [12] and Kelley [18] for details.

$C^*(X)$ is the set of bounded, continuous, real-valued functions on the space X. When a topology on $C^*(X)$ is referred to, it is the metric one

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induced by the norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$, the topology of uniform convergence on X. In 1.1(8), $C(X, C^*(Y))$ is the space of continuous functions from X to $C^*(Y)$, with the topology of uniform convergence on X; the subspace of bounded functions is normed by $\|\Psi\| \equiv \sup\{\|\Psi(x)\|_Y : x \in X\}$, and becomes a metric space. For $f \in C^*(X)$ and $S \subset X$, we set

$$\operatorname{osc}_S(f) = \sup\{|f(a) - f(b)| : a, b \in S\}.$$

A zero-set in the space X is a set of the form $f^{-1}(0)$, with $f \in C^*(X)$. βX denotes the Stone–Čech compactification of X, that compact space containing X densely such that each function in $C^*(X)$ extends continuously over X. When X and Y and Z are sets and f maps $X \times Y$ into Z, then f_y (for $y \in Y$) is defined on X by the rule $f_y(x) = f(x,y)$. Similarly, ${}_xf$ (for $x \in X$) is defined on Y by ${}_xf(y) \equiv f(x,y)$.

The equicontinuity condition referred to in 1.1(6) below is of the usual sort: if $f \in C^*(X \times Y)$, the family $\{f_y : y \in Y\}$ is said to be equicontinuous at the point x_0 of X provided that for each $\varepsilon > 0$ there is a neighborhood U of x_0 for which

$$|f_y(x_0) - f_y(x)| < \varepsilon$$

whenever $(x,y) \in U \times Y$.

1.1 THEOREM. *The following conditions in the product space $X \times Y$ are equivalent:*

(1) *The projection π_X from $X \times Y$ onto X carries zero-sets onto closed sets.*

(2) *If Z is a zero-set in $X \times Y$, then*

$$\operatorname{cl} Z = \bigcup\{\operatorname{cl}(Z \cap (\{x\} \times Y)) : x \in X\},$$

each closure being taken in $X \times \beta Y$.

(3) *Each function in $C^*(X \times Y)$ can be extended continuously over $X \times \beta Y$.*

(4) *If $f \in C^*(X \times Y)$, then*

$$F(x) \equiv \sup\{f(x,y) : y \in Y\}$$

defines a continuous function F on X (and similarly for $\inf\{f(x,y) : y \in Y\}$*).*

(5) *If $f \in C^*(X \times Y)$, then*

$$\gamma(x_1, x_2) \equiv \sup\{|f(x_1,y) - f(x_2,y)| : y \in Y\}$$

defines a continuous pseudometric γ for X.

(6) *If $f \in C^*(X \times Y)$, then $\{f_y : y \in Y\}$ is an equicontinuous family on X.*

(7) *If $f \in C^*(X \times Y)$, then*

$$\Phi(f)(x) \equiv {}_xf$$

defines a continuous mapping $\Phi(f)$ *from* X *into* $C^*(Y)$.

(8) Φ *(as defined in (7)) is a homeomorphism (indeed, an isometry) of* $C^*(X\times Y)$ *onto the space of bounded functions in* $C(X,C^*(Y))$.

(9) *If* $f\in C^*(X\times Y)$, $x_0\in X$ *and* $\varepsilon>0$, *then there is a neighborhood* U *of* x_0 *and* $g\in C^*(Y)$ *such that*

$$|f(x,y)-g(y)| < \varepsilon$$

whenever $x\in U$ *and* $y\in Y$.

(10) *If* $f\in C^*(X\times Y)$ *and* $\varepsilon>0$, *then there is an open cover* $\mathscr{U}$ *of* X, *and for each* U *in* $\mathscr{U}$ *a finite open cover* $\mathscr{V}(U)$ *of* Y, *such that* $\operatorname{osc}_{U\times V}(f)<\varepsilon$ *whenever* $U\in\mathscr{U}$ *and* $V\in\mathscr{V}(U)$.

