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MAJORIZING OPERATORS BETWEEN L^p SPACES AND AN OPERATOR EXTENSION OF LEBESGUE'S DOMINATED CONVERGENCE THEOREM

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Introduction.

We consider here some questions relating to majorizing operators between L^p spaces by positive operators and whether such operators map certain pointwise a.e. convergent sequences into a.e. convergent sequences. Significant aspects of majorizability have been studied by U. Krengel [9], who dealt with the more general case of operators between Banach lattices. As part of an earlier work, L. Kantorovitch [7] considered in more abstract terms the existence and properties of majorizable operators as well as their preservation of the convergence in question.

Let (X,Σ,μ) be a σ-finite positive measure space and T be a bounded linear operator on L^p into L^q, for fixed $p,q \in [1,\infty]$. Then if $p=1$ or $q=\infty$, Krengel [9] has shown that T is majorizable: there exists a positive linear operator P on L^p into L^q such that

$$|Tf| \le P|f| \text{ a.e. } (\mu) \quad \text{for all } f \in L^p .$$

(Similar results were presented in Chacon and Krengel [3] and Dunford and Schwartz [5, pp. 668–684], and all three articles used such results in attacking problems of ergodic theory.)

We show the above result to be best possible in a certain natural sense and in so doing give a characterization of majorizability. The majorizability of operators is connected with their preservation of a particular type of convergence of function sequences: One of the elementary properties of the conditional expectation operation $E\{\cdot\,|\mathscr{F}\}$ of probability theory is its preservation of bounded almost everywhere convergence, namely, given a sequence $\{f_n\}_1^\infty$ of functions integrable over a probability space (Ω,P), such that

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$$\sup_n |f_n| \in L^1(\Omega, P) \quad \text{and} \quad \lim_{n\to\infty} f_n = f \text{ a.e.},$$

then

$$\lim_{n\to\infty} E\{f_n|\mathscr{F}\} = E\{f|\mathscr{F}\} \text{ a.e.} \quad \text{and} \quad \sup_n |E\{f_n|\mathscr{F}\}| \in L^1(\Omega, P),$$

cf. [4]. We investigate here the natural extension of this preservation of bounded a.e. convergence to more general (even non-linear) operators on L^p into L^q, giving characterizations of those operators sharing the property. In the linear case, the effect of such hypotheses as majorizability, positivity, and weak* continuity is studied.

We now establish our definitions and notation.

Let (X, Σ, μ) be a σ-finite positive measure space unless otherwise noted. Equations and inequalities are understood to hold almost everywhere (a.e. (μ)), $\|f\|_p$ denotes the norm of f as an element of L^p, and χ_A denotes the function equal to 1 on A, zero elsewhere.

Unless otherwise specified, all operators are assumed linear. Only in Section 1 are not-necessarily-linear operators dealt with, and they are referred to there as non-linear operators. An operator T is said to be of *type* (p,q) if T is a bounded operator on L^p into L^q.

An operator P having domain and range in the functions measurable on (X, Σ, μ) is *positive* if it maps non-negative functions into functions non-negative a.e. Among the easily demonstrated properties of positive operators with domain some L^p, we shall use

i) $|Pf| \leq P|f|$,
ii) $\sup_{n\geq 1} Pf_n \leq P(\sup_{n\geq 1} f_n)$ if $f_n \geq 0$, $\sup_{n\geq 1} f_n \in L^p$,
iii) for any $p, q \in [1, \infty]$, a positive operator P on L^p into L^q is bounded, cf. [5, p. 682].

DEFINITION. An operator T on L^p into L^q is *majorizable* if there exists a positive operator P on L^p into L^q such that $|Tf| \leq P|f|$ for all $f \in L^p$. Any such P will be referred to as a *majorant* of T.

DEFINITION. A sequence $\{f_n\}_{n=1}^{\infty}$ of functions measurable on (X, Σ, μ) *converges boundedly* in L^p if

i) $\sup_{n\geq 1} |f_n| \in L^p$ and
ii) $\{f_n\}_{n=1}^{\infty}$ converges almost everywhere.

DEFINITION. A not-necessarily-linear operator T mapping L^p into L^q *preserves bounded convergence* if given a sequence $\{f_n\}_{n=1}^{\infty}$ boundedly convergent in L^p, then

i) $\sup_{n\geq 1}|Tf_n| \in L^q$ and
ii) $\lim_{n\to\infty} Tf_n = T(\lim_{n\to\infty} f_n)$ a.e. (μ).

