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INVARIANT PSEUDO-DIFFERENTIAL OPERATORS

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Introduction.

Let G be a Lie group that acts smoothly and transitively on the C^∞-manifold Ω. In this paper we study pseudo-differential operators that are defined on Ω and are invariant under the action of G. The term „pseudo-differential operator'' will denote the operators that L. Hörmander defined in [3].

We show that all G-invariant pseudo-differential operators on a homogeneous space G/H, where the closed subgroup H is a product of a compact and a normal subgroup, can be constructed in a way similar to the usual construction of left-invariant vector fields from a tangent vector. From that it follows that there are G-invariant pseudo-differential operators apart from the G-invariant differential operators.

We next examine the special case where $G\times G$ acts on G by

$$(g_1,g_2)\cdot g = g_1 g g_2^{-1} .$$

An operator is then $G\times G$-invariant if and only if it is bi-invariant, i.e. both right and left-invariant. The isotropy group K is here the diagonal of $G\times G$ so it is isomorphic to G. It is easily seen that K is a product of a compact and a normal subgroup of $G\times G$ if and only if G is a product of a compact and a central subgroup of G.

The surprising theorem is that every bi-invariant pseudo-differential operator on G is a sum of a bi-invariant differential operator and an operator of order $-\infty$, if G is not a product of a compact and a central subgroup, e.g., if G is semi-simple. If G is such a product, then we are dealing with the homogeneous space case, described above.

Finally we extend the theorem about bi-invariant operators to more general pseudo-differential operators by working with the distribution kernels of the operators. The essential property of these general operators is that they are pseudo-local, i.e. decrease singular support.

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hereby express my gratitude to them and to professor S. Helgason for their invaluable help and support. I must also mention that professor Singer suggested the problem that led to this paper.

1. Standard notations and definitions.

The following notations and definitions are used throughout the paper:

J $= (J_1, J_2, \ldots, J_n)$ is a multi-index, that is, $J_1, J_2, \ldots, J_n$ are non-negative integers;

$|J|$ $= J_1 + J_2 + \ldots + J_n$;

Ω: paracompact C^∞-manifold with a countable basis for the topology;

$T_x^*(\Omega)$: cotangent space of Ω at $x \in \Omega$.

Definition 1.1. Let U be an open subset of Ω.

$C^\infty(U)$ = set of all infinitely often differentiable complex-valued functions, defined in U, with the topology of uniform convergence of functions and their derivatives separately on compact sets.

$C_0^\infty(U)$ = set of all functions in $C^\infty(U)$ which have compact support in U, with the usual inductive limit topology.

That a function is smooth in U means that it belongs to $C^\infty(U)$.

G: a connected Lie group with identity element e, left Haar measure $d\mu(g)$ and Lie algebra $\mathfrak{G}$.

exp: the exponential mapping of $\mathfrak{G}$ into G.

Ad(ad): the adjoint representation of $G(\mathfrak{G})$ on $\mathfrak{G}$.

Definition 1.2. G is said to act smoothly on Ω if there is given a differentiable mapping $(g,x) \rightharpoonup g\cdot x$ of $G \times \Omega$ onto Ω such that

$$(g_1 g_2)\cdot x = g_1\cdot(g_2\cdot x) \quad \text{and} \quad e\cdot x = x$$

for all $g_1, g_2 \in G$ and for all $x \in \Omega$.

Ω is then said to be a G-manifold.

Definition 1.3. Suppose that G acts smoothly on Ω. If f is a real or complex valued function on Ω and $g \in G$, then ${}_gf$ denotes the function

$$({}_gf)(x) = f(g\cdot x) \quad \text{for every } x \in \Omega ,$$

and is called the translate of f by g. If $\Omega = G$ we call ${}_gf$ the left-translate of f by g, and we define the right-translate f_g of f by g by the formula

$$(f_g)(h) = f(hg) \quad \text{for all } h \in G .$$

DEFINITION 1.4. Suppose that G acts smoothly on Ω. An operator L mapping functions on Ω to functions on Ω is said to be *G-invariant* if

$$L({}_g f) = {}_g(Lf) \quad \text{for all } g \in G \text{ and all } f \text{ in the domain of definition of } L.$$

In the case $\Omega = G$, the operator L is said to be *left-invariant* if it is G-invariant; it is said to be *right-invariant* if

$$L(f_g) = (Lf)_g \quad \text{for all } g \in G \text{ and all } f \text{ in the domain of definition of } L;$$

and it is said to be *invariant*, if it is both right- and left-invariant.

We reserve the term pseudo-differential operator for the ones that are defined in Hörmander's paper [3] and use his notation concerning them.

DEFINITION 1.5. A symbol σ on the manifold Ω is a smooth mapping from the cotangent bundle of Ω minus the zero-section into R, which is positively homogeneous, i.e., there exists an $s \in \mathsf{R}$ such that

$$\sigma(x,t\xi) = t^s\,\sigma(x,\xi) \quad \text{for all } x \in \Omega, \text{ all } t > 0 \text{ and all } \xi \in T_x{}^*(\Omega) \setminus \{0\}\,.$$

NOTE. Let P be a pseudo-differential operator with asymptotic expansion

$$e^{-\lambda u}\,P(e^{i\lambda u}f) \sim \sum_{j=0}^{\infty} P_j(f,u)\,\lambda^{s_j} \quad \text{as } \lambda \to \infty\,.$$

It can be shown that $P_0(f,u)$ is a product of f and a smooth positively homogeneous function of degree s_0 on the co-tangent space (see [3, p. 512, note after Theorem 4.2]). The homogeneous function will be called the symbol of P and denoted σ_P or $\sigma(P)$. So the symbol of P is defined by the identity

$$P_0(f,u) = \sigma_P(x,du(x))\,f(x) \quad \text{for all } x \in \Omega,\ f \in C_0{}^\infty(\Omega) \text{ and } u \in C^\infty(\Omega) \text{ with } du(x) \neq 0\,.$$

It is well known that every symbol is the symbol of a pseudo-differential operator.

