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ROYDEN'S ALGEBRA ON RIEMANNIAN SPACES

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The purpose of the present paper is to rigorously develop a generalization to Riemannian spaces of Royden's algebra, Royden's compactification, and the harmonic boundary.

1.

A real-valued function $f(x^1,\ldots,x^n)$ on a rectangle $\prod_{i=1}^n(a^i,b^i)$ is called a *Tonelli* function, if

(T.1) f is continuous on $\prod_{i=1}^n(a^i,b^i)$,

(T.2) for each i, $f(\bar{x}^1,\ldots,\bar{x}^{i-1},x^i,\bar{x}^{i+1},\ldots,\bar{x}^n)$ is absolutely continuous with respect to x^i on (a^i,b^i) for almost all $(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \prod_{j=1,j\neq i}^n(a^j,b^j)$,

(T.3) $\partial f/\partial x^i$ is square integrable on each compact subset of the rectangle $\prod_{i=1}^n(a^i,b^i)$.

Note that (T.2) assures that $\partial f/\partial x^i$ exists and is finite almost everywhere in $\prod_{i=1}^n(a^i,b^i)$ with respect to the Lebesgue measure.

Let f be a real-valued function on a Riemannian space R, and Ω a parametric rectangle with a coordinate system $(x^1,\ldots,x^n)$ on $\prod_{i=1}^n(a^i,b^i)$.

LEMMA 1. *If f is a Tonelli function in terms of the coordinate system $(x^1,\ldots,x^n)$ on $\prod_{i=1}^n(a^i,b^i)$, then $D_K(f)$ is finite for each compact subset K of Ω.*

PROOF. Because of the positive definiteness of (g^{ij}) and the homogeneity of the following expression, there is, for each compact subset K of $\prod_{i=1}^n(a^i,b^i)$, a positive constant k, depending only on K, such that

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$$(1) \qquad k^{-1}\sum_{i=1}^{n}(y^i)^2 \leqq \sum_{i,j=1}^{n} g^{ij}y^iy^j \leqq k\sum_{i=1}^{n}(y^i)^2$$

for all $(y^1,\ldots,y^n) \in E^n$. Hence the square integrability of $\partial f/\partial x^i$ implies that

$$D_K(f) = \int_K g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\, dx^1 \wedge \ldots \wedge dx^n$$

is finite, for

$$g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \leqq g^{\frac{1}{2}} k \sum_{i=1}^{n} (\partial f/\partial x^i)^2$$

holds for almost all $(x^1,\ldots,x^n) \in K$ and $g^{\frac{1}{2}}$ is bounded on K.

A real-valued function f on a Riemannian space R is called a *Tonelli* function, if it is a Tonelli function in every parametric rectangle.

PROPOSITION 1. *If f and g are Tonelli functions on R, then so are $f\wedge g$ and $f\vee g$ defined by*

$$(f\wedge g)(p) = \min(f(p),g(p)) \quad \textit{and} \quad (f\vee g)(p) = \max(f(p),g(p))\ .$$

PROOF. In view of the identities

$$f\wedge g = \tfrac{1}{2}(f+g) - \tfrac{1}{2}|f-g|, \qquad f\vee g = \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g|\ ,$$

and the fact that the space of Tonelli functions on R forms a vector space over the reals with the point-wise addition and usual scalar multiplication, it suffices to prove that if h is a Tonelli function, then so is $|h|$.

Let Ω be a parametric rectangle with a coordinate system $(x^1,\ldots,x^n)$ on $\prod_{i=1}^n(a^i,b^i)$. For a pair of points

$$(\bar{x}^1,\ldots,\bar{x}^{i-1},x_1{}^i,\bar{x}^{i+1},\ldots,\bar{x}^n) \quad \text{and} \quad (\bar{x}^1,\ldots,\bar{x}^{i-1},x_2{}^i,\bar{x}^{i+1},\ldots,\bar{x}^n)$$

in $\prod_{i=1}^n(a^i,b^i)$, we have

$$\begin{aligned}&\big||h|(\bar{x}^1,\ldots,\bar{x}^{i-1},x_1{}^i,\bar{x}^{i+1},\ldots,\bar{x}^n) - |h|(\bar{x}^1,\ldots,\bar{x}^{i-1},x_2{}^i,\bar{x}^{i+1},\ldots,\bar{x}^n)\big| \\ &\qquad \leqq |h(\bar{x}^1,\ldots,\bar{x}^{i-1},x_1{}^i,\bar{x}^{i+1},\ldots,\bar{x}^n) - h(\bar{x}^1,\ldots,\bar{x}^{i-1},x_2{}^i,\bar{x}^{i+1},\ldots,\bar{x}^n)|\ .\end{aligned}$$

Therefore $|h|(\bar{x}^1,\ldots,\bar{x}^{i-1},x^i,\bar{x}^{i+1},\ldots,\bar{x}^n)$ is absolutely continuous with respect to x^i for almost all $(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \prod_{j=1,j\neq i}^n(a^i,b^i)$ and hence $\partial|h|/\partial x^i$ exists and is finite almost everywhere in $\prod_{i=1}^n(a^i,b^i)$. From the inequality we also obtain

$$\left|\partial|h|/\partial x^i\right| \leqq |\partial h/\partial x^i|$$

whenever $\partial|h|/\partial x^i$ and $\partial h/\partial x^i$ exist and are finite.

However, we need a stronger result to conclude that $D_K(|h|) \leqq D_K(h)$ for a compact subset K of Ω. Let E be a subset of $\prod_{i=1}^n (a^i, b^i)$ of measure zero such that $\partial|h|/\partial x^1, \ldots, \partial|h|/\partial x^n$, $\partial h/\partial x^1, \ldots, \partial h/\partial x^n$ all exist and are finite at every point of $L \equiv \prod_{i=1}^n (a^i, b^i) - E$.

We claim that

$$\begin{aligned}(\partial|h|/\partial x^1, \ldots, \partial|h|/\partial x^n) &= (\partial h/\partial x^1, \ldots, \partial h/\partial x^n) && \text{in } L^+ ,\\ &= 0 && \text{in } L^0 ,\\ &= -(\partial h/\partial x^1, \ldots, \partial h/\partial x^n) && \text{in } L^- ,\end{aligned}$$

where L^+, L^0, and L^- stand for the sets of points p in L at which $h(p)$ is positive, zero, and negative, respectively.

Observe that L^+, L^0, and L^- are all measurable.