We also say T *preserves the boundedness* of $\{f_n\}_{n=1}^{\infty}$ if i) holds, and T *preserves the pointwise convergence* of $\{f_n\}_{n=1}^{\infty}$ if ii) holds.

REMARKS. i) A majorizable operator T is clearly a bounded operator, since its majorant is positive, hence bounded.

ii) In vector lattice theory [10], the bounded convergence of $\{f_n\}$ is referred to as order convergence, and an operator preserving bounded convergence is known as a sequentially order continuous operator.

iii) It should be remarked that Blackwell and Dubins [2] have shown that conditional expectation operators need not preserve the pointwise convergence of unboundedly convergent sequences: given an a.e. convergent sequence $\{f_n\}_{n=1}^{\infty}$ of non-negative functions integrable over a probability space $(\Omega,\mathscr{F},P)$ such that $\sup_n |f_n| \notin L^1(\Omega,\mathscr{F},P)$, there exists a probability space $(\Omega',\mathscr{F}',P')$, a sequence $\{f_n'\}$ of functions measurable over $(\Omega',\mathscr{F}',P')$, and a subfield $\mathscr{C}'$ of $\mathscr{F}'$ such that

1° $\{f_n'\}$ and $\{f_n\}$ have the same joint distribution and
2° $\{E\{f_n' \mid \mathscr{C}'\}\}_{n=1}^{\infty}$ diverges a.e. on $(\Omega',\mathscr{F}',P')$.

If, in addition, the measure space $(\Omega,\mathscr{F},P)$ is non-atomic, then Al-Hussaini [1] has shown there exists a subfield $\mathscr{B}$ of $\mathscr{F}$ itself for which $\{E\{f_n \mid \mathscr{B}\}\}_{n=1}^{\infty}$ diverges a.e. (on $(\Omega,\mathscr{F},P)$).

In Section 1 we prove that certain non-linear and most majorizable operators preserve bounded convergence. We show by example that when the domain is L^∞ a positive linear operator may fail to preserve bounded convergence, and then prove that the additional hypothesis of weak* continuity is sufficient to guarantee the preservation of bounded convergence in this case. In Section 2 we give some conditions, largely due to Kantorovitch [7] and Krengel [9], under which an operator is majorizable. Section 3 contains examples of non-majorizable operators which do and which do not preserve the pointwise convergence of boundedly convergent sequences. These examples show Krengel's affirmative results on majorizing operators between individual L^p and L^q spaces (Theorem 2.2 below) to be best possible.

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1. Preservation of bounded convergence.

We consider non-linear operators possessing properties among the following:

(1.1) $0 \leq f \leq g$ implies $Pf \leq Pg$ for all $f, g \in L^p$

(1.2) $P(cf) \geq cPf$ for all $f \in L^p$, $f \geq 0$, and all constants $c > 0$

(1.3) $P(f+g) \leq Pf+Pg$ for all $f, g \in L^p$

(1.4) $P(f) \leq P(|f|)$ for all $f \in L^p$.

LEMMA 1.1. *Let P be a non-linear operator on L^p into L^q for some $p, q \in [1, \infty]$. Suppose* (1.1) *and* (1.2) *hold. Then there exists a real constant A such that $\|Pf\|_q \leqq A \|f\|_p$ for all $f \geq 0$.*

PROOF. The proof is due to S. Koshi [8].

REMARKS. i) This lemma is still valid if one replaces (1.2) with (1.2′): For every $g \in L^p$, $g \geq 0$, there exists a real sequence $\{a_n\}_1^\infty$ such that $\lim_{n\to\infty} a_n = \infty$ and $P(a_n g) \geq a_n Pg$.

ii) This lemma implies $P0 = 0$ and therefore $Pf \geq 0$ if $f \geq 0$.

LEMMA 1.2. *Let P be a non-linear operator on L^p into L^q for some $p, q \in [1, \infty]$. Suppose* (1.3) *and* (1.4) *hold. Then $|Ph - Pf| \leq P|h-f|$ for all $h, f \in L^p$.*

PROOF. The inequalities

$$Pf = P(h+f-h) \leq Ph + P(f-h) \leq Ph + P|f-h|$$

imply

$$Pf - Ph \leq P|f-h| .$$

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$$Ph - Pf \leq P|h-f| .$$

Thus $-P|f-h| \leq Ph - Pf \leq P|h-f|$, so $|Ph-Pf| \leq P|h-f|$.