2. G-invariant pseudo-differential operators.

DEFINITION 2.1. Let G act smoothly on Ω. A symbol σ on Ω is said to be G-invariant if

$$\sigma(x,du(x)) = \sigma(g\cdot x, d({}_{g^{-1}}u)(g\cdot x))$$

for all $x \in \Omega$, $g \in G$ and $u \in C^\infty(\Omega)$ with $du(x) \neq 0$.

Lemma 2.2. *Let G act smoothly on Ω. If the pseudo-differential operator P on Ω is G-invariant, then so is its symbol σ_P.*

Proof. It follows from the uniqueness of the asymptotic expansion

$$e^{-i\lambda u} P(e^{i\lambda u} f) \sim \sum_{j=0}^{\infty} P_j(f,u)\, \lambda^{s_j} \quad \text{as } \lambda \to \infty$$

and the invariance of P, that

$$P_j({}_g f, {}_g u)(x) = P_j(f,u)\,(g\cdot x)\,.$$

Then the formula

$$\sigma_P(x, du(x)) f(x) = P_0(f,u)(x)$$

yields the result.

Theorem 2.3 shows how G-invariant pseudo-differential operators can be constructed on the G-manifold Ω. The idea is to adapt the usual method of constructing left-invariant vector fields on the Lie group G: If ξ is a tangent vector to G at e, then a left-invariant vector field Ξ on G, satisfying $\Xi_e = \xi$, is given by the formula:

$$\Xi_g(f) = \xi({}_g f) \quad \text{for all } f \in C^\infty(G) \text{ and all } g \in G\,.$$

Theorem 2.3. *Let G act smoothly and transitively on Ω, and suppose that the isotropy group of $x_0 \in \Omega$ is a product of a normal subgroup and a compact subgroup H.*

If P is a pseudo-differential operator on Ω, we define for every $f \in C_0^\infty(\Omega)$ the function Qf on Ω by

$$(Qf)(g\cdot x_0) = \int_H P({}_{gh} f)(x_0)\, d\mu(h) \quad \text{for every } g \in G\,,$$

where $d\mu$ is Haar-measure on H.

Then Q is a G-invariant pseudo-differential operator on Ω with

$$\sigma_Q(x_0, du(x_0)) = \int_H \sigma_P(x_0, d({}_h u)(x_0))\, d\mu(h)$$

for every $u \in C^\infty(\Omega)$, $du(x_0) \neq 0$.

Corollary 2.4. *Let $P\colon C_0^\infty(G) \to C^\infty(G)$ be a pseudo-differential operator on G. For $f \in C_0^\infty(G)$ we denote by Qf the function defined by*

$$(Qf)(g) = P({}_g f)(e) \quad \text{for every } g \in G\,.$$

Then $f \to Qf$ is a left-invariant pseudo-differential operator on G, and

$$\sigma_Q(e,\cdot) = \sigma_P(e,\cdot)\,.$$

REMARK. The „translation construction" of Theorem 2.3 gives all G-invariant pseudo-differential operators on Ω.

PROOF OF THEOREM 2.3. The proof is routine work once we use the following idea here taken in the special case of Corollary 2.4:

We shall show that $e^{-i\lambda u}Q(e^{i\lambda u}f)$ has an asymptotic expansion

$$\sum_{j=0}^{\infty} Q_j(f,u)\,\lambda^{s_j} \quad \text{as } \lambda \to \infty .$$

Since

$$[e^{-i\lambda u}Q(e^{i\lambda u}f)](g) = e^{-i\lambda({}_gu)(e)}\,P(e^{i\lambda_g u}{}_gf)(e) ,$$

we see that $e^{-i\lambda u}Q(e^{i\lambda u}f)$ pointwise has an asymptotic expansion, namely with

$$Q_j(f,u)(g) = P_j({}_gf,{}_gu)(e) .$$

Using that the expansion of P is uniform for u in compact subsets of $C^\infty(G)$ and f in bounded subsets of $C_0^\infty(G)$ (Remark p. 510 in [5]), we find that $e^{-i\lambda u}Q(e^{i\lambda u}f)$ has the asymptotic expansion $\sum_j Q_j(f,u)\lambda^{s_j}$ in the $C(G)$ topology.

By the same arguments as in [5, p. 510] we finally show that the expansion actually is in the $C^\infty(G)$ topology.

For details, we refer to [9].

COROLLARY 2.5. *Suppose G acts smoothly and transitively on Ω, and that the isotropy group H of $x_0 \in \Omega$ is compact.*

To every G-invariant symbol σ on Ω there then exists a G-invariant pseudo-differential operator on Ω with symbol σ.

PROOF. By a standard result there is a pseudo-differential operator P on Ω with $\sigma_P=\sigma$. Let Q be constructed from P as in Theorem 2.3. Then

$$\begin{aligned}
\sigma_Q(x_0,du(x_0)) &= \int_H \sigma_P(x_0,d({}_hu)(x_0))\,d\mu(h) \\
&= \int_H \sigma(x_0,d({}_hu)(x_0))\,d\mu(h) \\
&= \int_H \sigma(h^{-1}\cdot x_0,du(h^{-1}\cdot x_0))\,d\mu(h) \\
&= \int_H \sigma(x_0,du(x_0))\,d\mu(h) = \sigma(x_0,du(x_0)) ,
\end{aligned}$$

so σ_Q and σ agree at x_0. By invariance they then agree everywhere.

Theorem 2.6. describes another way of finding invariant pseudo-differential operators, namely by averaging over the group. The result is known, but as far as I know, not written down in the literature. Atiyah and Singer mention in [1, footnote on p. 517] that the averaging construction when applied to a pseudo-differential operator, gives an element of the space $\mathscr{P}^m$, that consists of generalized pseudo-differential operators. But they do not need the fact that the element is an ordinary pseudo-differential operator.

The disadvantage of the averaging method is that G has to be compact. The advantage that G need *not* act transitively.