Proof. (1) $\Rightarrow$ (2). We denote the set $Z\cap(\{x\}\times Y\}$ by the symbol Z_x. If for some (x_0,q) in $X\times\beta Y$ we have

$$(x_0,q) \in \operatorname{cl}Z\setminus \bigcup_{x\in X}(\operatorname{cl}Z_x)\ ,$$

then in particular $(x_0,q)\notin \operatorname{cl}Z_{x_0}$, so there is a continuous function f on $X\times\beta Y$ for which $f\equiv 1$ on $\operatorname{cl}Z_{x_0}$ and $f\equiv 0$ on some neighborhood of (x_0,q).

Now let $Z'=Z\cap f^{-1}(0)$, so that Z' is a zero-set in $X\times Y$ and $x_0\notin\pi_X(Z')$. Since $(x_0,q)\in\operatorname{cl}Z'$, however, we have from (1) the contradiction

$$x_0 \in \pi_X(\operatorname{cl}Z') \subset \operatorname{cl}_X(\pi_X(Z')) = \pi_X(Z')\ .$$

(2) $\Rightarrow$ (3). [12; 6.4] asserts that if A is a dense subspace of the completely regular Hausdorff space B, then each function in $C^*(A)$ extends continuously to B if and only if disjoint zero-sets in A have disjoint closures in B. We verify this latter condition with $A=X\times Y$ and $B=X\times\beta Y$ as follows: if Z_1 and Z_2 are zero-sets in $X\times Y$ then

$$\begin{aligned}\operatorname{cl}Z_1\cap\operatorname{cl}Z_2 &= [\textstyle\bigcup_{x\in X}\operatorname{cl}(Z_1)_x]\cap[\bigcup_{x\in X}\operatorname{cl}(Z_2)_x]\\ &= \textstyle\bigcup_{x\in X}[\operatorname{cl}(Z_1)_x\cap\operatorname{cl}(Z_2)_x];\end{aligned}$$

now each $(Z_i)_x$ can be viewed as a zero-set in Y, and the characteristic property of βY and [12; 6.4] show that $\operatorname{cl}(Z_1)_x\cap\operatorname{cl}(Z_2)_x=\emptyset$.

(3) $\Rightarrow$ (4). Given f in $C^*(X\times Y)$, define F as in (4) and let f^* be the continuous extension of f over $X\times\beta Y$. To check the continuity of F at a point x_0 of X, let $\varepsilon>0$ be given and find for each point q of βY a rectangular neighborhood $U_q\times V_q$ of (x_0,q) on which f^* varies less than ε. The cover $\{V_q\}_{q\in\beta Y}$ admits a finite subcover $\{V_{q_k}\}_{k=1}^n$. It is easy to see that on $\bigcap_{k=1}^n U_{q_k}$, F varies by less than ε.

(4) $\Rightarrow$ (5). As with any pseudometric, the (joint) continuity of γ will follow from its separate continuity (see 15.G of [12]). To establish separate continuity, we fix x_2 and observe that the function

$$(x_1,y) \to |f(x_1,y)-f(x_2,y)|$$

is surely continuous on $X \times Y$. According to (4), then, the associated sup function, whose value at x_1 is just $\gamma(x_1,x_2)$, is continuous at each point x_1.

(5) $\Rightarrow$ (6). To check the equicontinuity of the family $\{f_y : y \in Y\}$ at a point x_0 in X, let $\varepsilon > 0$ and use (5) to choose for the desired neighborhood of x_0 a neighborhood U of x_0 for which

$$\sup\{|f(x,y)-f(x_0,y)| : y \in Y\} < \varepsilon$$

whenever $x \in U$.

(6) $\Leftrightarrow$ (7). This equivalence is clear, since each of the two conditions may be stated as follows: for each x_0 in X and each $\varepsilon > 0$ there is a neighborhood U of x_0 for which

$$|f(x,y)-f(x_0,y)| < \varepsilon$$

whenever $y \in Y$ and $x \in U$.