The first and the last cases are clear, because, for each point $(\bar{x}^1, \ldots, \bar{x}^n)$ of L^+ or L^-, there is a neighborhood of $(\bar{x}^1, \ldots, \bar{x}^n)$ where $|h|$ is h or $-h$, respectively. Let $(\bar{x}^1, \ldots, \bar{x}^n) \in L^0$. Then $h(\bar{x}^1, \ldots, \bar{x}^n) = 0$. By definition of L, the limits

$$\lim_{\Delta x^i \to 0} (\Delta x^i)^{-1} (|h|(\bar{x}^1, \ldots, \bar{x}^i + \Delta x^i, \ldots, \bar{x}^n) - 0) ,$$
$$\lim_{\Delta x^i \to 0} (\Delta x^i)^{-1} (h(\bar{x}^1, \ldots, \bar{x}^i + \Delta x^i, \ldots, \bar{x}^n) - 0)$$

must exist and be finite, and hence are both zero, for Δx^i can be positive and negative. It follows that

$$g^{ij} \frac{\partial|h|}{\partial x^i} \frac{\partial|h|}{\partial x^j} \leqq g^{ij} \frac{\partial h}{\partial x^i} \frac{\partial h}{\partial x^j}$$

holds a.e. in $\prod_{i=1}^n (a^i, b^i)$ and

$$D_K(|h|) \leqq D_K(h) .$$

2.

DEFINITION. Royden's algebra $M(R)$ of a Riemannian space R is the class of real-valued functions f on R such that

(M.1) f is bounded on R,

(M.2) f is a Tonelli function on R,

(M.3) the Dirichlet integral $D_R(f)$ is finite.

PROPOSITION 2. *$M(R)$ is a commutative algebra with identity over the reals.*

PROOF. We only have to verify that if f,g are in $M(R)$, then so is fg. First of all, $|fg| \leqq MN$, where M and N are bounds for $|f|$ and $|g|$ on R, respectively.

Let Ω be a parametric rectangle with a coordinate system $(x^1,\ldots,x^n)$ on $\prod_{i=1}^n (a^i,b^i)$. The inequality

$$|f(\bar{x}^1,\ldots,x_1{}^i,\ldots,\bar{x}^n)g(\bar{x}^1,\ldots,x_1{}^i,\ldots,\bar{x}^n) - f(\bar{x}^1,\ldots,x_2{}^i,\ldots,\bar{x}^n)g(\bar{x}^1,\ldots,x_2{}^i,\ldots,\bar{x}^n)|$$

$$\leqq \max(M,N)(|g(\bar{x}^1,\ldots,x_1{}^i,\ldots,\bar{x}^n) - g(\bar{x}^1,\ldots,x_2{}^i,\ldots,\bar{x}^n)| + |f(\bar{x}^1,\ldots,x_1{}^i,\ldots,\bar{x}^n) - f(\bar{x}^1,\ldots,x_2{}^i,\ldots,\bar{x}^n)|)$$

proves the absolute continuity of fg with respect to x^i for almost all $(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \prod_{j=1,j\neq i}^n (a^j,b^j)$. Hence

$$\frac{\partial(fg)}{\partial x^i} = \frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i}$$

a.e. in $\prod_{i=1}^n (a^i,b^i)$.

For a compact subset K of Ω, we have

$$D_K(gf) = \int_K g^{\frac{1}{2}} g^{ij} \left(\frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i}\right)\left(\frac{\partial f}{\partial x^j} g + f \frac{\partial g}{\partial x^j}\right) dx^1 \wedge \ldots \wedge dx^n$$

$$= \int_K g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} g^2 \, dx^1 \wedge \ldots \wedge dx^n +$$

$$+ \int_K g^{\frac{1}{2}} g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} f^2 \, dx^1 \wedge \ldots \wedge dx^n +$$

$$+ 2\int_K g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} fg \, dx^1 \wedge \ldots \wedge dx^n$$

$$\leqq N^2 D_K(f) + M^2 D_K(g) + 2MN(D_K(f) D_K(g))^{\frac{1}{2}} .$$

Here

$$\left|\int_K g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} fg \, dx^1 \wedge \ldots \wedge dx^n\right|$$

$$\leqq MN \int_K g^{\frac{1}{2}} \left| g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \right| dx^1 \wedge \ldots \wedge dx^n$$

$$\leqq MN \int_K g^{\frac{1}{2}} \left(g^{ij} \frac{df}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{\frac{1}{2}} \left(g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} \right)^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n$$

$$\leqq MN (D_K(f) D_K(g))^{\frac{1}{2}} ,$$

the last relation being an application of Schwarz's inequality. By a partition of unity, we obtain

$$D_R(fg) \leqq (M(D_R(g))^{\frac{1}{2}} + N(D_R(f))^{\frac{1}{2}})^2 .$$

PROPOSITION 3. *$M(R)$ is a lattice under the usual meet and join of two functions.*

This is clear by Proposition 1 and a partition of unity.

As for the division in $M(R)$, we state:

PROPOSITION 4. *Suppose that $f \in M(R)$. The function $1/f$ belongs to $M(R)$ if and only if $\inf_R |f| > 0$.*

The verification is straightforward.

We shall employ several modes of convergence of a sequence $\{f_m\}_{m=1}^{\infty}$ of functions on R:

(a) *C-convergence.* $f = C\text{-}\lim_{m \to \infty} f_m$ on R, if

$$\lim_{m \to \infty} \sup_K |f_m - f| = 0$$

for each compact subset K of R.

(b) *B-convergence.* $f = B\text{-}\lim_{m \to \infty} f_m$ on R, if $\{f_m\}_{m=1}^{\infty}$ is uniformly bounded on R and $f = C\text{-}\lim_{m \to \infty} f_m$ on R.

(c) *U-convergence.* $f = U\text{-}\lim_{m \to \infty} f_m$ on R, if

$$\lim_{m \to \infty} \sup_R |f_m - f| = 0 .$$

(d) *D-convergence.* $f = D\text{-}\lim_{m \to \infty} f_m$ on R, if

$$\lim_{m \to \infty} D_R(f_m - f) = 0 .$$

(e) *QD-convergence.* ($Q = C$, B, or U.) $f = QD\text{-}\lim_{m \to \infty} f_m$, if

$$f = Q\text{-}\lim_{m \to \infty} f_m \quad \text{and} \quad f = D\text{-}\lim_{m \to \infty} f_m \quad \text{on } R.$$

We reformulate UD-convergence by introducing in $M(R)$ the norm

$$\|f\| = \sup_R |f| + (D_R(f))^{\frac{1}{2}} .$$

It is easily verified that

(a) $\|f\| \geqq 0$, and $\|f\| = 0$ if and only if $f \equiv 0$ on R,
(b) $\|\alpha f\| = |\alpha| \, \|f\|$ for each real α,
(c) $\|f+g\| \leqq \|f\| + \|g\|$,
(d) $\|fg\| \leqq \|f\| \, \|g\|$,
(e) $\|1\| = 1$.