NOTATIONS. Let Ω be a G-manifold. If P: $C_0^\infty(\Omega) \to C^\infty(\Omega)$ is a continuous linear mapping and $g \in G$, then we denote by $g(P)$ the operator

$$u \to g(P)u = {}_{g^{-1}}[P({}_g u)] .$$

It is easily seen that the mapping $g \to g(P)u$ is continuous from G to $C^\infty(\Omega)$ for fixed $u \in C_0^\infty(\Omega)$. So for G compact the integral

$$\mathrm{Av}(P)u = \int_G g(P)u \, dg$$

converges in $C^\infty(\Omega)$. Observe that dg refers to Haar-measure on G.

THEOREM 2.6. *Let G be a compact Lie group, acting smoothly on the manifold Ω. Let P be a pseudo-differential operator on Ω. Then* $\mathrm{Av}(P)$ *is a pseudo-differential operator on Ω, invariant under the action of G, and its symbol is*

$$\sigma(\mathrm{Av}(P))\,(x,du(x)) = \int_G \sigma_P(g^{-1}\cdot x, d({}_g u)(g^{-1}\cdot x)) \, dg .$$

PROOF. The proof is analogous to the proof of Theorem 2.4. Instead of translating the terms in the expansion of P we integrate them over G. For details we refer to [9].

3. Invariant pseudo-differential operators on Lie groups.

DEFINITION 3.1. A symbol σ on G is said to be *left-invariant*, if

$$\sigma(g,du(g)) = \sigma(e,d({}_g u)(e))$$

for all $g \in G$ and all $u \in C^\infty(G)$ with $du(g) \neq 0$, and to be *right-invariant* if

$$\sigma(g,du(g)) = \sigma(e,d(u_g)(e))$$

for all $g \in G$ and all $u \in C^\infty(G)$ with $du(g) \neq 0$, and to be *invariant*, if it is both right- and left-invariant.

REMARK 3.2. If σ is an invariant symbol, then

$$\sigma(e,\xi) = \sigma(e, \mathrm{Ad}(g)^* \xi)$$

for all $g \in G$ and all $\xi \in T_e^*(G) \setminus \{0\}$.

Conversely, if ϱ is a smooth, positively homogeneous function on $T_e^*(G) \setminus \{0\}$, satisfying $\varrho(\xi) = \varrho(\mathrm{Ad}(g)^*\xi)$ for all $\xi \in T_e^*(G) \setminus \{0\}$ and all $g \in G$, then there is exactly one invariant symbol σ such that $\sigma(e, \cdot) = \varrho$.

PROPOSITION 3.3. *If a pseudo-differential operator is left-invariant, right-invariant or invariant, then so is its symbol.*

PROOF. Like the proof of Lemma 2.2.

PROPOSITION 3.4. *Let σ be an invariant symbol on G. If $\sigma(e, \cdot)$ is a polynomial function, then there exists an invariant differential operator D with $\sigma_D = \sigma$.*

PROOF. This is a well-known result from the theory of the universal enveloping algebra of a Lie algebra. See e.g. Helgason [2, p. 393].

In this section we will study pseudo-differential operators on G which are both right- and left-invariant. The main result (Theorem 3.5) states that on „most" groups all these operators are differential operators (modulo operators of order $-\infty$).

The procedure will be to show that the symbol σ_P at $e \in G$ of any such operator P is an invariant polynomial. Then we find D as in Proposition 3.4 and form the difference $P - D$, which then is an invariant pseudo-differential operator of strictly lower order than P. By finite induction we get that there are invariant differential operators $D_1, \ldots, D_k$ such that the symbol of $P - D_1 - \ldots - D_k$ has order less than zero. But the symbol is an invariant polynomial, hence it is zero, so $P - D_1 - \ldots - D_k$ has order $-\infty$.

THEOREM 3.5. a) *Every invariant pseudo-differential operator on G is a sum of an invariant differential operator and an invariant pseudo-differential operator of order $-\infty$, if $\mathfrak{G}$ is not a direct sum of a compact and an abelian Lie algebra.*

b) If *$\mathfrak{G}$ is a direct sum of a compact subalgebra and an abelian subalgebra, then there is an invariant pseudo-differential operator on G which is not a differential operator modulo operators of order $-\infty$.*

Part a) is a corollary of the following proposition.

PROPOSITION 3.6. *If there is a non-trivial nilpotent element $w \in \mathfrak{G}$, i.e.* $\mathrm{ad}\, w \neq 0$ *but* $(\mathrm{ad}\, w)^n = 0$ *for some n, then every invariant pseudo-differential* *operator on G is a sum of an invariant differential operator and an invariant pseudo-differential operator of order $-\infty$.*

PROOF THAT 3.6 IMPLIES 3.5a): We will show that $\mathfrak{G}$ has a non-trivial nilpotent element w.

Let $\operatorname{rad}\mathfrak{G}$ denote the radical of $\mathfrak{G}$ and Z the center of $\mathfrak{G}$. Obviously $\operatorname{rad}\mathfrak{G} \supseteq Z$, the radical being the maximal solvable ideal in $\mathfrak{G}$. There are the following cases:

1a) $\operatorname{rad}\mathfrak{G} \neq Z$ and $\operatorname{rad}\mathfrak{G}$ is not commutative. Let

$$\operatorname{rad}\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_{n-2} \supset \mathfrak{G}_{n-1} \supset \mathfrak{G}_n = \{0\}$$

be the derived series of the solvable Lie algebra $\operatorname{rad}\mathfrak{G}$. Then $n \geqq 2$ because $\operatorname{rad}\mathfrak{G}$ is not commutative.

If there is a $w \in \mathfrak{G}_{n-1}$ such that $[w, \operatorname{rad}\mathfrak{G}] \neq \{0\}$, then $\operatorname{ad}w \neq 0$ and $(\operatorname{ad}w)^2(\operatorname{rad}\mathfrak{G}) = \{0\}$ so $(\operatorname{ad}w)^3 = 0$.