(7) $\Leftrightarrow$ (8). That (8) $\Rightarrow$ (7) is obvious. That the map Φ does indeed take $C^*(X \times Y)$ into $C(X, C^*(Y))$ is guaranteed by (7) and it is trivial that each $\Phi(f)$ is bounded. To check that each bounded Ψ in $C(X, C^*(Y))$ has the form $\Psi = \Phi(f)$ for some f in $C^*(X \times Y)$, let Ψ be given and define f on $X \times Y$ by

$$f(x,y) = \Psi(x)(y) .$$

To see that f is continuous at, say, the point (x_0,y_0) in $X \times Y$, let $\varepsilon > 0$ and use the continuity of $\Psi(x_0)$ at y_0 to find a neighborhood V of y_0 for which

$$|\Psi(x_0)(y) - \Psi(x_0)(y_0)| < \tfrac{1}{2}\varepsilon$$

for $y \in V$. Since Ψ itself is continuous at x_0, there is a neighborhood U of x_0 for which $\|\Psi(x) - \Psi(x_0)\| < \frac{1}{2}\varepsilon$ whenever $x \in U$; it follows that on $U \times V$, f varies by $< \varepsilon$. Thus f is continuous, and evidently $\Phi(f) = \Psi$.

Finally, for f and g in $C^*(X \times Y)$,

$$\begin{aligned} \|f-g\|_{X\times Y} &= \sup\{|f(x,y)-g(x,y)| : (x,y) \in X \times Y\} \\ &= \sup\{\|{}_xf - {}_xg\|_Y : x \in X\} \\ &= \sup\{\|\Phi(f)(x) - \Phi(g)(x)\|_Y : x \in X\} \\ &= \|\Phi(f) - \Phi(g)\| , \end{aligned}$$

so that Φ is an isometry.

(6) $\Rightarrow$ (9). Given f, x_0, and ε, as in (9), choose the neighborhood U of x_0 so that

$$|f_y(x) - f_y(x_0)| < \varepsilon$$

whenever $(x,y) \in U \times Y$, and set $g(y) = f(x_0, y)$.

(9) $\Rightarrow$ (10). Given f and ε, as in (10), we are to produce, for a fixed point x_0 in X, a neighborhood U of x_0 and a finite open cover $\mathscr{V}$ of Y for which $\mathrm{osc}_{U \times V}(f) < \varepsilon$ whenever $V \in \mathscr{V}$: according to (9) there are U, a neighborhood of x_0, and g in $C^*(Y)$, with

$$|f(x,y) - g(y)| < \tfrac{1}{3}\varepsilon$$

whenever $(x,y) \in U \times Y$; and since g is bounded there is a finite open cover $\mathscr{V}$ of Y on each of whose elements g varies by less than $\frac{1}{3}\varepsilon$.

(10 $\Rightarrow$ (1). The trick here is to observe that if Z is a zero-set in $X \times Y$ and $x_0 \notin \pi_X(Z)$, then there is f in $C^*(X \times Y)$ for which $Z = f^{-1}(0)$ and for which $f \equiv 1$ on $\{x_0\} \times Y$. Indeed, if $Z = g^{-1}(0)$, let

$$f(x,y) = |g(x,y)/g(x_0,y)| \wedge 1 .$$

Now let $\varepsilon = \frac{1}{2}$ and apply (10): there is a neighborhood U of x_0 for which $f > \frac{1}{2}$ throughout $U \times Y$; thus U is a neighborhood of x_0 missing $\pi_X(Z)$, so that $x_0 \notin \mathrm{cl}_X \pi_X(Z)$.

The proof of Theorem 1.1 is now complete.

In precisely two of the conditions which appeared in 1.1 — specifically, in (2) and (3) — did the Stone–Čech compactification βY play a rôle. While no other compactification will do in (3), any compactification can serve in place of βY in (2). The first assertion is obvious; we formalize the second.

1.2 Proposition. *The following conditions on the product space $X \times Y$ are equivalent*:

(2) *If Z is a zero-set in $X \times Y$, then*

$$\mathrm{cl}Z = \bigcup \{\mathrm{cl}(Z \cap (\{x\} \times Y)) : x \in X\} ,$$

each closure being taken in $X \times \beta Y$.

$(2)_1$ *If cY is a space containing Y, and Z is a zero-set in $X \times Y$, then $\mathrm{cl}Z = \bigcup \{\mathrm{cl}(Z \cap (\{x\} \times Y)) : x \in X\}$, each closure being taken in $X \times cY$.*

$(2)_2$ *There is a compactification cY of Y such that if Z is a zero-set in $X \times Y$, then*

$$\mathrm{cl}Z = \bigcup \{\mathrm{cl}(Z \cap (\{x\} \times Y)) : x \in X\} ,$$

each closure being taken in $X \times cY$.