THEOREM 1.3. *Let P be a not-necessarily-linear operator defined on L^p into L^q, for some $p \in [1, \infty)$, $q \in [1, \infty]$, such that* (1.1), (1.2), (1.3) *and* (1.4) *hold. Then P preserves bounded convergence.*

PROOF. Let $f_n \in L^p$ and $\{f_n\}$ converge boundedly in L^p to f. Then

$\{\sup_{k>N}|f_k-f|\}_{N=0}^{\infty}$ is a monotone decreasing sequence, converging to zero boundedly in L^p. It follows ((1.1) and Remark ii) above) that $\{P(\sup_{k>N}|f_k-f|)\}_{N=0}^{\infty}$ converges monotonely to a non-negative function g, boundedly in L^q:

$$\lim_{N\to\infty} P(\sup_{k>N}|f_k-f|) = g .$$

Now

$$\begin{aligned} 0 &\le \sup_{k>N}|Pf_k-Pf| \\ &\le \sup_{k>N} P|f_k-f| && \text{(Lemma 1.2)} \\ &\le P(\sup_{k>N}|f_k-f|) && (1.1) \\ &\le P(\sup_{k>0}|f_k-f|) \in L^q . && (1.1) \end{aligned}$$

We show $g=0$ a.e., whence

$$\lim\sup|Pf_k-Pf| = 0 \text{ a.e. } ,$$

and the desired a.e. convergence holds. (The boundedness of the convergence is evident.)

Expressing $X=\bigcup_{n=1}^{\infty}B_n$, $\mu(B_n)<\infty$, it suffices to show

$$\int_{B_n} g\,d\mu = 0, \quad n=1,2,\ldots .$$

Now

$$\int_X \chi_{B_n} g\,d\mu \leqq \int_X \chi_{B_n} P(\sup_{k>N}|f_k-f|)\,d\mu < \infty , \tag{1.5}$$

since $\chi_{B_n}\in L^1\cap L^\infty$. Using Hölder's inequality and Lemma 1.1,

$$\begin{aligned} \int_{B_n} g\,d\mu &\leqq (1+\mu(B_n))\,\|P(\sup_{k>N}|f_k-f|)\|_q \\ &\leqq (1+\mu(B_n))\,A\,\|\sup_{k>N}|f_k-f|\|_p \quad \text{for all } N . \end{aligned}$$

But by hypothesis, $\lim_{N\to\infty}\sup_{k>N}|f_k-f|=0$ a.e. and boundedly in L^p, so $\|\sup_{k>N}|f_k-f|\|_p\to 0$ as $N\to\infty$. Hence $\int_{B_n} g\,d\mu=0$ and the proof is complete.

Remark. The above Theorem clearly applies to positive linear operators on L^p into L^q, and the following Corollary extends it to non-positive linear operators.

Corollary 1.4. *If an operator T on L^p into L^q for some $p\in[1,\infty)$, $q\in[1,\infty]$ is majorizable, then it preserves bounded convergence.*

PROOF. Let such an operator T be given. If $\{f_n\}_{n=1}^{\infty}$ is boundedly convergent in L^p to f, it suffices to show that $\{\sup_{k>N}|Tf_k - Tf|\}_{N=0}^{\infty}$ converges boundedly in L^q to zero. Let P be a majorant of T of type (p,q). Then

$$\sup_{k>N}|Tf_k - Tf| \leq \sup_{k>N} P(|f_k - f|) \leq P(\sup_{k>N}|f_k - f|) .$$

By the proof of Theorem 1.3, $\{P(\sup_{k>N}|f_k - f|)\}_{N=0}^{\infty}$ converges boundedly in L^q to zero, completing the present proof.

REMARK. In a much more general setting, Kantorovitch has proved the above Corollary as well as its converse [7, Theorem 8, p. 227]. (The regular operations of Kantorovitch are our majorizable operators, while his (o)-continuous operations are, in our case, those that preserve bounded convergence.) We present in Section 2 (Theorem 2.1, ii)) a slightly strengthened converse.

EXAMPLE 1.5. The following example shows Theorem 1.3 to be sharp, with respect to the indices p,q. Let (X_n, Σ_n, μ_n), $n = 1, 2, \ldots$, each be the unit interval with Lebesgue measure. Form the product measure space

$$(X, \Sigma, \mu) = (\times_{i=1}^{\infty} X_i, \times_{i=1}^{\infty} \Sigma_i, \times_{i=1}^{\infty} \mu_i)$$

[6, p. 137]. For $g \in L^1(X, \Sigma, \mu)$, define

$$Pg(x_1, x_2, \ldots, x_n, \ldots) = g(x_1, x_2/2, \ldots, x_n/n, \ldots) .$$

Letting

$$f_N(x_1, x_2, \ldots) = \prod_{k=1}^{N} \chi_{[0,1/k)}(x_k) ,$$

it is clear that $Pf_N = 1$ a.e. (μ) yet $\lim_{N\to\infty} f_N = 0$ a.e. (μ). Thus P, which is clearly a positive operator of type (∞, q) for each $q \in [1, \infty]$, fails to preserve the pointwise convergence of boundedly convergent sequences in L^∞. Note that P is not of type (p,q) for any $p \in [1, \infty)$, $q \in [1, \infty]$.