In case $[\mathfrak{G}_{n-1}, \operatorname{rad}\mathfrak{G}] = \{0\}$ we take a $w \in \mathfrak{G}_{n-2}$ such that $[w, \mathfrak{G}_{n-2}] \neq \{0\}$. Then $\operatorname{ad}w \neq 0$ and

$$(\operatorname{ad}w)^3(\operatorname{rad}\mathfrak{G}) \subseteq (\operatorname{ad}w)^2(\mathfrak{G}_{n-2}) \subseteq (\operatorname{ad}w)(\mathfrak{G}_{n-1}) = \{0\},$$

so $(\operatorname{ad}w)^4 = 0$.

1b) $\operatorname{rad}\mathfrak{G} \neq Z$ and $\operatorname{rad}\mathfrak{G}$ is commutative. Any element $w \in \operatorname{rad}\mathfrak{G} \setminus Z$ has $\operatorname{ad}w \neq 0$ and $(\operatorname{ad}w)^2 = 0$.

2) $\operatorname{rad}\mathfrak{G} = Z$. The Levi-decomposition theorem [8, Cor. 1, p. LA 6.10] says that $\mathfrak{G}$ can be written as a semi-direct product

$$\mathfrak{G} = \operatorname{rad}\mathfrak{G} \oplus_{\text{semi}} \mathfrak{G}_s$$

of its radical and a semi-simple subalgebra $\mathfrak{G}_s$. But since $\operatorname{rad}\mathfrak{G} = Z$, we see that $\mathfrak{G}$ is actually a direct sum

$$\mathfrak{G} = Z \oplus \mathfrak{G}_s .$$

Here $\mathfrak{G}_s$ is not compact by the assumption of Theorem 3.5a). It is then well known from the theory of restricted roots that $\mathfrak{G}_s$ has a non-trivial nilpotent element: Just take a non-zero vector in a restricted-root space. But when $\mathfrak{G}_s$ has a non-trivial nilpotent element, then so does

$$\mathfrak{G} = Z \oplus \mathfrak{G}_s .$$

We are now going to prove Proposition 3.6. By the remarks right after Proposition 3.4 it suffices to show that every invariant symbol σ on G is a polynomial. We use Lemma 3.7 below to see that σ is a polynomial in some of its variables, and then this will imply that it is a polynomial in all of its variables.

Lemma 3.7. *Suppose $H \in C^\infty(\mathsf{R}^n)$, and that there exists an $s \in \mathsf{R}$ and strictly positive numbers $s_1,\ldots,s_n$ such that*

$$(*) \qquad H(t^{s_1}x_1,\ldots,t^{s_n}x_n) = t^s H(x_1,\ldots,x_n)$$

for all $t>0$ and all $(x_1,\ldots,x_n) \in \mathsf{R}^n$.

Then H is a polynomial in $x_1,\ldots,x_n$ of degree $\leqq s/(\min s_i)$.

Proof. Differentiation of the equation (*) above N times with respect to x_1 gives

$$t^{s_1N} \frac{\partial^N H}{\partial {x_1}^N}(t^{s_1}x_1,\ldots,t^{s_n}x_n) = t^s \frac{\partial^N H}{\partial {x_1}^N}(x_1,\ldots,x_n)$$

so

$$\frac{\partial^N H}{\partial {x_1}^N}(x_1,\ldots,x_n) = t^{s_1N-s} \frac{\partial^N H}{\partial {x_1}^N}(t^{s_1}x_1,\ldots,t^{s_n}x_n)\,.$$

Choosing N so big that $s_1N-s>0$ and letting $t \to 0$ we obtain

$$\frac{\partial^N H}{\partial {x_1}^N}(x_1,\ldots,x_n) = 0$$

for all $(x_1,\ldots,x_n) \in \mathsf{R}^n$.

Similar considerations hold for all other derivatives, so all derivatives of H of a sufficiently high order are identically 0. Hence, H is a polynomial.

Let k be the degree of H. If $s<0$, we see by letting $t \to 0$ in (*) that $H \equiv 0$, so $k=-\infty$.

Let now $s \geqq 0$, and assume H is not identically 0, so that we may write

$$H(x) = \textstyle\sum_{|J|=k} c_J x^J + \text{lower order terms}\,,$$

where $c_J \neq 0$ for some J with $|J|=k$.

From (*) it follows that

$$\begin{aligned}\textstyle\sum_{|J|=k} c_J t^{s_1J_1+\ldots+s_nJ_n} x^J &+ \text{lower order terms} \\ &= t^s \textstyle\sum_{|J|=k} c_J x^J + \text{lower order terms}\,,\end{aligned}$$

so

$$s = s_1J_1+s_2J_2+\ldots+s_nJ_n \quad \text{for some } J \text{ with } |J|=k\,.$$

Then

$$\begin{aligned}k = |J| = J_1+\ldots+J_n &= s_1J_1s_1^{-1}+\ldots+s_nJ_ns_n^{-1} \\ &\leqq (s_1J_1+\ldots+s_nJ_n)/\min s_i = s/\min s_i\,.\end{aligned}$$

Proof of proposition 3.6. The following proof is communicated to me by professor L. Carleson, and simplifies the author's original proof.

By Lemma 3.7 it is enough to show that every invariant symbol σ can be extended so that the extension is C^∞ also at the origin. The invariance of σ,

$$\sigma(\xi) = \sigma(\mathrm{Ad}(\exp sw)^* \xi) \quad \text{for all } s \in \mathsf{R}^1 \text{ and all } \xi \in \mathfrak{G}^* \setminus \{0\},$$

may here be expressed as

$$\sigma(\xi) = \sigma(\exp(sB)\xi),$$

where $B = (\mathrm{ad}\, w)^*$ is nilpotent, that is, $B^n = 0$ for some $n \geqq 2$.

Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{G}^*$ such that $\{x_1, \ldots, x_p\}$ is a basis of $\ker B$. We will first show that for fixed $y \notin \ker B$, the function $\sigma(\xi_1 x_1 + \ldots + \xi_p x_p + y)$ is a polynomial in $\xi_1, \xi_2, \ldots, \xi_p$.