PROOF. The implications $(2)_1 \Rightarrow (2) \Rightarrow (2)_2$ are clear, and the implication $(1) \Rightarrow (2)_1$ can be proved precisely as was the implication $(1) \Rightarrow (2)$ of Theorem 1.2. Thus it suffices to prove that $(2)_2 \Rightarrow (1)$. To do this, we recall the easily-proved fact that if B is any compact space, then the projection from $B \times A$ onto A takes closed sets to closed sets. In the present case, taking $B = cY$ and $A = X$, denoting by $\bar{\pi}_X$ the projection from $X \times cY$ onto X and assuming $(2)_2$, we have $\pi_X(Z) = \bar{\pi}_X(\mathrm{cl}\, Z)$ for each zero-set Z.

2. Mappings into arbitrary spaces.

While each of the conditions of 1.1 and 1.2 refers explicitly or implicitly to real-valued continuous functions, some of the more interesting uses of these conditions involve functions mapping into arbitrary topological and uniform spaces. We shall formulate and sketch the proof of a general analogue of Theorem 1.1. Some terminology is required first, details of which are available in [17].

A uniform space A is called fine if the uniformity is the finest (largest) compatible with the uniform topology. Among all uniformities inducing a given topology, the fine uniformity is characterized by this statement about functions: each function from A to a uniform space which is continuous (relative to the uniform topologies on A and B) is uniformly continuous; and by this about pseudometrics: each continuous pseudometric on A is uniformly continuous; and by this about covers: each normal cover of A is uniform.

If A and B are topological spaces, $C(A,B)$ denotes the set of all continuous functions from A into B equipped with the topology of uniform convergence on A (relative to the fine uniformities on A and B); specifically, then, for each continuous pseudometric φ on B and each function f in $C(A,B)$ the set

$$N(f,\varphi) = \{g \in C(A,B) : \sup\{\varphi(f(a),g(a)) : a \in A\} < 1\}$$

is a neighborhood of f, and the collection of all such open sets is basic for $C(A,B)$.

When φ is a pseudometric for B, $f \in C(A,B)$, and $S \subset A$, we write

$$\varphi\text{-osc}_S(f) = \sup\{\varphi(f(x),f(y)) : x \in S \text{ and } y \in S\}.$$

A subset $\mathscr{F}$ of $C(A,B)$ is called equicontinuous if it is equiuniformly continuous when A and B are equipped with their fine uniformities. This means, specifically, that if φ is continuous pseudometric on B and $a_0 \in A$, then there is a neighborhood U of a_0 such that

$$\varphi(f(a), f(a_0)) < 1$$

whenever $f \in \mathscr{F}$ and $a \in U$.

When A and B are uniform spaces, the semi-uniform product $A*B$ is the set $A \times B$ with the uniformity whose uniformly continuous functions (to arbitrary uniform spaces) are the so-called "semi-uniform" functions, i.e., those functions f for which ${}_af$ is uniformly continuous for each $a \in A$, and the family $\{f_b : b \in B\}$ is equi-uniformly continuous. This uniformity is, in general, larger than the usual product uniformity, and it is compatible with the product topology.

When X and Y are topological spaces, we say "$X*Y$ is fine" if upon equipping X and Y with their fine uniformities, the uniformity of $X*Y$ coincides with the fine uniformity on the topological product. According to the preceeding discussion, this means that if Z is a topological space and $f \in C(X \times Y, Z)$, then $\{f_y : y \in Y\}$ is equi-continuous. (This is part of [17,VII, exercise 7 (a)].)

Finally, specializing immediately to fine spaces: if X and Y are topological spaces, a cover of $X \times Y$ is "semi-uniform" if it has the form $\{U_\alpha \times V^\alpha{}_\beta\}_\beta$, where $\{U_\alpha\}_\alpha$ is a normal cover of X and for each α, $\{V^\alpha{}_\beta\}_\beta$ is normal cover of Y.

2.1. Theorem. *The following conditions on the product space $X \times Y$ are equivalent, and equivalent to the conditions of* 1.1.