Thus $M(R)$ is a normed algebra.

THEOREM 1. *$M(R)$ with the norm is a Banach algebra.*

PROOF. Only completeness needs attention. Let $\{f_m\}_{m=1}^{\infty} \subset M(R)$ be a Cauchy sequence in the above norm, that is,

$$\|f_m - f_k\| = \sup_R |f_m - f_k| + (D_R(f_m - f_k))^{\frac{1}{2}} \to 0$$

as $m, k \to \infty$. The fact that $\sup_R |f_m - f_k| \to 0$ implies that there is a bounded continuous function f on R such that

$$\sup_R |f_m - f| \to 0 \quad \text{as } m \to \infty .$$

Let α be a 1-form with local representation

$$\alpha = a_1(x^1, \ldots, x^n) \, dx^1 + \ldots + a_n(x^1, \ldots, x^n) \, dx^n$$

in terms of a coordinate system $(x^1, \ldots, x^n)$ where $a_1(x^1, \ldots, x^n), \ldots, a_n(x^1, \ldots, x^n)$ are measurable and $\int_R \alpha \wedge * \alpha < \infty$. The totality of such 1-forms is a Hilbert space with inner product $(\alpha, \beta) = \int_R \alpha \wedge * \beta$ (cf. Springer [9]). Therefore

$$D_R(f_m - f_k) = \int_R d(f_m - f_k) \wedge * d(f_m - f_k) \to 0$$

as $m, k \to \infty$ implies that df_m converges to a 1-form α on R, that is,

$$\lim_{m \to \infty} \int_R (df_m - \alpha) \wedge * (df_m - \alpha) = 0 ,$$

and

$$\int_R \alpha \wedge * \alpha < \infty .$$

We will show that f is a Tonelli function and that $df=\alpha$. Let Ω be a parametric rectangle with a coordinate system $(x^1,\ldots,x^n)$ on $\prod_{i=1}^n (a^i,b^i)\equiv C$. Take a rectangle $C'=\prod_{i=1}^n (c^i,d^i)$ with $\bar{C}'\subset C$. Suppose that α has the form

$$\alpha = a_1(x^1,\ldots,x^n)dx^1+\ldots+a_n(x^1,\ldots,x^n)dx^n .$$

Let $\theta(x^1,\ldots,x^n)$ be a function on C such that it is continuous and continuously differentiable, with a compact support in C, and $\equiv 1$ on C'. We introduce:

$$g_m(x^1,\ldots,x^n) = \theta(x^1,\ldots,x^n)\, f_m(x^1,\ldots,x^n) ,$$

$$g(x^1,\ldots,x^n) = \theta(x^1,\ldots,x^n)\, f(x^1,\ldots,x^n) ,$$

$$b_i(x^1,\ldots,x^n) = \theta(x^1,\ldots,x^n)\, a_i(x^1,\ldots,x^n) + \frac{\partial\theta}{\partial x^i} f(x^1,\ldots,x^n) ,$$

$$g^i(x^1,\ldots,x^i,\ldots,x^n) = \int_{a^i}^{x^i} b_i(x^1,\ldots,t,\ldots,x^n)\, dt .$$

Observe that $\partial g_m/\partial x^i=(\partial\theta/\partial x^i)f_m+\theta(\partial f_m/\partial x^i)$ converges, in L^2-norm of $L^2(C)$, to b_i. In fact, $(\partial\theta/\partial x^i)f_m$ converges uniformly to $(\partial\theta/\partial x^i)f$, for f_m converges uniformly on C to f and $\partial\theta/\partial x^i$, which is continuous and has compact support in C, is thus bounded. To see that $\theta(\partial f_m/\partial x^i)$ also converges, in L^2-norm, to θa_i, let K' be the support of θ in C. We have

$$\begin{aligned}
&\int_C \left(\theta\,\frac{\partial f_m}{\partial x^i} - \theta a_i\right)^2 dx^1\wedge\ldots\wedge dx^n \\
&\leqq \int_{K'} \sum_{j=1}^n \theta^2 \left(\frac{\partial f_m}{\partial x^j} - a_j\right)^2 dx^1\wedge\ldots\wedge dx^n \\
&\leqq \int_{K'} \theta^2 k\, g^{ij}\left(\frac{\partial f_m}{\partial x^i} - a_i\right)\left(\frac{\partial f_m}{\partial x^j} - a_j\right) dx^1\wedge\ldots\wedge dx^n \\
&= \int_{K'} \theta^2 k g^{-\frac12}\left[g^{\frac12} g^{ij}\left(\frac{\partial f_m}{\partial x^i} - a_i\right)\left(\frac{\partial f_m}{\partial x^j} - a_j\right)\right] dx^1\wedge\ldots\wedge dx^n \\
&\leqq c \int_{K'} (df_m-\alpha)\wedge *(df_m-\alpha) \;\to\; 0 ,
\end{aligned}$$

as $m\to\infty$. Here the second inequality follows from formula (1) and c is a bound for $(k\theta^2)/g^{\frac12}$ on K', which exists on account of the positiveness of g and the compactness of K'.

For each m,

$$g_m(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n) = \int_{a^i}^{x^i} \frac{\partial}{\partial t} g_m(\bar{x}^1,\ldots,\bar{x}^{i-1},t,\bar{x}^{i+1},\ldots,\bar{x}^n)\, dt$$

for almost all

$$(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \textstyle\prod_{j=1,j\neq i}^n (a^j,b^j)\,,$$

because $g_m(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)$ is absolutely continuous with respect to x^i and $\lim_{t\to a^i} g_m(\bar{x}^1,\ldots,t,\ldots,\bar{x}^n)=0$. By Schwarz's inequality, we obtain

$$\begin{aligned}
&|g_m(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)-g^i(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)|^2 \\
&= \Bigg|\int_{a^i}^{x^i} [(\partial/\partial t)g_m(\bar{x}^1,\ldots,\bar{x}^{i-1},t,\bar{x}^{i+1},\ldots,\bar{x}^n) - \\
&\qquad - b_i(\bar{x}^1,\ldots,\bar{x}^{i-1},t,\bar{x}^{i+1},\ldots,\bar{x}^n)]\, dt\Bigg|^2 \\
&\leqq (b^i-a^i)\int_{a^i}^{x^i} [(\partial/\partial t)g_m(\bar{x}^1,\ldots,\bar{x}^{i-1},t,\bar{x}^{i+1},\ldots,\bar{x}^n) - \\
&\qquad - b_i(\bar{x}^1,\ldots,\bar{x}^{i-1},t,\bar{x}^{i+1},\ldots,\bar{x}^n)]^2\, dt
\end{aligned}$$