As shown by Theorem 1.3 and Example 1.5, a positive operator P on L^p into L^q preserves bounded convergence if $p \in [1, \infty)$, $q \in [1, \infty]$, but need not do so in case $p = \infty$, $q \in [1, \infty]$. Difficulty in the case $p = \infty$ arises because bounded convergence in L^∞ does not imply norm convergence, so Hölder's inequality is of no help. If one tries to use the adjoint of P, the fact that the dual of L^∞ is not L^1 gives trouble. With the additional hypothesis of weak* continuity of P (that is, continuity from L^p with the $L^{p'}$ topology to L^q with the $L^{q'}$ topology $(1/p + 1/p' = 1/q + 1/q' = 1)$, cf. [5, p. 462]), these problems are avoided:

Corollary 1.6. *Let $q \in [1,\infty]$ and let P be a positive operator on L^∞ into L^q which is weak* continuous. Then P preserves bounded convergence.*

Proof. Let $f_n \in L^\infty$, $\{f_n\}$ converge boundedly in L^∞ to f. Then $\{\sup_{k>N}|f_k-f|\}_{N=0}^\infty$ converges boundedly to 0 in L^∞. Hence this sequence is also weak* convergent to 0 in L^∞. Hence $\{P(\sup_{k>N}|f_k-f|)\}_{N=0}^\infty$ converges to $P0=0$ in the weak* topology on L^q. Now following the proof of Theorem 1.3 up through (1.5), we see by weak* convergence that

$$\lim_{N\to\infty}\int_X \chi_{B_n} P(\sup_{k>N}|f_k-f|)\, d\mu = 0$$

since $\chi_{B_n} \in L^{q'}$. Thus

$$\int_X \chi_{B_n} g\, d\mu = 0,$$

so $g=0$ a.e. on B_n for all n. Therefore $g=0$ a.e. as desired, completing the proof.

Remark 1.7. Note that an operator of type (p,q) for fixed $p \in [1,\infty)$, $q \in [1,\infty]$ is weak* continuous, since

$$\int gTf = \int fT^*g \quad \text{for } f \in L^p,\ g \in L^{q'}$$

$(1/q+1/q'=1)$. Now fix $q \in [1,\infty]$. A common class of weak* continuous operators of type (∞,q) is provided by those operators which are simultaneously of type (p,q) for all $p \in [r,\infty]$, for some $r<\infty$. To see this, fix $g \in L^{q'}$, and denote the adjoint of T regarded as an operator on L^∞ into L^q by T_∞^*. Recall that $L^{\infty *}$, the dual of $L^\infty(X,\Sigma,\mu)$, is the space of bounded additive functions on Σ, absolutely continuous with respect to μ, cf. [5]. Denoting action of the functional $T_\infty^* g$ on $f \in L^\infty$ by $(T_\infty^* g, f)$, let $\{E_n\}_1^\infty$ be a disjoint family of sets in Σ of finite measure, whose union is a set E in Σ, also of finite measure. Now $\sum_{k=1}^N \chi_{E_k}$ converges to χ_E in the L^r norm, so $T(\sum_{k=1}^N \chi_{E_k})$ converges to $T(\chi_E)$ in the L^q norm. Thus $T(\sum_{k=1}^N \chi_{E_k})$ converges to $T(\chi_E)$ in the weak* topology on L^q; so, as $N\to\infty$,

$$(T_\infty^* g, \textstyle\sum_{k=1}^N \chi_{E_k}) = (g, T\sum_{k=1}^\infty \chi_{E_k}) \to (g, T\chi_E) = (T_\infty^* g, \chi_E).$$

It follows that $T_\infty^* g$ is countably additive on the ring of sets in Σ of finite measure, and by the extension theory of measures, is countably additive on Σ. Clearly, $T_\infty^* g$ is a finite measure on Σ. Thus by the Radon–Nikodym Theorem, $\int gTf\,d\mu = \int g_* f\,d\mu$ for some $g_* \in L^1$, so T is weak* continuous as an operator on L^∞ into L^q.