Let $k \geqq 1$ be so that $B^k y \neq 0$, but $B^{k+1} y = 0$, and consider

$$H(\xi_1, \ldots \xi_p, r) = \sigma(\xi_1 x_1 + \ldots + \xi_p x_p + r^k y).$$

Then H is C^∞ except possibly at the origin, and

$$H(t^k \xi_1, \ldots, t^k \xi_p, tr) = t^{ks} H(\xi_1, \ldots, \xi_p, r),$$

where s = the degree of σ. Furthermore,

$$\begin{aligned} H(\xi, r) &= \sigma(e^{sB}(\xi_1 x_1 + \ldots + \xi_p x_p + r^k y)) \\ &= \sigma(\xi_1 x_1 + \ldots + \xi_p x_p + r^k \textstyle\sum_{j=0}^k s^j (j!)^{-1} B^j y). \end{aligned}$$

If $r \neq 0$, we get with $s = r^{-1}$ that

$$H(\xi, r) = \sigma(\xi_1 x_1 + \ldots + \xi_p x_p + r^k y + \ldots + (k!)^{-1} B^k y).$$

By continuity this formula also holds when $r = 0$ and $|\xi|$ is small, because then

$$\xi_1 x_1 + \ldots + \xi_p x_p + (k!)^{-1} B^k y \neq 0.$$

Since the right hand side for $\xi = 0$ and $r = 0$ is $\sigma((k!)^{-1} B^k y)$ and $B^k y \neq 0$, then H can be extended over the origin to a C^∞-function. According to lemma 3.7, $H(\xi, r)$ is a polynomial in ξ and r of degree

$$\leqq ks/\min(k, 1) = ks \leqq ns,$$

so that

$$H(\xi, 1) = \sigma(\xi_1 x_1 + \ldots + \xi_p x_p + y)$$

is a polynomial in ξ of degree $\leqq ns$. Now consider

$$\sigma(\xi + \xi^{(0)}, \eta) = \sigma(\textstyle\sum_1^p (\xi_i + \xi_i^{(0)}) x_i + \sum_{p+1}^n \eta_i x_i),$$

which by what we have just proved, is a polynomial in ξ of degree less than or equal to ns if $\eta \neq 0$, that is,

$$\sigma(\xi+\xi^{(0)},\eta) = \sum_{|J|\leq ns}\sigma_J(\xi^{(0)},\eta)\,\xi^J$$

or

$$\sigma(\xi,\eta) = \sum_{|J|\leq ns}\sigma_J(\xi^{(0)},\eta)(\xi-\xi^{(0)})^J .$$

Choosing $\xi^{(0)}\neq 0$ we get that

$$\sigma_J(\xi^{(0)},\eta) = (J!)^{-1}D_\xi{}^J\sigma(\xi^{(0)},\eta)$$

which is C^∞ even when $\eta=0$.

Hence $\sigma(\xi,\eta)$ is C^∞ also at the origin.

PROPOSITION 3.8. *To every invariant symbol σ on G, there exists an invariant pseudo-differential operator with symbol σ.*

PROOF. In the proof of Theorem 3.5a) and Proposition 3.6 it was shown that every invariant symbol is a polynomial, if $\mathfrak{G}$ is not a direct sum of a compact and an abelian subalgebra. In that case we can apply Proposition 3.4.

So we may now assume that $\mathfrak{G}=\mathfrak{H}\oplus\mathfrak{A}$ is a direct sum of a compact Lie algebra $\mathfrak{H}$ and an abelian Lie algebra $\mathfrak{A}$. It is easy to see that G then can be written as $G=HA$, where H is a compact subgroup of G, and where A is contained in the center of G.

By Corollary 2.5 there exists a right-invariant pseudo-differential operator P on G with symbol $\sigma_P=\sigma$.

H is compact and acts smoothly on G by left-multiplication, so $\mathrm{Av}(P)$ makes sense (Theorem 2.6) and is an H-invariant pseudo-differential operator on G. With $dh=$Haar measure on H we have that

$$\begin{aligned}\sigma(\mathrm{Av}(P))(g,du(g)) &= \int_H \sigma_P(h^{-1}g,\, d_hu)(h^{-1}g)\,dh\\ &= \int_H \sigma(h^{-1}g,\, d({}_hu))(h^{-1}g)\,dh \quad (\text{since } \sigma_P=\sigma)\\ &= \int_H \sigma\big(e, d({}_{h^{-1}g}({}_hu))(e)\big)\,dh \quad (\text{since } \sigma \text{ is left-invariant})\\ &= \int_H \sigma(e, d({}_gu)(e))\,dh\\ &= \sigma(e,d({}_gu)(e)) = \sigma(g,du(g))\,,\end{aligned}$$

so $\sigma(\mathrm{Av}(P))=\sigma$. Furthermore, $\mathrm{Av}(P)$ is right-invariant, because

$$[\mathrm{Av}(P)(u_{g'})](g) = \int_H P({}_h u_{g'})(h^{-1}g)\,dh$$
$$= \int_H P({}_h u)(h^{-1}gg')\,dh \qquad \text{(since } P \text{ is right-invariant)}$$
$$= [\mathrm{Av}(P)\mathrm{u}](gg') = [\mathrm{Av}(P)u]_{g'}(g)\,.$$

By construction $\mathrm{Av}(P)$ is invariant under left-multiplication by H. But is is also invariant under left-multiplication by A, because A commutes with everything, so that ${}_a u = u_a$, which reduces the case to the right-invariant one.

Hence $\mathrm{Av}(P)$ is also left-invariant.

PROOF OF THEOREM 3.5 b). By Proposition 3.8 it suffices to produce an invariant symbol, which is not a polynomial. Let as there

$$G = HA\,,$$

where H is a compact subgroup and A is contained in the center of G. Since A commutes with everything, $\mathrm{Ad}(a) = \mathrm{Id}$ for all $a \in A$, so invariance under $(\mathrm{Ad}\,G)^*$ reduces to invariance under $(\mathrm{Ad}_G H)^*$. Since H is compact, there exists on $\mathfrak{G}^*$ a strictly positive definite quadratic form $Q(\cdot,\cdot)$, which is invariant under $(\mathrm{Ad}_G(H))^*$. Hence

$$\sigma(\xi) = (Q(\xi,\xi))^{\frac{1}{2}}$$

is an invariant symbol on $\mathfrak{G}^*$, and it is not a polynomial.