for almost all $(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \prod_{j=1,j\neq i}^n (a^j,b^j)$, and therefore

$$\begin{aligned}
&\int_C |g_m(x^1,\ldots,x^n)-g^i(x^1,\ldots,x^n)|^2\, dx^1\wedge\ldots\wedge dx^n \\
&\leqq (b^i-a^i)\int_C \Bigg\{\int_{a^i}^{b^i} [(\partial/\partial t)g_m(x^1,\ldots,x^{i-1},t,x^{i+1},\ldots,x^n) - \\
&\qquad - b_i(x^1,\ldots,x^{i-1},t,x^{i+1},\ldots,x^n)]^2\, dt\Bigg\}\, dx^1\wedge\ldots\wedge dx^n \\
&= (b^i-a^i)^2\int_C [(\partial/\partial t^i)g_m(x^1,\ldots,x^n) - \\
&\qquad - b_i(x^1,\ldots,x^n)]^2\, dx^1\wedge\ldots\wedge dx^n\,.
\end{aligned}$$

This implies that g_m converges, in L^2-norm of $L^2(C)$, to g^i in $L^2(C)$. Hence there is a subsequence g_{m_k} of g_m which converges almost uniformly on C. On the other hand, g_{m_k} converges uniformly to g on C. Therefore, for almost all $(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \prod_{j=1,j\neq i}^n (a^j,b^j)$,

$$\begin{aligned}\theta(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)f(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n) &= g(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)\\ &= g^i(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)\\ &= \int_{a^i}^{x^i} b^i(\bar{x}^1,\ldots,\bar{x}^{i-1},t,\bar{x}^{i+1},\ldots,\bar{x}^n)\,dt,\end{aligned}$$

because both functions are continuous in x^i.

Since $\theta(x^1,\ldots,x^n)$ is arbitrary, $f(\bar{x}^1,\ldots,x^i,\ldots,\bar{x}^n)$ is an absolutely continuous function of x^i for almost all

$$(\bar{x}^1,\ldots,\bar{x}^{i-1},\bar{x}^{i+1},\ldots,\bar{x}^n) \in \textstyle\prod_{j=1,j\neq i}^{n}(a^j,b^j)\ ,$$

and $\partial f/\partial x^i = a^i$ a.e. in $\prod_{i=1}^n (a^i,b^i)$.

With a slight modification, the above proof also gives

THEOREM 2. *$M(R)$ is BD-complete.*

3.

The following generalization to Riemannian spaces of an approximation theorem by Nakai [8] plays an important role:

THEOREM 3. *Let f be a Tonelli function on R with $D_R(f)<\infty$. For a positive number ε, there exists a C^∞-function f_ε such that $\|f-f_\varepsilon\|<\varepsilon$. Moreover, if f has compact support in an open set G, then f_ε can be chosen to have its support in G.*

PROOF. Let us first assume that Ω is a parametric ball with a coordinate system $(x^1,\ldots,x^n)$ on

$$B_2 = \{x\in E^n \mid |x| = ((x^1)^2+\ldots+(x^n)^2)^{\frac{1}{2}} < 2\}$$

and that f has its support in Ω; more precisely, the support is in

$$B_1 = \{x\in E^n \mid |x|<1\}$$

in terms of the coordinate system $(x^1,\ldots,x^n)$. We define a C^∞-function ϱ_m on R by setting

$$\frac{n\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n+1)}\int_0^{n^{-2}} e^{-t^{-1}}\,dt\,\varrho_m(p)$$

equal to $\exp(-(m^{-2}-(x^1)^2-\ldots-(x^n)^2)^{-1})$ if $p \in \Omega$ and $p=(x^1,\ldots,x^n) \in B_{1/m}$, and equal to 0 otherwise; here Γ is the gamma function. Clearly

$$\int_R \varrho_m \, dx^1 \wedge \ldots \wedge dx^n = 1$$

and ϱ_m is nonnegative on R.

Let

$$f_m(q) = \int_R \varrho_m(q-p) f(p) \, dx^1 \wedge \ldots \wedge dx^n$$

$$= \int_R \varrho_m(p) f(q-p) \, dx^1 \wedge \ldots \wedge dx^n .$$

From the first equality, we deduce that f_m is C^∞, f_m vanishes outside of Ω, and

$$f = U\text{-}\lim_{m\to\infty} f_m$$

on R. From the second equality, we obtain

$$\frac{\partial}{\partial y^i} f_m(q) = \int_R \varrho_m(p) \frac{\partial}{\partial y^i} f(q-p) \, dx^1 \wedge \ldots \wedge dx^n$$

a.e. in B_2. By the definition of $\varrho_m(p)$, the supports of all f_m are contained in a compact subset K' of B_2. Schwarz's inequality and formula (1) yield

$$g^{ij}\left(\frac{\partial}{\partial y^i} f_m(q) - \frac{\partial}{\partial y^i} f(q)\right)\left(\frac{\partial}{\partial y^j} f_m(q) - \frac{\partial}{\partial y^j} f(q)\right)$$

$$\leqq k \sum_{i=1}^n \left(\int_R \varrho_m(p) \frac{\partial}{\partial y^i} f(q-p) dx^1 \wedge \ldots \wedge dx^n - \right.$$

$$\left. - \int_R \varrho_m(p) \frac{\partial}{\partial y^i} f(q) dx^1 \wedge \ldots \wedge dx^n\right)^2$$

$$\leqq k \int_R \varrho_m(p) dx^1 \wedge \ldots \wedge dx^n \cdot$$

$$\cdot \int_R \varrho_m(p) \sum_{i=1}^n \left(\frac{\partial}{\partial y^i} f(q-p) - \frac{\partial}{\partial y^i} f(q)\right)^2 dx^1 \wedge \ldots \wedge dx^n$$

$$\leqq k \int_R \varrho_m(p) \sum_{i=1}^n \left(\frac{\partial}{\partial y^i} f(q-p) - \frac{\partial}{\partial y^i} f(q)\right)^2 dx^1 \wedge \ldots \wedge dx^n .$$