2. Majorizability.

Condition i) of the following theorem is due to Kantorovitch [7].

THEOREM 2.1. *Let T be an operator on L^p into L^q for some fixed $p, q \in [1,\infty]$. Each of the following two conditions is equivalent to T being majorizable:*

i) *for any non-negative $f \in L^p$, $\sup_{|g|\leq f}|Tg| \in L^q$, and*
ii) *T preserves the boundedness of sequences boundedly convergent in L^p to zero.*

PROOF. i) See Kantorovitch [7, pp. 222–223, Remarks 4 and 5].

ii) If T is majorizable then i), hence ii) must hold. If T is not majorizable, it follows that there exists $f \in L^p$ such that there is no $h \in L^q$ for which $|Tg| \leq |h|$ for all $|g| \leq |f|$. It is a property of the lattice structure of the space of functions measurable on (X,Σ,μ) that there exists a sequence of functions g_n such that $|g_n| \leq |f|$ and $\sup_n |Tg_n| \notin L^q$, cf. [5, p. 336].

Now for each integer $k > 0$ there exists an integer $n_k > 0$ such that $\|\sup_{n\leqq n_k}|Tg_n|\|_q > 2^{2k}$. Defining $\Psi_j = g_j/2^k$ if $n_{k-1} < j \leqq n_k$, it follows that $\{\Psi_j\}$ converges boundedly to zero in L^p, yet $\sup_n |T\Psi_n| \notin L^q$.

The following result is due to Krengel [9, Theorems 4.1, 4.2]. Part of it, the majorizability of type $(1,q)$ operators, was earlier essentially proved (independently) by Chacon and Krengel [3] and Kantorovitch [7].

THEOREM 2.2. *Any operator T of type $(1,q)$ for some $q \in [1,\infty]$ or of type (p,∞) for some $p \in [1,\infty]$ is majorizable.*

PROOF. The proof of [3] that an operator of type $(1,1)$ is majorizable also shows that an operator of type $(1,q)$ for $q \in (1,\infty]$ is majorizable, with only very minor changes in the argument.

Now let T be an operator of type (p,∞) for $p \in [1,\infty]$ and let $\|T\|_{p,\infty}$ denote the norm of T. Then if f and $g \in L^p$ and $|g| \leq |f|$, it follows that

$$|Tg| \leq \|T\|_{p,\infty}\|g\|_p \leq \|T\|_{p,\infty}\|f\|_p ,$$

hence $\sup_{|g|\leqq|f|}|Tg| \in L^\infty$. Theorem 2.1 now implies that T is majorizable.

3. Some non-majorizable operators.

Our first example yields an operator of type (p,q) for all $p \in [1,\infty]$, $q \in [1,\infty)$, but which is only majorizable as an operator from L^1 to L^q

for $q \in [1, \infty)$. In a sense this shows Krengel's result (Theorem 2.2 above) to be sharp. (Remarks: (i) A weaker result in this direction is provided by the Hilbert transform H, which is of type (p,p) for $1<p<\infty$ [11], but which is clearly not majorizable:

$$\sup_{|g|\leqq f}|Hg| = \int_{-\infty}^{\infty} (|f(t)|/|x-t|)\, dt .$$

(ii) A related result of interest is the following theorem of Krengel [9]: The bounded linear operators on a real L^2 space into itself form a lattice if and only if L^2 is finite dimensional.) We also show that given any $p \in (1, \infty]$, the operator of this example fails to preserve the pointwise convergence of sequences boundedly convergent in L^p.

EXAMPLE 3.1. Let (X, Σ, μ) be the positive real numbers with Lebesgue measure. Let φ_n, $n=1,2,\ldots$, denote the n^{th} Rademacher function: $\varphi_n(x) = \mathrm{sign}(\sin(2^{n+1}\pi x))$ on $(0,1)$, $\varphi_n(x)=0$ elsewhere. Let S_n denote the positive linear functional on each $L^p(X)$, $1 \leqq p \leqq \infty$, defined by

$$S_n f = \int_n^{n+1/n} f(t)\, dt = \int_0^{\infty} \chi_{[n,n+1/n]}(t) f(t)\, dt .$$

Note that

$$|S_n f| \leqq n^{-1/p'} \|f\|_p, \quad \text{where } 1/p + 1/p' = 1 . \tag{3.1}$$

Define

$$Tf = \textstyle\sum_{n=1}^{\infty} (S_n f)\varphi_n .$$

A standard result of Rademacher series [12, pp. 123–4] guarantees that if $\sum_1^\infty (S_n f)^2 < \infty$ then the series defining Tf converges a.e. and for any $q \in [1, \infty)$ there exists a positive constant B_q such that

$$\|Tf\|_q \leqq B_q [\textstyle\sum_{n=1}^{\infty} (S_n f)^2]^{\frac{1}{2}} . \tag{3.2}$$