(5′) *If Z is a topological space, φ is a continuous pseudometric for Z, and $f \in C(X \times Y, Z)$, then*

$$\gamma(x_1, x_2) \equiv \sup\{\varphi(f(x_1, y), f(x_2, y)) : y \in Y\}$$

defines a continuous pseudometric[1] *γ for X.*

(6′) *$X*Y$ is fine.*

(7′) *If Z is a topological space and $f \in C(X \times Y, Z)$, then*

$$\Phi(f)(x) \equiv {}_xf$$

defines a continuous mapping $\Phi(f)$ from X into $C(Y, Z)$.

(8′) *If Z is a topological space, then Φ (defined in (7′)) is a homeomorphism of $C(X \times Y, Z)$ onto $C(X, C(Y, Z))$.*

(9′) *If Z is a topological space, φ a continuous pseudometric for Z, $f \in C(X \times Y, Z)$, and $x_0 \in X$, then there is a neighborhood U of x_0 and $g \in C(Y, Z)$ such that $\varphi(f(x, y), g(y)) < 1$ whenever $x \in U$ and $y \in Y$.*

[1] γ might take the value $+\infty$, so "continuous" in (5′) is meant as a map of $X \times X$ into $[0, +\infty]$. The point is ancillary: the statement obtained by requiring in (5′) that the φ's be bounded (so the γ's take only real values) is equivalent to (5′), as is readily seen by a device such as is used in the proof that 1.1(4) $\Rightarrow$ (5′).

(10′) *If Z is a topological space and φ is a continuous pseudometric for Z, and if $f \in C(X \times Y, Z)$, then there is a semi-uniform cover $\mathscr{W}$ of $X \times Y$ such that φ-$\mathrm{osc}_W(f) < \varepsilon$ whenever $W \in \mathscr{W}$.*

PROOF. The implications (5′) ⇒ (6′) ⇒ (7′) ⇒ (8′) ⇒ (9′) ⇒ (10′) are proved by modifying the corresponding proofs in Theorem 1.1, while the implication (10′) ⇒ (5′) follows from the form of the cover of (10′). Since evidently (5′) ⇒ (5), it suffices to establish the implication 1.1(4) ⇒ (5′). To do this, suppose (5′) fails: for some Z, f in $C(X \times Y, Z)$, and continuous pseudometric φ on Z, the pseudometric γ defined on $X \times X$ as in (5′) is not continuous. As before, then, according to 15.G of [12], we may suppose that γ is not continuous in the first variable (say at the point x_1) when the second variable is fixed at the point x_2.

For $s \in [0, +\infty)$, let $h(s) = s/(1+s)$, set $h(+\infty) = 1$, and define $g(x,y) = h\big(\varphi(f(x,y), f(x_2,y))\big)$. Then g is bounded, and violates 1.1(4) (at x_1).

Concerning the relation between the conditions of 2.1 and those of 1.1: the statements obtained in 2.1 by replacing, where appropriate, Z by the real line and φ by the usual metric are superficially stronger than the corresponding statements in 1.1, exactly because the functions in 2.1 are not required to be bounded. In addition to this, (10′) is stronger than (10) because no normality property is required of the cover of (10).

Conditions on $X \times Y$ analogous to (5′) and (7′)–(10′) can be formulated for mappings into uniform spaces. The changes to be made are these: Z and φ, where appearing, become respectively: a uniform space and a uniformly continuous pseudometric. $C(X \times Y, Z)$ becomes the set of uniformly continuous functions from $X \times Y$ to Z, $X \times Y$ being given the fine uniformity; where appropriate, $C(X \times Y, Z)$ is to have the uniformity of uniform convergence on $X \times Y$. Similar remarks apply to $C(Y, Z)$ and to $C(X, C(Y, Z))$. In (5′), γ is to be uniformly continuous where X is given its fine uniformity (that is, γ is just continuous). In (7′), $\Phi(f)$ is to be uniformly continuous, and in (8′), Φ is to be a uniform isomorphism.

These changes having been made, the statements which result are all equivalent, and equivalent to (6′). The proofs are very similar to those above. The resulting theorem is stronger than 2.1, because 2.1 treats the special case of fine spaces Z.

3. Origins of the conditions.

We now review some occurrences in the literature of the conditions of the theorems. Many of these involve pseudocompactness (the condition on a space that each real-valued continuous function is bounded). In particular, in perhaps the first paper, [10], where the conditions play a serious (albeit technical) rôle, Glicksberg has shown in connection with a theorem which we reprove below that pseudocompactness of $X \times Y$ implies (6), and that (6) $\Rightarrow$ (7) $\Rightarrow$ (3). The relation of pseudocompactness with, say, the conditions of 1.1, goes further; see the theorem of Tamano and Lemma 4.2 below.