From this and Fubini's theorem, we obtain

$$D_R(f-f_m)$$

$$= \int_R g^{\frac{1}{2}} g^{ij} \left(\frac{\partial}{\partial y^i} f(q) - \frac{\partial}{\partial y^i} f_m(q)\right) \left(\frac{\partial}{\partial y^j} f(q) - \frac{\partial}{\partial y^j} f_m(q)\right) dx^1 \wedge \ldots \wedge dx^n$$

$$\leqq kN \int_{K'} \left[\int_{B_{1/m}} \varrho_m(p) \sum_{i=1}^n \left(\frac{\partial}{\partial y^i} f(q-p) - \frac{\partial}{\partial y^i} f(q)\right)^2 \cdot \right.$$

$$\left. \cdot\, dx^1 \wedge \ldots \wedge dx^n \right] dy^1 \wedge \ldots \wedge dy^n$$

$$= kN \int_{B_{1/m}} \varrho_m(p) \left[\int_{K'} \sum_{i=1}^n \left(\frac{\partial}{\partial y^i} f(q-p) - \frac{\partial}{\partial y^i} f(q)\right)^2 \cdot \right.$$

$$\left. \cdot\, dy^1 \wedge \ldots \wedge dy^n \right] dx^1 \wedge \ldots \wedge dx^n ,$$

where N is a bound for $g^{\frac{1}{2}}$ in K'.

Since $\partial f(q)/\partial y^i$ is square integrable over K', by Lebesgue's theorem

$$\lim_{p\to 0} \int_{K'} \sum_{i=1}^n \left(\frac{\partial}{\partial y^i} f(q-p) - \frac{\partial}{\partial y^i} f(q)\right)^2 dy^1 \wedge \ldots \wedge dy^n = 0 .$$

A fortiori $D_R(f-f_m) \to 0$ as $m \to \infty$, and hence we conclude that $\lim_{m\to\infty} \|f-f_m\| = 0$.

Next, let $\{\varphi_m\}_{m=1}^{\infty}$ be a sequence of C^{∞}-function on R such that the support of φ_m is contained in a parametric ball Ω_m, $\{\Omega_m\}_{m=1}^{\infty}$ is a locally finite covering of R, and $\sum_{m=1}^{\infty} \varphi_m \equiv 1$ on R. Clearly $f\varphi_m$ is a function of the type just considered. Hence we can find a C^{∞}-function f_m such that the support of f_m is compact in Ω_m and $\|f\varphi_m - f_m\| < \varepsilon/2^{m+1}$. If f has compact support in an open set G, we can get the desired f_m with its support in G.

Let $f_\varepsilon = \sum_{m=1}^{\infty} f_m$. By local finiteness of Ω_m, $f_\varepsilon \in C^{\infty}(R)$, and we also have

$$\|f-f_\varepsilon\| \leqq \sum_{m=1}^{\infty} \|f\varphi_m - f_m\| < \varepsilon .$$

As an application of the approximation theorem, we prove two useful generalizations of Green's formula.

LEMMA 2. *If f is a Tonelli function with $D_R(f) < \infty$, and $u \in H(\bar{\Omega})$, with Ω a regular region, then*

$$D_\Omega(f,u) = \int_{\partial\Omega} f * du . \tag{2}$$

PROOF. By the approximation theorem, there is a sequence $\{f_m\}$ of C^∞-functions such that $\|f-f_m\| \to 0$ as $m \to \infty$. Therefore $D_\Omega(f_m, u) = \int_{\partial\Omega} f_m * du$ for each m. On letting $m \to \infty$, we obtain (2).

LEMMA 3. *Suppose that f is a Tonelli function with $D_R(f) < \infty$ and $u \in HD(\Omega)$, with Ω a regular region. Let γ_1 be a union of some components of $\partial\Omega$ and $\gamma_2 = \partial\Omega - \gamma_1$. If $f = 0$ on γ_1, and u is harmonic on γ_2, then*

$$D_\Omega(f,u) = \int_{\gamma_2} f * du . \tag{3}$$

PROOF. First, let us assume that g is such that $g\,|\,\bar{\Omega}$ is a Morse function on the manifold triad $(\bar{\Omega}; \gamma_1, \gamma_2)$ (Milnor [2]). Then there are only finitely many non-degenerate critical points in Ω. Let $\Omega_r = \{p \in \Omega \mid r < g(p) < 1\}$ for $r > 0$, and $\beta_r = \partial\Omega_r - \gamma_2$. By formula (2),

$$\begin{aligned} D_{\Omega_r}(g,u) &= \int_{\partial\Omega_r} g * du = \int_{\beta_r} g * du + \int_{\gamma_2} g * du \\ &= r \int_{\beta_r} * du + \int_{\gamma_2} g * du = r \int_{\gamma_2} * du + \int_{\gamma_2} g * du . \end{aligned}$$

For $r \to 0$, we obtain (3).

Next, we observe that it suffices to verify (3) for nonnegative functions $-f \wedge 0$ and $f \vee 0$. In fact, in view of the case just considered, it is enough to prove (3) for a positive function f on $\Omega \cup \gamma_2$, for we can add g to f.

Let $f_c(p) = (f(p) - c) \vee 0$ for $0 < c < \min_{\gamma_2} f$. For a sufficiently small r, $f_c \,|\, \beta_r = 0$ and $f_c = f - c$ on γ_2.

By virtue of formula (2), we have

$$\begin{aligned} D_\Omega(f_c, u) = D_{\Omega_r}(f_c, u) &= \int_{\beta_r} f_c * du + \int_{\gamma_2} f_c * du \\ &= \int_{\gamma_2} f_c * du = \int_{\gamma_2} f * du - c \int_{\gamma_2} * du . \end{aligned}$$

On the other hand,

$$\begin{aligned} |D_\Omega(f_c,u)-D_\Omega(f,u)| &= |D_\Omega(f_c-f,u)| \\ &= |D_{\{p\in\Omega|f(p)\leqq c\}}(f,u)| \\ &\leqq D_\Omega(f)D_{\{p\in\Omega|f(p)\leqq c\}}(u) \end{aligned}$$

and consequently $D_\Omega(f_c,u)\to D_\Omega(f,u)$ as $c\to 0$, for $D_\Omega(u)<\infty$. We conclude that (3) holds.

Finally, if $\gamma_2=\emptyset$, then we take a parametric ball B in Ω and apply (3) to $\Omega-B$ with $-\partial B$ as γ_2, and (2) to B.