Noting that

$$(\textstyle\sum_1^\infty (S_n f)^2)^{\frac{1}{2}} \leqq \sum_1^\infty |S_n f| \leqq \sum_1^\infty S_n |f| \leqq \|f\|_1 ,$$

T is of type $(1,q)$ for any $q \in [1,\infty)$. Setting $p=\infty$ in (3.1) it follows from (3.2) that

$$\|Tf\|_q \leqq B_q \pi 6^{-\frac{1}{2}} \|f\|_\infty ,$$

for any $q \in [1,\infty)$. Now by the Riesz convexity theorem, T is of type (p,q) for any $p \in [1,\infty]$, $q \in [1,\infty)$. (It is easy to show T is not of type (p,∞) for any $p \in [1,\infty]$.) It is clear that for $f \geq 0$,

$$\text{(3.3)} \qquad \sup_{|h|\leqq f}|Th| = \chi_{(0,1)} \textstyle\sum_1^\infty S_n f = \chi_{(0,1)} \displaystyle\int_0^\infty (\textstyle\sum_1^\infty \chi_{[n,n+1/n]}(t))f(t)\,dt\,.$$

Since $\sum_1^\infty \chi_{[n,n+1/n]} \in L^\infty(X)$ but $\notin L^r(X)$ for $r<\infty$, the integral in (3.3) is finite for all non-negative $f \in L^p$ if and only if $p=1$. Thus by Theorem 2.1, T is not majorizable as an operator from L^p to L^q for any $p \in (1,\infty]$, $q \in [1,\infty)$. By either of Theorems 2.1 or 2.2, of course, T is majorizable as an operator from L^1 to L^q for any $q \in [1,\infty)$.

REMARK. By Remark 1.7 the operator T is weak* continuous as an operator from L^p to L^q for each $p \in (1,\infty]$, $q \in [1,\infty)$. The above example thus shows that weak* continuity does not imply majorizability in these cases.

We now show that the operator T fails to preserve the almost everywhere convergence of sequences boundedly convergent in L^p, for any $p \in (1,\infty]$.

We shall select a sequence $\{g_n\}_1^\infty$ of functions measurable over $(0,\infty)$ such that

$$\text{(3.4)} \qquad \lim_{n\to\infty} g_n = 0 \text{ a.e.,}$$

(3.5) $\sup_n |g_n| \in L^p(0,\infty)$ for all $p \in (1,\infty]$, and

(3.6) there exists an increasing sequence $\{n_k\}_{k=0}^\infty$ of indices such that $\sup_{n_k<n\leqq n_{k+1}} Tg_n > 1$ on $(0,1)$ and

$$\inf_{n_k<n\leqq n_{k+1}} Tg_n < -1 \text{ on } (0,1)\,.$$

Such a sequence $\{g_n\}_1^\infty$ then provides the desired behavior for it is boundedly convergent in every L^p, $1<p\leqq\infty$, yet $\{Tg_n\}_{n=1}^\infty$ does not converge anywhere on $(0,1)$.

To construct $\{g_n\}$, consider the operation

$$\sup_{|h|\leqq f}|Th| = \chi_{[0,1)} \int_0^\infty (\textstyle\sum_{n=1}^\infty \chi_{[n,n+1/n]})f\,dx$$

defined on non-negative functions f measurable over $(0,\infty)$. This operation corresponds to integration over the infinite measure space $Y = \bigcup_{n=1}^\infty [n,n+1/n)$ with relative Lebesgue measure, also denoted dx. There exists a non-negative function g defined on Y, belonging to $L^p(Y,dx)$ for every $p \in (1,\infty]$ but not for $p=1$. Extend g to $(0,\infty)$ by defining it to be zero on $(0,\infty)-Y$.