Among other things in [6], Frolík has given a more direct proof of Glicksberg's theorem, and shown that the pseudocompactness of $X \times Y$ implies (4), (5) and (7). He has noted also that (5) $\Rightarrow$ (4). ((4) was considered earlier by Mrówka, in showing that a product is pseudocompact if both spaces are, and one is compact [20].) Frolík also has shown in [6] that for infinite spaces X and Y, pseudocompactness of $X \times Y$ is equivalent to the condition: given $f \in C^*(X \times Y)$ and $\varepsilon > 0$, there are finitely many open rectangles covering $X \times Y$ on each of which $\operatorname{osc}(f) < \varepsilon$. In [13], it is noted that this condition characterizes functions extendable over $\beta X \times \beta Y$. Condition (10) is, of course, a "one-directional analogue" of this covering condition (and characterizes functions extendable over $X \times \beta Y$, as do (6), (7) and (9)).

Condition (9) has a similar source. Using Glickberg's Theorem, Tamano [29] has shown that for infinite spaces X and Y, pseudocompactness of $X \times Y$ is equivalent to: the functions of the form $\sum_{i=1}^{n} f_i(x) g_i(y)$ are uniformly dense in $C^*(X \times Y)$. It is noted in [13] that the property of being uniformly approximable by such functions characterizes extendability over $\beta X \times \beta Y$. Here, (9) is the "one-directional analogue."

Condition (1) is one of the more simple and tractable of the ten conditions. Perhaps the first occurrence is in [29] (see 4.1 below). The condition is mentioned by Stephenson in [25], in connection with a generalization of this theorem of Tamano, and it is used in [26] in connection with the question of when "the Stone–Weierstrass theorem holds in $X \times Y$"; a generalization of (4) appears in [26].

Other uses of (1) involve conditions (6′) and (8′), and the uniform space version of (8′) mentioned after 2.1. Now, the semi-uniform product seems to have been invented exactly so conditions like (8′) hold [17; III.26]. Isbell has pointed out in [17; VII.39] that (6′) and (3) are equivalent. [14] concerns the question of what topological properties of X and Y make (6′) hold; this is approached *via* (1): it is shown that (1) $\Rightarrow$ (3), and hence that (1) $\Leftrightarrow$ (6′). In [21], Noble has shown rather neatly that (1), (3), (6′), (8), and (8′) are equivalent; his primary concerns are general "exponential laws" (e.g., (8) and (8′)) and applications to theorems of Ascoli type (this last being carried out in [22]). In [23], (1) $\Rightarrow$ (4) is noted.

In [21], Noble also shows the equivalence of these three conditions:

(i) π_X and π_Y both carry zero-sets to closed sets;

(ii) the uniform product of the fine spaces X and Y is fine;

(iii) each $f \in C^*(X \times Y)$ extends over both $X \times \beta Y$ and $\beta X \times Y$.

Combining this with other results, Noble reproved a rather remarkable theorem of Isbell (and Glicksberg–Frolík–Onucic) (cf. [17; Ch. VII]) completely classifying the circumstances under which (ii) holds. ([14] is an attempt at a similar classification for $X^* Y$.) The property (iii) was earlier considered in [5], where a weak form of (1) $\Rightarrow$ (3) was shown, together with the fact that (3) implies that the Hewitt realcompactification $v(X \times Y)$ is $vX \times vY$ if Y has nonmeasurable power.

The cozero-set analogue of (2), and the trick used in the proof (2) $\Rightarrow$ (3), originated in [15]; this has been exploited in [4]. A condition bearing roughly the same relation to (3) that $(2)_2$ bears to (2) appears in Lemma 1.4 of [6].

There is, of course, a relation between, say, (1) and the condition that π_X be closed. We won't go into this at all, except to note that Noble has related the condition to exponential laws. A reasonably complete discussion can be found in [14] and [21].

In addition to the preceding remarks, we want to emphasize the fact that Theorem 2.1 (in particular) does not represent much of an original contribution on our part. Its formulation, based on 1.1 and results in [17] and [21], is not difficult; the equivalence (6′) $\Leftrightarrow$ (10′) is clear from the definition and [17; III.23].