As a direct consequence of (3), we derive the Dirichlet principle:

THEOREM 4. *Let Ω be a regular region of R. If f is a Tonelli function with $D_R(f)<\infty$, and $u\in H(\Omega)$ with $u|\partial\Omega=f|\partial\Omega$, then*

$$D_\Omega(f) = D_\Omega(u)+D_\Omega(f-u)\,. \tag{4}$$

PROOF. By the approximation theorem, there is a sequence of C^∞-functions f_m with $D_R(f_m)<\infty$ such that $\|f-f_m\|<1/m$. The Dirichlet principle for the boundary C^∞-function $f_m|\partial\Omega$ and the regular region Ω gives

$$D_\Omega(f_m) = D_\Omega(u_m)+D_\Omega(f_m-u_m)\,.$$

Clearly $u=U\text{-}\lim_{m\to\infty}u_m$ on $\bar{\Omega}$. Since

$$\begin{aligned} (D_\Omega(u_m-u_k))^{\frac12} &= (D_\Omega(f_m-f_k)-D_\Omega(f_m-f_k+u_k-u_m))^{\frac12} \\ &\leqq (D_\Omega(f_m-f))^{\frac12}+(D_\Omega(f_k-f))^{\frac12}\,, \end{aligned}$$

$u=D\text{-}\lim_{m\to\infty}u_m$ on Ω and therefore $u\in HD(\Omega)$.

By Lemma 3 with $\gamma_1=\partial\Omega$, we obtain

$$\begin{aligned} D_\Omega(f) = D_\Omega(f-u+u) &= D_\Omega(f-u)+2D_\Omega(f-u,u)+D_\Omega(u) \\ &= D_\Omega(u)+D_\Omega(f-u)\,, \end{aligned}$$

for $f-u$ vanishes on $\partial\Omega$.

4.

In this number we will use the conjugate space $M(R)^*$ of $M(R)$ with the weak* topology to imbed R.

THEOREM 5. *For a Riemannian space R there exists a compactification R^*, unique up to a homeomorphism leaving R element-wise fixed, with respect to the following properties:*

(a) *R^* is a compact Hausdorff space,*
(b) *R is an open dense subset of R^*,*
(c) *every function in $M(R)$ can be continuously extended to R^*,*
(d) *$\overline{M(R)}$, the class of continuous extensions of functions in $M(R)$, separates points of R^*.*

R^* will be called *Royden's compactification* of R.

The proof is broken into several lemmas. Let R^* be the set of multiplicative continuous linear functionals x on $M(R)$ with $x(1)=1$.

Clearly R^* is a subset of the dual space $M(R)^*$ of $M(R)$.

LEMMA 4. *If $x \in R^*$, then $\|x\|=1$.*

PROOF. By definition $\|x\|=\sup_{\|f\|\leqq 1}|x(f)| \geqq 1$, for $x(1)=1$. On the other hand, if $\|x\|>1$, then there exists a function f in $M(R)$ such that $\|f\|\leqq 1$ and $|x(f)|=1+\delta$, with $\delta>0$. For each m,

$$\|f^m\| \leqq \|f\|^m \leqq 1, \quad |x(f^m)| = |x(f)|^m = (1+\delta)^m .$$

Therefore, $\|x\|=\infty$, a contradiction.

LEMMA 5. *$\bar{R}^*=R^*$, where the closure $\bar{R}^*$ of R^* is taken with respect to the weak* topology of $M(R)^*$.*

PROOF. Assume that $y \in \bar{R}^*$. Then y is multiplicative, for, given any $f,g \in M(R)$ and $\varepsilon>0$, by the definition of the weak* topology, there is an $x \in R^*$ such that

$$|x(fg)-y(fg)| < \frac{\varepsilon}{3},$$

$$|x(g)-y(g)| < \frac{\varepsilon}{3(|y(f)|+1)},$$

$$|x(f)-y(f)| < \frac{\varepsilon}{3\left[|y(g)|+\dfrac{\varepsilon}{3(|y(f)|+1)}\right]}.$$

Hence

$$|y(f)y(g)-y(fg)|$$

$$\leqq |y(g)y(f)-x(g)y(f)|+|x(g)y(f)-x(g)x(f)|+|x(fg)-y(fg)|$$

$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon .$$

By the same token, $y(1)=1$ and hence $y \in R^*$.

As is well known, R^*, a closed subset of the unit sphere of $M(R)^*$, is compact in the weak* topology. For every $p \in R$, let $x_p(f)=f(p)$. Clearly $x_p \in R^*$.

LEMMA 6. *The mapping $\tau: p \to x_p$ is an imbedding of R into R^*.*

PROOF. The mapping τ is continuous. To see this let

$$O = \{x \in R^* \mid |x(f)| < \varepsilon\},$$

a typical subbasic open set. The set

$$\tau^{-1}(0) = \{p \in R \mid |x_p(f)| = |f(p)| < \varepsilon\}$$

is open in R, for f is continuous on R.

The mapping τ is one-to-one: if $p \neq q$, then $\tau(p) \neq \tau(q)$, that is, there is a function f in $M(R)$ such that $f(p) \neq f(q)$. To see this and a claim we will make in the next paragraph, we make the observation that R is locally Euclidean and normal. Therefore, for any closed set C and a point q of R, not in C, there is a C^∞-function f such that f has compact support in any given neighborhood of q, and $f(q)=1$, $f(C)=0$. Such a function is in $M(R)$. As a consequence, τ is one-to-one.

To prove that τ is closed, let C be any closed subset of R, and

$$x_q \in \overline{\{x_p \in R^* \mid p \in C\}} \cap \{x_p \in R^* \mid p \in R\},$$

where the closure of $\{x_p \in R^* \mid p \in C\}$ is taken in R^* with the induced topology. Then, for any $\varepsilon > 0$ and $g \in M(R)$, there must be a point p of C such that $|x_q(g) - x_p(g)| < \varepsilon$, or $|g(p) - g(q)| < \varepsilon$. We claim that, by the above observation, the latter inequality is absurd when $g=f$ and $\varepsilon = \frac{1}{2}$.

LEMMA 7. *Every function f in $M(R)$ can be extended continuously to R^*.*

PROOF. Let $f \in M(R)$. We define $\bar{f}(x)=x(f)$ for $x \in R^*$. For $p \in R$, $\bar{f}(\tau(p))=\bar{f}(x_p)=x_p(f)=f(p)$. To see the continuity of $\bar{f}$, take a net x_α converging to x in the weak* topology, that is, $x_\alpha(g) \to x(g)$ for any $g \in M(R)$. In particular, $x_\alpha(f) \to x(f)$, whence $\bar{f}(x_\alpha) \to \bar{f}(x)$.

LEMMA 8. *$\overline{M(R)} = \{\bar{f} \mid \bar{f}(x)=x(f)$ for $x \in R^*$ and $f \in M(R)\}$ is dense in the class $C(R^*)$ of continuous functions on R with the sup-norm topology.*

PROOF. The fact that $M(R)$ is a separating subalgebra of $C(R^*)$ containing constants is in the definition of R^*. From this and the Stone–Weierstrass thorem we conclude that $\overline{M(R)}$ is dense in $C(R^*)$.