Let $\{N_k\}_{k=0}^{\infty}$ be a sequence of positive integers satisfying $N_0=1$, $N_{k-1}<N_k$, and

$$\int_{N_{k-1}}^{N_k} g\,dx > 1, \quad k=1,2,\ldots .$$

Define

$$g_1 = \chi_{[1,N_1)}g, \quad g_2 = -\chi_{[1,2)}g+\chi_{[2,N_1)}g,\ldots,g_{2^{N_1-1}} = -\chi_{[1,N_1)}g\,,$$

where all 2^{N_1-1} possible changes of sign of g over the N_1-1 disjoint intervals with integer endpoints contained in $[1,N_1)$ have been effected. Set $n_0=0$, $n_1=2^{N_1-1}$. It follows that

$$\begin{aligned}\sup_{n_0<n\leqq n_1} Tg_n &= \sup_{n_0<n\leqq n_1} \textstyle\sum_{j=1}^{N_1-1}\left(\displaystyle\int_j^{j+1/j} g_n\,dx\right)\varphi_j \\ &= \chi_{[0,1)}\int_1^{N_1} g\,dx \geq \chi_{[0,1)}\,.\end{aligned}$$

The first equality uses just the definitions of T and of the g_n. The second equality follows from the definitions of the Rademacher functions φ_j and the functions g_n. The inequality is due to the choice of g and N_1.

Similarly $\inf_{n_0<n\leqq n^1} Tg_n \leq -\chi_{[0,1)}$.

Continue this procedure: setting $n_k=\sum_{j=1}^{k}2^{N_j-N_{j-1}}$, for $n_k<n\leqq n_{k+1}$ define g_n by

$$g_n = \textstyle\sum_{j=1}^{N_{k+1}-N_k}(-1)^{j_n}\chi_{[N_k+j-1,N_k+j)}\,g\,,$$

with the j_n chosen in such a fashion that all $2^{N_{k+1}-N_k}$ possible changes of sign of g over the $N_{k+1}-N_k$ disjoint intervals with integer endpoints contained in $[N_k,N_{k+1})$ have been effected. Again, the supremum over the $2^{N_{k+1}-N_k}$ functions Tg_n is equal to

$$\int_{N_k}^{N_{k+1}} g\,dx\,,$$

which is greater than 1 on $(0,1)$. Similarly, the corresponding infimum is less than -1 on $(0,1)$. In other words, (3.6) holds.

Since $|g_n|\leq g$, $\sup_n|g_n|\in L^p(0,\infty)$ for all $p\in(1,\infty]$ so (3.5) holds. Finally, since the support intervals $[N_k,N_{k+1})$ for the g_n march out to ∞ as $n\to\infty$, it is clear that $\lim_{n\to\infty}g_n=0$ a.e., so (3.4) holds. This completes the construction of the example.

Slight modification of the above example yields a class of non-majorizable operators which, however, preserve the pointwise convergence of boundedly convergent sequences. (Of course the boundedness of the convergence will not be preserved (Theorem 2.1).) First we prove the following lemma, giving bounds on the L^p norms of Rademacher series with terms truncated at dyadic rationals, cf. [12, p. 124].

LEMMA 3.2. *Let φ_k, $k=1,2,\ldots$, denote the k^{th} Rademacher function, let $\{i_k\}_{k=1}^{\infty}$ be a non-decreasing sequence of positive integers, and let $a_k = 2^{-i_k}$. Setting*

$$\psi = \textstyle\sum_1^\infty c_k \varphi_k \chi_{[0,a_k)},$$

then for each $q \in [1,\infty)$ there exists a positive constant A_q such that

$$\|\psi\|_q \leqq A_q (\textstyle\sum_1^\infty c_k^2)^{\frac{1}{2}}.$$

PROOF. Assume for the present that $i_k < k$. Now

$$\begin{aligned} \psi &= \textstyle\sum_{k=1}^\infty (c_k \varphi_k \sum_{n=i_k}^\infty \chi_{[2^{-n-1},2^{-n})}) \\ &= \textstyle\sum_{k=1}^\infty (c_k \varphi_k \sum_{n=k}^\infty \chi_{[2^{-n-1},2^{-n})}) + \sum_{k=1}^\infty (c_k \varphi_k \sum_{n=i_k}^{k-1} \chi_{[2^{-n-1},2^{-n})}) \\ &= \alpha + \beta, \end{aligned}$$

for simplicity. Observe that

$$\begin{aligned} \|\alpha\|_q &\leqq \textstyle\sum_{k=1}^\infty \|c_k \varphi_k (\sum_{n=k}^\infty \chi_{[2^{-n-1},2^{-n})})\|_q \\ &= \textstyle\sum_{k=1}^\infty |c_k| 2^{-k/q} \leq (\sum_{k=1}^\infty c_k^2)^{\frac{1}{2}} (\sum_{k=1}^\infty 2^{-2k/q})^{\frac{1}{2}}. \end{aligned}$$