4. Theorems of Tamano and Glicksberg.

The fact that the product of two pseudocompact spaces need not be pseudocompact, and the question of what extra conditions on the factors make it so, have been discussed frequently from various points of view: see for example [12; Chapter 9], [10], [16], [1], [6], [29], [3], [7], [8], [17; Chapter VII], [28], [25], [24]. One of the better results in this vein is due to Tamano [29; Proposition 2]: the product of two spaces is pseudocompact if each is, and one is a k-space. This generalizes simultaneously the observations that for a product of two pseudocompact spaces to be pseudocompact it suffices that one of the spaces be locally compact [10; Theorem 3] or first-countable [16]. Tamano derived his theorem quite quickly from Theorem 4.1 below, which he proved in turn by invoking Glicksberg's theorem (4.5 below). We shall prove 4.1 directly, and derive 4.5 from it.

In the proofs which follow we shall use without mention the equivalence of these three conditions:

(a) X is pseudocompact;

(b) if $f \in C^*(X)$, then f assumes its supremum and infimum;

(c) each sequence of nonvoid open sets in X has a cluster point.

These equivalences are easily proved (and are used in most of the papers on pseudocompactness mentioned above); see [1] and [19].

4.1. THEOREM (Tamano). *The following are equivalent:*

(a) *$X \times Y$ is pseudocompact;*

(b) *X and Y are pseudocompact, and π_X carries zero-sets onto closed sets.*

PROOF. (a) $\Rightarrow$ (b). Assume (a). Then X and Y are pseudocompact, as continuous images of $X \times Y$. Now, if $x_0 \in \mathrm{cl}(\pi_X(Z)) \setminus \pi_X(Z)$, where $Z = f^{-1}(0)$, we may suppose (as in the proof (10) $\Rightarrow$ (1) of 1.2) that $f(x_0, y) = 1$ whenever $y \in Y$. Arguing inductively, we define for each integer n a point (x_n, y_n) in Z, a neighborhood $W_n = U_n \times V_n$ of (x_n, y_n) throughout which $f < \frac{1}{3}$, a neighborhood $W_n' = U_n' \times V_n$ of (x_0, y_n) throughout which $f > \frac{2}{3}$, all with $U_{n+1} \cup U'_{n+1} \subset U_n'$. Since $X \times Y$ is pseudocompact, there is a cluster point $(\bar{x}, \bar{y})$ of the sequence $\{W_n\}_{n=1}^{\infty}$, and by continuity, $f(\bar{x}, \bar{y}) \leqq \frac{1}{3}$. If a rectangular neighborhood of (x,y) meets each of the sets $\{W_{n_k}\}_{k=1}^{n}$ then, because

$$U_{n_k} \subset U'_{n_{k-1}} \subset \ldots \subset U'_{n_{k-1}},$$

this neighborhood meets each of the sets $\{W'_{n_k}\}_{k=2}^{\infty}$. Thus $(\bar{x}, \bar{y})$ is a cluster point of the sequence $\{W_n'\}_{n=1}^{\infty}$, so that $f(\bar{x}, \bar{y}) \geqq \frac{2}{3}$.

(b) $\Rightarrow$ (a). To show that $X \times Y$ is pseudocompact, it suffices to show that if $f \in C^*(X \times Y)$ and $f(x,y) > 0$ for each (x,y), then

$$\inf\{f(x,y) : (x,y) \in X \times Y\} > 0 .$$

Set $F(x) = \inf\{f(x,y) : y \in Y\}$; assuming (b), $F \in C^*(X)$, from 1.1 (4). For each $x \in X$, $F(x) > 0$, because $f(x,y) > 0$, for each y, and $\{x\} \times Y$ is pseudocompact. Since X is pseudocompact, $\inf\{F(x) : x \in X\} > 0$. We have, then,

$$\inf\{f(x,y) : (x,y) \in X \times Y\} = \inf\{F(x) : x \in X\} > 0 ,$$

which was to be shown.

The proof above that $(a) \Rightarrow (b)$ is almost exactly that used by Frolík in [6; 1.3] to prove $(a) \Rightarrow$ 1.1(4). The argument is similar to, though simpler than, Glicksberg's proof in [10; Lemma 1] that $(a) \Rightarrow$ 1.1(6).