4. Invariant pseudo-local operators.

DEFINITION 4.1. (See [7, p. 69]). A strictly positive smooth measure on Ω is a Radon measure μ on Ω such that if $\varphi\colon 0 \to \mathsf{R}^n$ is a chart in Ω, there is a strictly positive smooth function ϱ on R^n such that

$$\int_\Omega f(x)\,d\mu(x) = \int_{\mathsf{R}^n} (f\circ\varphi^{-1})(\xi)\,\varrho(\xi)\,d\xi$$

for any continuous complex-valued function f on Ω, having as support a compact subset of 0.

REMARK. A Riemannian structure on Ω gives rise to such a measure with

$$\varrho(\xi) = \big(\det(g_{ij}(\xi))\big)^{\frac{1}{2}}\,.$$

Since a paracompact manifold has a Riemannian structure, it also has a strictly positive smooth measure.

NOTATIONS. $\mathscr{E}'(\Omega)=(C^\infty(\Omega))'$ denotes the set of all distributions on Ω with compact support, while $\mathscr{D}'(\Omega)=(C_0^\infty(\Omega)'$ denotes the set of all distributions on Ω. We give $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$ the strong topologies.

Given a strictly positive smooth measure μ on Ω we will identify $C_0^\infty(\Omega)$ with a subspace of $\mathscr{E}'(\Omega)$ and $C^\infty(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ by means of μ in the usual way.

DEFINITION 4.2. Let μ be a strictly positive smooth measure on Ω. If $u \in \mathscr{D}'(\Omega)$ then the singular support of u, denoted sing supp u, is defined as the set of points in Ω, having no neighbourhood where $u \in C^\infty$.

REMARK. It is easy to see that sing supp u does not depend on the choice of the measure μ, but only on u.

DEFINITION 4.3. A continuous linear operator $P\colon \mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega)$ is said to be *pseudo-local*, if

$$\operatorname{sing\,supp} Pu \subseteq \operatorname{sing\,supp} u \quad \text{for all } u \in \mathscr{E}'(\Omega)\,.$$

REMARK. It is easy to verify that if P is pseudo-local with respect to μ, then P is also pseudo-local with respect to any other strictly positive smooth measure on Ω. So the property of being pseudo-local depends only on the operator and not on the choice of measure.

REMARK. It follows from the closed graph theorem that

$$P\,|\,C_0^\infty(\Omega)\colon\ C_0^\infty(\Omega) \to C^\infty(\Omega)$$

is continuous, when P is pseudo-local.

REMARK. J. J. Kohn and L. Nirenberg were apparently the first to observe that pseudo-differential operators are pseudo-local ([5, p. 293]). But also more general spaces consist of pseudo-local operators. For example, operators in $L_{\varrho,\delta}(\Omega)$ (defined in [4, p. 153]) are pseudo-local.

From now on we restrict ourselves to the case where $\Omega = G$ is a connected Lie group. As a strictly positive, smooth measure on G we take left Haar measure $d\mu$. The modular function will be denoted by Δ.

ASSUMPTION 4.4. Every real-valued function k which is defined and smooth on $G\setminus\{e\}$ and which satisfies

$$k(aga^{-1}) = \Delta(a)k(g) \quad \text{for all } a \in G \text{ and } g \in G\setminus\{e\}$$

can be extended to a smooth function defined on all of G.

REMARK. Assumption 4.4 is not satisfied for all groups, for example not for abelian groups. But it is satisfied for large classes of groups, in particular for all complex semi-simple groups, as will be shown in the proof of Corollary 4.6.

THEOREM 4.5. *If the assumption 4.4 holds for G, then every invariant pseudo-local operator on G is a sum of an invariant differential operator and an invariant integral operator with smooth kernel.*

COROLLARY 4.6. *Every invariant pseudo-local operator on G is a sum of an invariant differential operator and an invariant integral operator with smooth kernel, if*

$$G \text{ is not unimodular},$$

or if

$$G \text{ is complex reductive},$$

or if

$$\mathfrak{G} \text{ has an element } x \text{ such that } \dim((\operatorname{ad} x)(\mathfrak{G})) = 1,$$

or if

$$G = \text{the connected component of } \mathrm{GL}(2, \mathbf{R}),$$

or if

$$G = \mathrm{SL}(2, \mathbf{R}).$$

REMARK. If $\mathfrak{G}$ is the Heisenberg algebra or more generally a nilpotent Lie algebra with one-dimensional center, then there is $x \in \mathfrak{G}$ such that $\dim(\operatorname{ad} x)(\mathfrak{G})) = 1$.

REMARK. An inspection of the proof of Theorem 4.5 shows that the assumption sing supp $Pu \subseteq$ sing supp u can be weakened. We only need that sing supp $P\delta \subseteq \{e\}$, where δ is the Dirac δ-function with support at e.

PROOF OF THEOREM 4.5. If $\varphi \in C_0^\infty(G)$ we let $\check{\varphi} \in C_0^\infty(G)$ denote the function $\check{\varphi}(g) = \varphi(g^{-1})$.