Define $k^{-1}(n)$ as the greatest integer for which $i_{k^{-1}(n)} \leq n$. Next observe that

$$\beta = \textstyle\sum_{n=1}^\infty (\sum_{k=n+1}^{k^{-1}(n)} c_k \varphi_k) \chi_{[2^{-n-1},2^{-n})},$$

so

$$\|\beta\|_q \leqq \textstyle\sum_{n=1}^\infty \|(\sum_{k=n+1}^{k^{-1}(n)} c_k \varphi_k) \chi_{[2^{-n-1},2^{-n})}\|_q.$$

Since when $k \geq n+1$, φ_k is periodic with period 2^{-n-1}, it is clear that each q-norm over $[2^{-n-1}, 2^{-n})$ in the above series bounding $\|\beta\|_q$ can be replaced by 2^{-n-1} times a q-norm over $[0,1)$ as follows:

$$\begin{aligned} \|\beta\|_q &\leqq \textstyle\sum_{n=1}^\infty 2^{-n-1} \|\sum_{k=n+1}^{k^{-1}(n)} c_k \varphi_k\|_q \\ &\leqq \textstyle\sum_{n=1}^\infty 2^{-n-1} B_q (\sum_{k=n+1}^{k^{-1}(n)} c_k^2)^{\frac{1}{2}} \\ &\leqq \tfrac{1}{2} B_q (\textstyle\sum_{k=1}^\infty c_k^2)^{\frac{1}{2}}, \end{aligned}$$

where we have used the full Rademacher series estimate of [12, p. 124].

Removing the assumption $i_k < k$ has at most the effect of removing some of the positive terms in the q-norm estimates of α and β, so that the desired inequalities still hold. Thus we may set $A_q = \frac{1}{2} B_q + (\sum_{k=1}^\infty 2^{-2k/q})^{\frac{1}{2}}$.

EXAMPLE 3.3. Using the same measure space and notations as in Example 3.1 above, fix $r \in (1,\infty)$ and let $s=(1-1/r)^{-1}$. Define

$$T_r f = \sum_{n=1}^{\infty}\left(\int_n^{n+1/n} f(t)\,dt\right)\chi_{[0,a_n)}\varphi_n\,,$$

with $a_n = 2^{-i_n}$, where $\{i_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers tending to ∞ in such a way that $\sum_{n=1}^{\infty}[a_n{}^s/n]$ diverges.

With this choice of $\{a_n\}$, and using Lemma 3.2, it follows as in Example 3.1 that T_r is of type (p,q) for $p \in [1,\infty]$, $q \in [1,\infty)$.

To determine that T_r is not majorizable as an operator from L^p to L^q for any fixed $p \in [r,\infty]$, $q \in [1,\infty)$, we use the criterion of Theorem 2.2: we show there exist $f \in L^p(0,\infty)$ for which $f \geq 0$ and

$$\sup_{|g|\leq f}|T_r g| = \sum_{n=1}^{\infty}\left(\int_n^{n+1/n} f(t)\,dt\right)\chi_{[0,a_n)} \tag{3.7}$$

is not in $L^1(0,\infty)$. (This is sufficient since the expressions in (3.7) are supported on (0,1).) Integrating the right side of (3.7) gives

$$\sum_{n=1}^{\infty} a_n \int_n^{n+1/n} f(t)dt = \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n\chi_{[n,n+1/n)}\right) f\,dx\,. \tag{3.8}$$

Since $\sum_{n=1}^{\infty}[a_n{}^q/n]$ diverges for all $q \leqq s$,

$$\sum_{n=1}^{\infty} a_n\chi_{[n,n+1/n)} \notin L^q(0,\infty) \quad \text{for any } q \leqq s\,.$$

Thus for any $p \geqq r$ there exists $f \in L^p(0,\infty)$, $f \geq 0$, for which the expressions in (3.8) are undefined, hence those in (3.7) are non-integrable.

Now let $\{h_j\}_{j=1}^{\infty}$ converge to h boundedly in $L^p(0,\infty)$ for some $p \in [1,\infty]$. Since for each fixed n, the definition of $T_r f$ over the interval $[2^{-i_n},1)$ involves at most a finite number of terms of the series defining $T_r f$, it is clear that $\lim_{j\to\infty} T_r h_j = T_r h$ a.e. on $[2^{-i_n},1)$. Since $i_n \to \infty$ as $n \to \infty$, it follows that $\lim_{j\to\infty} T_r h_j = T_r h$ a.e. on $(0,1)$ (hence on $(0,\infty)$).

(It is in this final stage that the convergence of $\{i_n\}$ to ∞ is used. Monotonicity of $\{i_n\}$ was used in Lemma 3.2 to simplify the change of order of summation.)

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