LEMMA 9. *The set $\tau(R)$ is dense in R^*.*

PROOF. If not, then the closure $\bar{R}$ of $\tau(R)$ is not R^*, that is, there is an $\tilde{x}$ in $R^*-\bar{R}$. By Uryshon's lemma, there is a continuous function g on R^* such that $g(\bar{R})=1$ and $g(\tilde{x})=0$. In view of Lemma 8, there is an $\bar{f}$ in $\overline{M(R)}$ such that $\bar{f}(\tilde{x})=0$ and $\bar{f}(\bar{R})>0$. Let f be a function in $M(R)$ whose extension to R^* is $\bar{f}$. Then $\inf_R|f|>0$. By Proposition 4, f has an inverse $g\in M(R)$, that is, $fg=1$ on R. A fortiori $\overline{fg}=1$ on R^*. On the other hand, $\overline{fg}(\tilde{x})=\bar{f}(\tilde{x})\,\bar{g}(\tilde{x})=0$, a contradiction.

Putting the six lemmas together, we conclude that the desired compactification R^* of R exists.

LEMMA 10. *If X is any compactification of R with the properties* (a)-(d), *then the mapping from X into R^*, given for each $p\in X$ by $\sigma(p)(f)=f(p)$ for all $f\in\overline{M(R)}$, leaves points of R element-wise fixed and is a homeomorphism onto R^*.*

PROOF. It is clear that σ leaves points of R fixed. By an argument similar to the one in Lemma 6, σ is an imbedding of X. But the image of X contains R and is compact. Hence, by the denseness of R, σ is onto, and the proof is complete.

THEOREM 6. *For any two nonempty disjoint compact subsets K_1 and K_2 of R^* there is a real-valued function $\bar{f}$ in $\overline{M(R)}$ such that $\bar{f}(R^*)\subset[0,1]$ and $\bar{f}(K_1)=0$, $\bar{f}(K_2)=1$.*

PROOF. By Urysohn's lemma, there is a continuous function g in $C(R^*)$ such that $g(R^*)\subset[-2,3]$, and $g(K_1)=-2$, $g(K_2)=3$. By the denseness of $\overline{M(R)}$ in $C(R^*)$, there is a function h in $M(R)$ whose extension $\bar{h}$ is such that $|\bar{h}-g|<1$ on R^*.

Let $\bar{f}=(\bar{h}\wedge 1)\vee 0$. Then $\bar{f}$ has the desired property, for the operations of taking the meet, the join, and the extension to R^* are interchangeable.

We shall make the following convention: When there is no likelihood of confusion, f shall stand for both $\bar{f}$ and f.

5.

The set of functions in $M(R)$ with compact supports will be denoted by $M_0(R)$, and the set of functions on R which are BD-limits of $M_0(R)$, by $M_\Delta(R)$. By Theorem 2, $M_\Delta(R)\subset M(R)$.

PROPOSITION 5. *$M_0(R)$ is an ideal of $M_\Delta(R)$ and $M(R)$. Furthermore, $M_\Delta(R)$ is an ideal of $M(R)$.*

PROOF. The first statement is clear. For the second statement, let $f \in M_\Delta(R)$ and $g \in M(R)$. Then there is a sequence of functions f_m in $M_0(R)$ such that $f = \text{BD-lim}_{m\to\infty} f_m$ on R. Clearly the sequence of functions gf_m is uniformly bounded on R, belongs to $M_0(R)$, and $gf_m \to gf$ on each compact subset of R:

$$gf = \text{B-lim}_{m\to\infty} gf_m \,.$$

Furthermore,

$$\begin{aligned}
(5) \qquad & \int_R g^{\frac{1}{2}} g^{ij} \left(\frac{\partial g}{\partial x^i} f + g \frac{\partial f}{\partial x^i} - \frac{\partial g}{\partial x^i} f_m - g \frac{\partial f_m}{\partial x^i}\right) \cdot \\
& \quad \cdot \left(\frac{\partial g}{\partial x^j} f + g \frac{\partial f}{\partial x^j} - \frac{\partial g}{\partial x^j} f_m - g \frac{\partial f_m}{\partial x^j}\right) dx^1 \wedge \ldots \wedge dx^n \\
& \leqq 2 \int_R g^{\frac{1}{2}} g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} (f - f_m)^2 \, dx^1 \wedge \ldots \wedge dx^n + \\
& \quad + 2 \int_R g^{\frac{1}{2}} g^{ij} \left(\frac{\partial f}{\partial x^i} - \frac{\partial f_m}{\partial x^i}\right) \left(\frac{\partial f}{\partial x^j} - \frac{\partial f_m}{\partial x^j}\right) g^2 \, dx^1 \wedge \ldots \wedge dx^n \\
& \leqq 2 \int_K g^{\frac{1}{2}} g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} (f - f_m)^2 \, dx^1 \wedge \ldots \wedge dx^n + \\
& \quad + 2 \int_{R-K} g^{\frac{1}{2}} g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} (f - f_m)^2 \, dx^1 \wedge \ldots \wedge dx^n + \\
& \quad + 2 \int_R g^{\frac{1}{2}} g^{ij} \left(\frac{\partial f}{\partial x^i} - \frac{\partial f_m}{\partial x^i}\right) \left(\frac{\partial f}{\partial x^j} - \frac{\partial f_m}{\partial x^j}\right) g^2 \, dx^1 \wedge \ldots \wedge dx^n \\
& \leqq 2 \int_K g^{\frac{1}{2}} g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} (f - f_m)^2 \, dx^1 \wedge \ldots \wedge dx^n + \\
& \quad + 2N \int_{R-K} g^{\frac{1}{2}} g^{ij} \frac{\partial g}{\partial x^i} \frac{\partial g}{\partial x^j} \, dx^1 \wedge \ldots \wedge dx^n + \\
& \quad + 2M \int_R g^{\frac{1}{2}} g^{ij} \left(\frac{\partial f}{\partial x^i} - \frac{\partial f_m}{\partial x^i}\right) \left(\frac{\partial f}{\partial x^j} - \frac{\partial f_m}{\partial x^j}\right) dx^1 \wedge \ldots \wedge dx^n \,,
\end{aligned}$$

where K is any compact subset of R, M is a bound for g^2 on R, and N is a bound for $\{|f-f_m|^2, m=1,2,\ldots\}$ on R; N exists on account of B-convergence.