Let P be an invariant pseudo-local operator on G. The mapping

$$T: \varphi \to (P\check{\varphi})(e)$$

from $C_0^\infty(G)$ into $\mathbf{C}$ is a distribution on G.

$$(T^*\varphi)(g) = \langle T, (\varphi_g)^\vee \rangle = P(\varphi_g)(e) = (P\varphi)(g),$$

because P is right-invariant, so

$$T^*\varphi = P\varphi \quad \text{for all } \varphi \in C_0^\infty(G).$$

There is a sequence $(\varphi_n) \subset C_0^\infty(G)$ such that $\varphi_0 \to \delta$ in $\mathscr{E}'(G)$, so going to the limit we get that

$$T = T^*\delta = P\delta.$$

Let $k = P\delta \mid G \setminus \{e\}$. Then $k \in C^\infty(G \setminus \{e\})$ because P is pseudo-local. Furthermore

$$(P\varphi)(a) = \int_G k(g)\, \varphi(g^{-1}a)\, d\mu(g) \quad \text{if } a \notin \operatorname{supp}\varphi .$$

Since P is left-invariant,

$$(P\varphi)(a) = P({}_a\varphi)(e) = \int_G k(g)\, \varphi(ag^{-1})\, d\mu(g) .$$

Changing variables, we find that

$$(P\varphi)(a) = \int_G \Delta(a)\, k(a^{-1}ga)\, \varphi(g^{-1}a)\, d\mu(g) ,$$

so k must satisfy the identity

$$k(g) = \Delta(a)\, k(a^{-1}ga) \quad \text{for all } a \in G \text{ and } g \in G \setminus \{e\} .$$

By hypothesis the assumption 4.4 holds for G, so k can be extended to a smooth function defined on all of G. The extension will also be denoted k. We can now define the operator $K\colon C_0^\infty(G) \to C^\infty(G)$ by

$$(K\varphi)(a) = \int_G k(g)\, \varphi(g^{-1}a)\, d\mu(g) .$$

It is easily seen that K is invariant, because k satisfies the identity $k(g) = \Delta(a) k(a^{-1}ga)$.

If $\varphi \in C_0^\infty(G)$ and $e \notin \operatorname{supp}\varphi$, then

$$(P\varphi)(e) - (K\varphi)(e) = \int_G k(g)\, \varphi(g^{-1})\, d\mu(g) - \int_G k(g)\, \varphi(g^{-1})\, d\mu(g) = 0$$

so the distribution

$$\varphi \to (P\varphi)(e) - (K\varphi)(e)$$

has support contained in the single point e. Hence it is a linear combination of the Dirac δ-function and its derivatives, i.e., there is a differential operator D_e such that

$$(P\varphi)(e) - (K\varphi)(e) = (D_e\varphi)(e) \quad \text{for all } \varphi \in C_0^\infty(G) .$$

Defining D by $(D\varphi)(g) = D_e({}_g\varphi)(e)$, we see that D is a left-invariant differential operator on G, that agrees with D_e at e. Since both P and K are left-invariant and $P - K = D$ at e, it follows that $P - K = D$ everywhere.

Since P and K are invariant, so is $D = P - K$.

Now $P = D + K$ is the desired decomposition of P into an invariant differential operator and an invariant integral operator with smooth kernel.

Proof of Corollary 4.6. In each of 4 cases we will show that the assumption 4.4 is valid for the group G.

1. *G is not unimodular.* Observe that k satisfies the identity

$$k(aga^{-1}) = \Delta(a)\,k(g) \quad \text{for all } a \in G \text{ and } g \in G \setminus \{e\}\,.$$

In particular $k(g) = \Delta(g)k(g)$, so that $\Delta(g) = 1$ when $k(g) \neq 0$.

The modular function Δ is a continuous homomorphism of the Lie group G into the Lie group R^+. Hence it is an analytic mapping ([2, Theorem 2.6, p. 107]). If $k \neq 0$ then $\Delta = 1$ on an open set, so $\Delta = 1$ everywhere, because it is analytic. But G is not unimodular, so we conclude, that $k \equiv 0$.

We may and will assume that the modular function $\Delta \equiv 1$ in the remaining cases. Then k satisfies the identity

$$k(aga^{-1}) = k(g) \quad \text{for all } a \in G \text{ and } g \in G \setminus \{e\}\,.$$

Let $U = \{x \in \mathfrak{G} \mid \exp x \neq e\}$. Then U is an open set, and $V = U \cup \{0\}$ is a neighbourhood of $0 \in \mathfrak{G}$. The function h, defined by

$$h(x) = k(\exp x) \quad \text{for } x \in U\,,$$

is well-defined and smooth in U. Since exp is a diffeomorphism close to $0 \in \mathfrak{G}$, it obviously suffices to show that h can be extended to a smooth function defined on all of V. To prove that it turns out we only need to know that

(α) h is smooth in $V \setminus \{0\}$

and

(β) $h(\mathrm{Ad}(g)x) = h(x)$ for all $x \in V \setminus \{0\}$ and $g \in G$.

It is easy to prove (β):

$$h(\mathrm{Ad}(g)x) = k\big(\exp(\mathrm{Ad}(g)x)\big) = k(g(\exp x)g^{-1}) = k(\exp x) = h(x)\,.$$

2. *G is complex reductive.* We will use some results from the paper [6], so in this proof the notations will be as in [6]. In particular, an orbit means an orbit of an element in $\mathfrak{G}$ under the action of the adjoint group.

Let $u_1, \ldots, u_k$ be algebraically independent homogeneous polynomials generating the ring of invariants. We define the mapping $u \colon \mathfrak{G} \to \mathsf{C}^k$ by

$$u(x) = \big(u_1(x), \ldots, u_k(x)\big)\,.$$

Let

$$P(\xi) = \{x \in \mathfrak{G} \mid u(x) = \xi\} \quad \text{for } \xi \in \mathsf{C}^k\,.$$

According to Theorem 3, p. 365 in [6], there exists an orbit $O^r(\xi)$ such that $P(\xi)=\overline{O^r(\xi)}$. h is constant on $O^r(\xi)$ by (β). Since h is continuous, it is constant on $\overline{O^r(\xi)}=P(\xi)$. So $h(x)$ depends only on the value of $u(x)$.

