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BOREL STRUCTURES IN GROUPS AND SEMIGROUPS

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This paper deals with some connections between the topology and the Borel structure of certain groups and semigroups. Results in this direction were obtained by Banach and a number of subsequent authors (see [1], [4], [5], [7]) by methods using the Baire category theorem. Our method, making use of a theorem from classical harmonic analysis, seems to be simpler, and in some cases it yields stronger results.

Universally measurable means measurable with respect to the universal completion of the Borel field of the topology under consideration.

THEOREM 1. *Let* $(S,+)$ *be an abelian semigroup with neutral element* 0. *Suppose* S *is equipped with a topology generated by a complete metric* d *such that all translations* $\tau_a\colon x \to a+x$ *are continuous. Let* $S=\bigcup_{i=1}^{\infty} A_i$ *be a denumerable covering of* S. *Then there is an* $i_0 \in \mathsf{N}$ *such that for every universally measurable set* $U \supseteq A_{i_0}$ *the set*

$$U-U \;=\; \{x \in S \mid (x+U) \cap U \neq \emptyset\}$$

is a 0*-neighbourhood.*

PROOF. If this were not true, we could choose universally measurable sets B_i, $i \in \mathsf{N}$, with $A_i \subseteq B_i$ and $B_i - B_i$ not a neighbourhood of 0. We choose a sequence of natural numbers $i_n \in \mathsf{N}$ such that every value $i \in \mathsf{N}$ is assumed for infinitely many n. By induction on n we choose a sequence $s_n \in S$ with $s_n \notin B_{i_n} - B_{i_n}$ and $d(r, r+s_n) \leqq 2^{-n}$, where r is any sum of different s_ν, $\nu = 1,\ldots,n-1$. (No condition for $n=1$.)

Consider the space $K=\{0,1\}^{\mathsf{N}}$. With the product topology and the usual group structure it is a compact metrizable abelian group. We define the mapping $\varphi\colon K \to S$ by

$$\varphi(x) \;=\; \textstyle\sum_{n=1}^{\infty} x_n s_n, \qquad x=(x_n)_{n\in \mathsf{N}}\,,$$

which is continuous. The sets $\varphi^{-1}(B_i)$ form a countable covering of K by universally measurable sets. Hence at least one of them, $\varphi^{-1}(B_m)$

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say, has non zero Haar measure. Then $\varphi^{-1}(B_m)-\varphi^{-1}(B_m)$ is a neighbourhood of 0 in K (see [3, p. 296]). Consequently there is an $M \in \mathsf{N}$ such that $e_\nu \in \varphi^{-1}(B_m)-\varphi^{-1}(B_m)$ for every $\nu \geqq M$ where

$$e_\nu = (0,\ldots,0,1,0,\ldots) \quad (1 \text{ on the } \nu\text{-th place})\,.$$

Then for every $\nu \geqq M$ there exists $x \in \varphi^{-1}(B_m)$ with $x_\nu = 0$ such that $x+e_\nu \in \varphi^{-1}(B_m)$. Thus $s_\nu \in B_m - B_m$ for every $\nu \geqq M$. Since this contradicts the properties of the sequence s_n, Theorem 1 is proved.

With suitable assumptions and modifications the following results will also be valid for semigroups, however, to avoid complications we now consider groups.

Let $(M,\mathscr{U})$ be a space M with a uniform structure $\mathscr{U}$ (see [2, p. 201]). We call the space *σ-bounded* if

1) for every $U \in \mathscr{U}$ there is a sequence $x_n \in M$ with

$$M = \bigcup_{n=1}^{\infty} U[x_n];$$

or, equivalently, if

2) any subset N of M such that, for every $U \in \mathscr{U}$ and all $x,y \in N$, $x \neq y$ implies $U[x] \cap U[y] = \emptyset$ is countable.

THEOREM 2. *Let G be an abelian topological group which is metrizable with a complete metric d. Let H be an abelian topological group which is σ-bounded in the uniform structure defined by the group operations. Then every universally measurable homomorphism θ from G to H is continuous.*

PROOF. Let U be a 0-neighbourhood in H. Choose an open 0-neighbourhood W with $W-W \subseteqq U$. Choose a sequence $h_n \in H$ with $H = \bigcup_{n=1}^{\infty}(h_n+W)$. Then the sets $\theta^{-1}(h_n+W)$ form a countable covering of G by universally measurable sets. Hence there is an n such that $\theta^{-1}(h_n+W)-\theta^{-1}(h_n+W)$ is a 0-neighbourhood. But

$$\theta^{-1}(h_n+W)-\theta^{-1}(h_n+W) \subseteqq \theta^{-1}(W-W) \subseteqq \theta^{-1}(U)\,.$$

Hence $\theta^{-1}(U)$ is a 0-neighbourhood in G for every 0-neighbourhood U in H. This completes the proof.

Let $(M,\mathscr{U})$ be a uniform space as before. Let $\varphi\colon P \to M$ be a Borel measurable surjection from P to M, where P is a Polish space. Suppose (Lusin's hypothesis) that $2^{\aleph} \gg 2^{\aleph_0}$ for $\aleph \gg \aleph_0$, where $\aleph_0$ is the cardinal number of N and $\aleph$ is a transfinite cardinal number or, alternatively, suppose φ is injective and M is separated. Then $(M,\mathscr{U})$ is σ-bounded.