The technique has been exploited further in [17; VII.34]. The proof above that $(b) \Rightarrow (a)$ follows [25; 4.8].

For our proof of Glicksberg's theorem, we require a lemma. For present purposes, we shall say that a (completely regular Hausdorff) space X is a P-space provided that each function in $C^*(X)$ is locally constant, that is, provided that whenever $x \in X$ and $f(x) = r$ and $f \in C^*(X)$, there is a neighborhood U of x throughout which f assumes only the value r. We recall that no infinite P-space is pseudocompact [12; 4K.2]. (A proof: if X is infinite then there is f in $C^*(X)$ with infinite range; if X is a P-space, each member of the cover $\{f^{-1}(r) : r \in R\}$ is open; these sets are all disjoint and infinitely many of them are non-void, and it is easy to construct an unbounded continuous function.)

4.2 Lemma. *Let $\pi_X : X \times Y \to X$ carry zero-sets onto closed sets. Then either X is a P-space or Y is pseudocompact.*

Proof. If X is not a P-space then there is a function f in $C^*(X)$ and a point x_0 in X for which $f(x_0) = 0$ but f is identically 0 on no neighborhood of x_0. If Y is not pseudocompact then there is an everywhere positive g in $C^*(Y)$ with $\inf\{g(y) : y \in Y\} = 0$. Define $h \in C^*(X \times Y)$ by $h(x,y) = g(y)^{f(x)}$. With $H(x) = \inf\{h(x,y) : y \in Y\}$, we have $H(x_0) = 1$, while each neighborhood of x_0 contains, say, a point x for which $H(x) < \frac{1}{2}$. Thus 1.1(4) fails, so that (1) does also.

(4.2) is stated and proved (differently) in [5; 2.1], and was noticed independently by one of us and S. G. Mrówka (see [14; 2.1]); 4.2 has been generalized by Noble [21; 3.1]. The proof above is due to Nathan J. Fine, and is used in another connection in [13; Prop. 7].)

We observe in passing that Lemma 4.2 permits the formulation of a result closely akin to Theorem 4.1.

4.3 Theorem. *For X infinite, the following are equivalent.*

(a) *$X \times Y$ is pseudocompact;*

(b) *X is pseudocompact and $\pi_X : X \times Y \to X$ carries zero-sets onto closed sets.*

Proof. In view of 4.1, we need only show that Y is pseudocompact whenever (b) holds. Since the (infinite) space X cannot be a P-space, this follows from 4.2.

While the theorem we are about to prove is generally considered the major result of [10], it is, in fact, a special case: Glicksberg shows that for infinite spaces $\{X_\alpha\}_{\alpha \in A}$, the space $\prod_{\alpha \in A} X_\alpha$ is pseudocompact if and only if $\prod_{\alpha \in A} \beta X_\alpha = \beta(\prod_{\alpha \in A} X_\alpha)$.

4.4 THEOREM (Glicksberg). *For infinite X and Y the following are equivalent.*

(a) *$X \times Y$ is pseudocompact;*

(b) *Each function in $C^*(X \times Y)$ extends continuously over $\beta X \times \beta Y$, that is, $\beta(X \times Y) = \beta X \times \beta Y$.*

PROOF. (a) $\Rightarrow$ (*b*). According to 4.1 and the implication (1) $\Rightarrow$ (3) of 1.1, each function in $C^*(X \times Y)$ extends continuously over $X \times \beta Y$. Since this latter space contains $X \times Y$ densely, it is itself pseudocompact. Again applying 4.1 and 1.1 (with βY and X playing the roles of X and Y respectively) we see that each function in $C^*(X \times \beta Y)$ extends continuously over $\beta X \times \beta Y$.

(*b*) $\Rightarrow$ (*a*). It follows from (*b*) that each function in $C^*(X \times Y)$ extends continuously over the (intermediate) spaces $X \times \beta Y$ and $\beta X \times Y$. From Theorem 1.1, both π_X and π_Y carry zero-sets in $X \times Y$ to closed sets, so from Lemma 4.2: X is pseudocompact or βX is a P-space. Since neither βX nor βY is a P-space (being infinite and compact), both X and Y are pseudocompact. By 4.1, then, $X \times Y$ is pseudocompact.

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