Next choose a principal nilpotent element $e_- \in \mathfrak{G}$, (see 4.2.1, p. 370 in [6]) so small that $e_- \in V$. By Theorem 7, p. 381 in [6] there exists a transversal k-plane

$$v = e_- + \mathfrak{G}^{e+}$$

such that

$$t = u|v:\ v \to \mathbf{C}^k$$

is a global coordinate system on v. Furthermore there is a basis $(z_1,\ldots,z_k)$ of $\mathfrak{G}^{e+}$ such that if

$$x = e_- + s_1 z_1 + \ldots + s_k z_k \in e_- + \mathfrak{G}^{e+},$$

then for $i=1,\ldots,k$

$$u_i(x) = s_i + p_i(s_1,\ldots,s_{i-1}),$$

where p_i is a polynomial in $i-1$ variables without constant term. We see that $u_i(e_-)=0+p_i(0,\ldots,0)=0$, so $u(e_-)=0$.

The mapping $x \to h\big(t^{-1}(u(x))\big)$ is smooth in a neighbourhood of 0, because h is smooth close to $e_- \in V$. But since $h(x)$ only depends on the value of $u(x)$, we have

$$h\big(t^{-1}(u(x))\big) = h(x),$$

so h is the restriction of the smooth function $h\big(t^{-1}(u(x))\big)$ close to $0 \in \mathfrak{G}$.

3. *$\mathfrak{G}$ has an element x such that* $\dim[\mathrm{ad}\,x(\mathfrak{G})]=1$. Let $y \in [x,\mathfrak{G}]$ be non-zero. Then there are two cases:

(a) $[x,y]=0$,

and

(b) $[x,y] = y$ (normalize x).

(a) Let $(x_1=y,x_2,\ldots,x_{n-1})$ be a basis of $(\mathrm{ad}\,x)^{-1}(0)$ and choose x_n such that $(\mathrm{ad}\,x)(x_n)=y$. Then $(x_1,\ldots,x_n)$ is a basis for $\mathfrak{G}$.

$$\mathrm{Ad}(\exp(tx)) = e^{\mathrm{ad}(tx)} = \sum_0^\infty \frac{t^k}{k!}(\mathrm{ad}\,x)^k = I + t\,\mathrm{ad}\,x$$

has with respect to the basis $(x_1,\ldots,x_n)$ the matrix

$$\mathrm{Ad}(\exp(tx)) = \begin{Bmatrix} 1 & 0 & \ldots & 0 & t \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \end{Bmatrix}$$

Therefore

$$h(\lambda_1,\ldots,\lambda_n) = h(\lambda_1+t\lambda_n,\lambda_2,\ldots,\lambda_n)$$

for all $t \in \mathsf{R}$ and $(\lambda_1,\ldots,\lambda_n) \neq 0$, so h does not depend on its first argument:

$$h(\lambda_1,\lambda_2,\ldots,\lambda_n) = h(1,\lambda_2,\ldots,\lambda_n) .$$

The right hand side — and hence the left hand side — is the restriction to $V \setminus \{0\}$ of the function

$$(\lambda_1,\ldots,\lambda_n) \to h(1,\lambda_2,\ldots,\lambda_n) ,$$

which is smooth on V, because h is smooth outside 0.

(b) In this case we choose a basis $(y,x_1,\ldots,x_n)$ of $\mathfrak{G}$ such that $(\mathrm{ad}\,x)(x_i) = 0$ for $i = 1,\ldots,n$. The operator

$$\mathrm{Ad}(\exp(tx)) = \sum_0^\infty \frac{t^k}{k!}(\mathrm{ad}\,x)^k$$

has with respect to the basis $(y,x_1,\ldots,x_n)$ the matrix

$$\mathrm{Ad}(\exp(tx)) = \begin{Bmatrix} e^t & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{Bmatrix}$$

so

$$h(\lambda,\mu_1,\ldots,\mu_n) = h(e^t\lambda,\mu_1,\ldots,\mu_n) .$$

Letting $t \to -\infty$ we get

$$h(\lambda,\mu_1,\ldots,\mu_n) = h(0,\mu_1,\ldots,\mu_n) ,$$

so again h does not depend on its first argument, and we can use the finishing arguments of case (a).

4. *G = the connected component of* GL(2, R) *or* G = SL(2, R). The proofs are practically the same, so we will only treat the case, where G is the connected component of GL(2, R).

If

$$x = \begin{Bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{Bmatrix} \in G \quad \text{and} \quad x_{21} > 0 ,$$

we let

$$g = \begin{Bmatrix} x_{21}^{\frac{1}{2}} & -x_{11}x_{21}^{-\frac{1}{2}} \\ 0 & x_{21}^{-\frac{1}{2}} \end{Bmatrix} .$$

Then

$$\mathrm{Ad}(g)x = gxg^{-1} = \begin{Bmatrix} 0 & -\det x \\ 1 & \operatorname{tr} x \end{Bmatrix},$$

so

$$h(x) = h\left(\begin{Bmatrix} 0 & -\det x \\ 1 & \operatorname{tr} x \end{Bmatrix}\right),$$

when $x_{21} > 0$. Hence h coincides for $x_{21} > 0$ with the function f_+, defined by

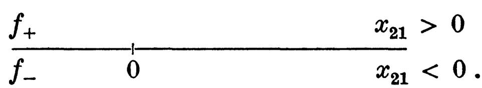
$$f_+(x) = h\left(\begin{Bmatrix} 0 & -\det x \\ 1 & \operatorname{tr} x \end{Bmatrix}\right),$$

and f_+ is smooth close to 0.

For $x_{21} < 0$ we get similarly that h coincides with the function f_-, where

$$f_-(x) = h\left(\begin{Bmatrix} 0 & \det x \\ -1 & \operatorname{tr} x \end{Bmatrix}\right),$$

and f_- is smooth close to 0. The situation is illustrated by



Observe that f_+ and f_- and all their derivatives agree on $x_{21} = 0$, because h is smooth. The function

$$\begin{aligned} f(x) &= f_+(x), \quad \text{when } x_{21} \geqq 0, \\ &= f_-(x), \quad \text{when } x_{21} \leqq 0, \end{aligned}$$

is therefore smooth in a neighbourhood of 0, and extends h.

This finishes the proof of Corollary 4.6.

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