By U-convergence on K and D-convergence of $\{f_m\}$ on R, the formula (5) gives

$$\limsup_{m\to\infty} D_R(gf-gf_m) \leqq 2N D_{R-K}(g) .$$

Since K is any compact subset,

$$\lim_{m\to\infty} D_R(gf-gf_m) = 0 .$$

Hence $gf = \text{BD-}\lim_{m\to\infty} gf_m$ on R and $gf \in M_\Delta(R)$.

The following concept plays an important role in our approach.

DEFINITION. For a Riemannian space R with its Royden's algebra $M(R)$ and Royden's compactification R^*, the set

$$\Delta = \{p \in R^* \mid f(p)=0 \text{ for each } f \in M_\Delta(R)\}$$

is the harmonic boundary of R.

Clearly, Δ is a closed subset of R^* and is disjoint from R.

6.

We are ready to generalize to Riemannian spaces the fundamental harmonic decomposition theorem (cf. Royden [7], Nakai [6]):

THEOREM 7. *Let $f \in M(R)$. Then there exist functions $u \in \text{HBD}(R)$ and $g \in M_\Delta(R)$ such that $f=u+g$ on R. Moreover, the decomposition is unique, if Δ is nonempty.*

PROOF. Let $\{\Omega_m\}_{m=1}^{\infty}$ be a regular exhaustion of R. For each $m=1,2,\ldots$, let u_m be the continuous function on R defined by

$$u_m \mid R-\Omega_m = f \mid R-\Omega_m, \quad u_m \mid R_m \in H(\Omega_m) .$$

For $m<k$, by Green's formula, $D_{\Omega_k}(u_m-u_k, u_k)=0$, and therefore

$$D_R(u_m) = D_R(u_k)+D_R(u_m-u_k) .$$

Thus

$$D_R(u_m) \geqq D_R(u_k) \geqq 0$$

for $m<k$. We infer that $D_R(u_m)$ converges and $D_R(u_m-u_k)$ tends to zero, as $m,k \to \infty$. Since $\{u_m\}$ is uniformly bounded, by compactness, we may

assume, without loss of generality, that u_m converges uniformly in compact subsets to a harmonic function u on R. Owing to BD-completeness of $M(R)$, $u \in M(R)$, and $u \in \mathrm{HBD}(R)$. Let $g_m = f - u_m$, $m = 1, 2, \ldots$. Clearly the g_m's are in $M_0(R)$ and BD-converge to $g = f - u$.

To prove the uniqueness of the decomposition, we suppose that $f = u + g = \bar{u} + \bar{g}$, with $u, \bar{u} \in \mathrm{HBD}(R)$ and $g, \bar{g} \in M_\Delta(R)$. Clearly $u - \bar{u} = \bar{g} - g \in M_\Delta(R)$. Let $v = u - \bar{u}$. Then $v \in \mathrm{HBD}(R) \cap M_\Delta(R)$. Hence there is a sequence of functions $v_m \in M_0(R)$ such that $v = \text{BD-lim}_{m\to\infty} v_m$. Therefore

$$\begin{aligned} D_R(v) &= \lim_{m\to\infty} D_R(v_m, v) \\ &= \lim_{m\to\infty} D_{\Omega_k}(v_m, v) = \lim_{m\to\infty} \int_{\partial\Omega_k} v_m * dv = 0\,, \end{aligned}$$

where $k > m$ is so large that the support of v_m is in Ω_k. Thus v is constant. If Δ is not empty, then we conclude that $v \equiv 0$ on R.

Henceforth we use the notation $\pi(f)$ for u. In view of the proof, we can state:

COROLLARY 1. *If $f \in M(R)$ and $f \leqq 0$ on R, then $\pi(f) \leqq 0$ on R.*

COROLLARY 2. *If $f \in M(R)$, then*

$$\sup_R |f| \geqq \sup_R |\pi(f)|\,.$$

COROLLARY 3. *If $f \in M(R)$ is subharmonic (or superharmonic), then $\pi(f) \geqq f$ (or $\pi(f) \leqq f$) on R.*

COROLLARY 4. *If $f \in M(R)$ and for some superharmonic (or subharmonic) function v on R. $v \geqq f$ (or $v \leqq f$) on R, then $v \geqq \pi(f)$ (or $\pi(f) \geqq v$) on R.*

The following theorem shows the importance of the harmonic boundary (Mori–Ôta [3]).

THEOREM 8 (*maximum principle*). *Let u be an* HBD-*function on R. Then*

$$\inf_R u = \min_\Delta u, \quad \sup_R u = \max_\Delta u\,.$$

PROOF. If Δ is empty, it is easily seen that u is constant and hence the theorem is trivial. We therefore assume that $\Delta \neq \emptyset$.

The statement is a direct consequence of the denseness of R in R^* and the fact that $u(\Delta) \leqq 0$ implies $u \leqq 0$ in R. To see this, consider the set $A=\{p\in R^* \mid u(p)\geqq\varepsilon\}$ for any $\varepsilon>0$. Clearly $A\cap\Delta=\emptyset$, and hence, for each $p\in A$, there is a $f_p\in M_\Delta(R)$ such that $f_p(p)\geqq 2$ and $f_p\geqq 0$ on R^*. Therefore

$$\{U_p=\{q\in R^* \mid f_p(q)>1\}\mid p\in A\}$$

forms an open covering of A. But A is compact in R^*, and thus there exist points $p_1,\ldots,p_m$ in A such that $\bigcup_{k=1}^m U_{p_k}\supset A$. Hence $f=\sum_{k=1}^m f_{p_k}$ is in $M_\Delta(R)$ and $f>1$ on A. Since u is bounded from above, there exists an M such that $u-Mf\leqq 0$ on A. Then $u-Mf-\varepsilon\leqq 0$ on R^*. By the decomposition theorem,

$$u-Mf-\varepsilon = v+g$$

with $v\in \mathrm{HBD}(R)$ and $g\in M_\Delta(R)$, and $u-\varepsilon=v$, for $\Delta\neq\emptyset$. Corollary 1 gives $v\leqq 0$ on R^*, and thus $u-\varepsilon\leqq 0$ on R^*. Thus we obtain $u\leqq+\varepsilon$ on R^*. A fortiori $u\leqq 0$ on R^*.

COROLLARY 5. *Let u be a subharmonic (or superharmonic) function in $M(R)$. Then*

$$\sup_R u = \max_\Delta u \quad (\textit{or } \inf_R u = \min_\Delta u)\,.$$

PROOF. If u is subharmonic, the above reasoning applies except that we use Corollary 3 to obtain $u-\varepsilon\leqq v$.

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