For suppose there is an $U \in \mathscr{U}$ and an uncountable $N \subseteqq M$ such that for all $x, y \in N$,

$$x \neq y \Rightarrow \mathring{U}[x] \cap \mathring{U}[y] = \emptyset .$$

Then we obtain a number of Borel sets in P greater than $2^{\aleph_0}$. But it is well known that the cardinality of the Borel field of a Polish space is $2^{\aleph_0}$. If instead of Lusin's hypothesis we suppose φ injective and M separated, then N is closed in M. Hence $\varphi^{-1}(N)$ is an uncountable Borel set in P and thus has cardinality $2^{\aleph_0}$ (see [6, p. 12]). But then as above we can conclude that the Borel field has cardinality at least $2^{2^{\aleph_0}} \gg 2^{\aleph_0}$.

Now we are able to prove

THEOREM 3. *Let G be an abelian metrizable group which can be equipped with a complete metric. Let θ be a Borel measurable homomorphism from G onto an arbitrary topological group H. Then θ is continuous.*

PROOF. Clearly it is sufficient to prove the statement for separable G. The preceding discussion shows that H is σ-bounded and Theorem 2 can be applied. If θ is injective and H is separated, the proof is independent of Lusin's hypothesis.

THEOREM 4. *Let the abelian topological group H be a surjective Borel measurable image of a Polish space P. Then H is σ-bounded. (If H is a bijective Borel measurable image of a Polish space P and H is separated, this holds independently of Lusin's hypothesis.) Let θ be a surjective Borel measurable homomorphism from H onto a Polish group G. Then θ is open.*

PROOF. Let U be a neighbourhood in H. We choose an open neighborhood W in H with $W - W \subseteqq U$. There is a sequence $h_n \in H$ such that $H = \bigcup_{n=1}^{\infty} (h_n + W)$. For each $n, \theta(h_n + W)$ is an analytic subset of G, in particular universally measurable. An argument similar to the proof of Theorem 2 completes the proof.

The usual measurable graph theorems are easily derived from Theorem 4.

The following result makes it possible to prove all the preceding results for the special class of non-abelian groups H satisfying:

(R) There is a 0-neighbourhood base $\mathscr{B}$ such that $hBh^{-1} = B$ for every $B \in \mathscr{B}$ and every $h \in H$.

THEOREM 5. *Let G be a topological group with the topology arising* *from a complete metric d. Let* $G = \bigcup_{i=1}^{\infty} A_i$ *be a countable covering of* G *by universally measurable sets. Then for every neighbourhood* U *there are finitely many elements* $g_1, \ldots, g_p \in U$ *and an* $i_0 \in \mathsf{N}$ *such that*

$$\bigcup_{\nu=1}^{p} g_\nu A_{i_0} A_{i_0}^{-1} g_\nu^{-1}$$

is a neighbourhood of 1.

PROOF. Suppose the statement is false for a neighbourhood U. By induction we choose a sequence $g_n \in G$ such that for every $x \in K = \{0,1\}^{\mathsf{N}}$ the product

$$\varphi(x) = \prod_{n=1}^{\infty} g_n^{x_n} = \lim_{n\to\infty} g_1^{x_1} \cdots g_n^{x_n}$$

exists and is in U, and furthermore such that

$$\forall\, i \in \mathsf{N}\ \forall\, N \in \mathsf{N}\ \exists\, n > N:\ g_n \notin \varphi(x)^{-1} A_i A_i^{-1} \varphi(x)$$

for every $x \in K$ with $x_\nu = 0$ for $\nu \geqq n$. The mapping $\varphi\colon K \to G$ is measurable, and a contradiction is obtained in a similar way as in the proof of Theorem 1.

Now it is easy to prove that Theorems 2,3,4 hold in the non-abelian case if H satisfies (R).

We state some immediate corollaries of the preceding results.

COROLLARY 1. *Let* G *be a topological group with a complete metric and* H *a universally measurable normal subgroup whose index is at most denumerable. Then* H *is open and closed.*

COROLLARY 2. *Let* F *be a complete metrizable topological vector space and* $A \subseteqq F$ *a convex, absorbing, universally measurable set. Then* A *is a* 0*-neighbourhood.*

ADDED IN PROOF. The results of Theorems 3 and 4 seem to be more or less known, but our method is new.

A set A has the Baire property if there exists an open set U such that $(A \setminus U) \cup (U \setminus A)$ is of the first category. The family of sets having the Baire property is closed under the Souslin operation. Hence every analytic set has the Baire property (see K. Kuratowski, *Topologie* I, 2. édition, Warszawa, 1948, Chap. 1, pp. 62–63). Let G be an arbitrary topological group. If $A \subseteq G$ is not of the first category and has the Baire property, then $A \cdot A^{-1}$ is a neighborhood (see B. J. Pettis, *On continuity and* *openness of homomorphisms in topological groups*, Ann. of Math. (2) 52 (1950), 293–308). Combining these remarks with our arguments we obtain Theorems 3 and 4 even without the assumptions that the groups are abelian.

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